UN 802

Reactor Physics

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Atomic and Nuclear Constituents

Atomic and Nuclear Constituents

Atom

- Electrons
- Nucleus - made up of nucleons:
  - protons
  - neutrons

The nucleus is “held together” by nuclear attraction forces. These have to be stronger than the repulsive electrostatic forces.

For neutral atoms, the number of protons in the nucleus equals the number of electrons in orbit.

Some subatomic particles

- proton
- neutron
- electron (beta particle)
- positron
- photon (gamma particle)
- neutrino
- antineutrino
- alpha particle (2 protons + 2 neutrons)

Properties (quantities) characterizing subatomic particles

- Mass (rest mass)
- charge
- spin (denoted by s)
- parity
  - property resulting from Quantum Mechanics.
  - describes the parity of the wave function

$$+ \Leftrightarrow \psi(\vec{r}) = \psi(-\vec{r})$$

$$- \Leftrightarrow \psi(\vec{r}) = -\psi(-\vec{r})$$

All these quantities are important because they are conserved in nuclear reactions.

Properties of Nuclei

- Atomic number – Z = number of protons
- Mass number – A = total number of nucleons (protons and neutrons)
- Number of neutrons – N

The atomic number Z identifies the nuclear species.

Two nuclei with the same Z but different N are called **isotopes**.

Notation: ${}^{A}_{Z}X$, where X is the chemical symbol.

Other properties of nuclei (parallel those of particles)

- Mass
- charge (+Ze)
- spin (s)
- parity

Atomic Mass Unit (amu)

- Defined as 1/12 of the mass of a ^{12}C **atom**
  That means that it is 1/12 of the ^{12}C nucleus, plus the mass of 1/2 electron.
- Atomic weight = Mass of an atom expressed in amu
- Molecular weight = Mass of a molecule expressed in amu
- 1 Mole – Quantity of a pure substance that has the same mass expressed in grams as the atom's (or molecule's) mass expressed in amu.
- 1 Mole Has $N_A = 6.023 \times 10^{23}$ atoms (molecules)
- N_A is the ratio between 1g and 1 amu.
  (There are N_A amus in a gram.)

How many Kg does an amu have?

N_A atoms of ^{12}C weigh 12 g. It follows that 1 amu weighs $1/N_A$ grams.

$$1\,\text{amu} = \frac{12(g)}{12 \times N_A} = \frac{1}{N_A}(g) = \frac{1}{6.023 \times 10^{23}}(g) = 1.66 \times 10^{-24}(g) = 1.66 \times 10^{-27}(kg)$$

Expressing mass using energy

Because of the mass-energy equivalence expressed by Einstein's formula $E = mc^2$, mass can also be expressed in units of energy over c^2.

For example:

$$1kg = \frac{1kg \times c^2}{c^2} = \frac{1 \times \left(3 \times 10^8\right)^2 (J)}{c^2} = 9 \times 10^{16} \left(\frac{J}{c^2} \right)$$

Expressing mass using energy

In nuclear physics the energy is often measured in MeV, and the mass in MeV/c^2. To find the relation between 1kg and one MeV/c^2 we write:

$$1\frac{MeV}{c^2} = \frac{10^6 eV}{\left(3\times10^8 (m/s)\right)^2} = \frac{10^6 \times 1.602\times10^{-19} C\times V}{\left(3\times10^8 (m/s)\right)^2}$$

$$= \frac{10^6 \times 1.602\times10^{-13} J}{\left(3\times10^8 (m/s)\right)^2} = 1.78\times10^{-24} Kg$$

Instead of saying that the mass if a particle is X MeV/c^2 it is customary to just say that the mass is x MeV. What is really meant is that the total energy of that particle is X MeV, and hence its mass is X MeV/c^2. One just omits mentioning c^2.

MeV Equivalent of 1 amu

$$E_{amu} = \frac{1}{N_A} 1g \times c^2 = \frac{1}{6.023 \times 10^{23}} \times 10^{-3} Kg \times \left(3 \times 10^8 \frac{m}{s} \right)^2 =$$

$$\frac{1}{6.023 \times 10^{23}} \times 10^{-3} \times \left(3 \times 10^8 \right)^2 J =$$

$$1.494 \times 10^{-7} J \frac{J}{MeV} MeV = .494 \times 10^{-7} \frac{C \times V}{10^6 e \times V} MeV =$$

$$1.494 \times 10^{-7} \frac{C \times V}{10^6 \times 1.6 \times 10^{-19} C \times V} MeV = 933 MeV$$

Examples of elementary particle mass

particle	mass		
	kg	amu	MeV/c^2
proton	1.6726E-27	1.007276	938.28
neutron	1.6750E-27	1.008665	939.57
electron	9.1090E-31	5.486E-4	0.511

Atomic Weight for a Mixture of Atoms

Consider a mixture of 30% (by atom) C and 70% (by atom) Al. What is the average atomic weight of the mixture?

Answer

- Assume there are N atoms in total
- of these
  - N_C=0.3N are C
  - N_{Al}=0.7N are Al

Atomic Weight for a Mixture of Atoms

The total mass of the mixture (in amu) is:

$$m = N_C M_C + N_{Al} M_{Al} = 0.3NM_C + 0.7NM_{Al}\,(amu)$$

The average mass of one atom (in amu) is:

$$\overline{M} = \frac{m}{N} = \frac{0.3NM_C + 0.7NM_{Al}}{N} =$$

$$= 0.3M_C + 0.7M_{Al} = 0.3 \times 12 + 0.7 \times 13 = 12.7\,(amu)$$

Atomic Weight for a Mixture of Atoms

In general

For a mixture of n types of atoms, each with atomic fraction $X_i=N_i/N$, the average atomic weight is:

$$M = \sum_{i=1}^{n} X_i M_i$$

If the different types of atoms are isotopes of the same atom, the atomic fractions are called *isotopic abundances*.

Properties and Structure of Nuclei

Nuclear Radius

Assume that nuclei are made of “nuclear material” of the same density ρ for all species of nuclei.

It follows that the mass of the nucleus is given by:

$$m = \rho V = \rho \frac{4\pi R^3}{3}$$

Nuclear Radius

The mass of the nucleus is given also by the mass of its constituents (neutrons and protons)

$$m = Nm_n + Zm_p$$

Because the mass of the proton and the one of the neutron are almost equal to 1 amu, we can write:

$$m = Nm_n + Zm_p \cong N\,amu + Z\,amu =$$

$$(N + Z)\,amu = A\,amu$$

Nuclear Radius

By writing the equality between the two masses, we have:

$$A\,amu = \rho \frac{4\pi R^3}{3}$$

Solving for R^3, we obtain

$$R^3 = A\left(\frac{3}{4\pi\rho}\,amu\right)$$

Nuclear Radius

Solving for R, by taking the cube root on both sides, we have:

$$R = \sqrt[3]{A}\sqrt[3]{\frac{3}{4\pi\rho}amu}$$

It turns out that:

$$\sqrt[3]{\frac{3}{4\pi\rho}amu} = 1.25\times10^{-15}\,m$$

So:

$$R = 1.25\times10^{-15}\times\sqrt[3]{A}\ \text{meters}$$

Binding Energy

Since particles that constitute the nucleus stay together (held by nuclear interaction forces), the total (rest) energy of the nucleus must be lower than the total (rest) energy of the particles if they were separated.

$$B = \left[\left(A - Z \right) \times E_{neutron} + Z \times E_{proton} \right] - E\left({}^{A}_{Z}X \right)$$

This is called the **Binding Energy**

In the above, E denotes rest energy.

Binding Energy

Einstein's energy formula translates into:

$$E_{neutron} = m_{neutron}c^2$$

$$E_{proton} = m_{proton}c^2$$

$E\left({}^{A}_{Z}X\right) = M\left({}^{A}_{Z}X\right)c^2$ (M is rest mass of the nucleus.)

$$B = \left[\left(A-Z\right)\times m_{neutron}c^2 + Z\times m_{proton}c^2\right] - M\left({}^{A}_{Z}X\right)c^2$$

$$B = c^2\left\{\left[\left(A-Z\right)\times m_{neutron} + Z\times m_{proton}\right] - M\left({}^{A}_{Z}X\right)\right\} = c^2\Delta$$

The mass of the nucleus is smaller than the sum of the masses of its constituents

The difference, Δ, is called the ***mass defect***

Alternate expression for the mass defect

Using the nuclear mass to calculate the mass defect can be difficult because, most of the time, what is given in tables is the mass of neutral atoms, rather than the mass of their nuclei.

To use the atomic masses instead of the nuclear masses, we can add and subtract the mass of electrons. (We will also ignore the binding energy of the electrons. However, that energy is much smaller that the nuclear binding energy, so we can safely neglect it.) Hence:

$$\Delta = \left[\left(A-Z\right)\times m_{neutron} + Z\times m_{proton}\right] - M\left({}^{A}_{Z}X\right) =$$
$$\left[\left(A-Z\right)\times m_{neutron} + Z\times\left(m_{proton} + m_{e}\right)\right] - \left(M\left({}^{A}_{Z}X\right) + Z\times m_{e}\right) =$$
$$\left(A-Z\right)\times m_{neutron} + Z\times \mathrm{M}\left({}^{1}_{1}H\right) - \mathrm{M}\left({}^{A}_{Z}X\right)$$

Where $\mathrm{M}^{0}\left({}^{A}_{Z}X\right)$ is the **atomic** mass of element ${}^{A}_{Z}X$

Question Period

- Q: If I climb to the top of the CN tower (approximately 550 m) will my body mass be larger?
- A: Yes, but not enough for people to notice.

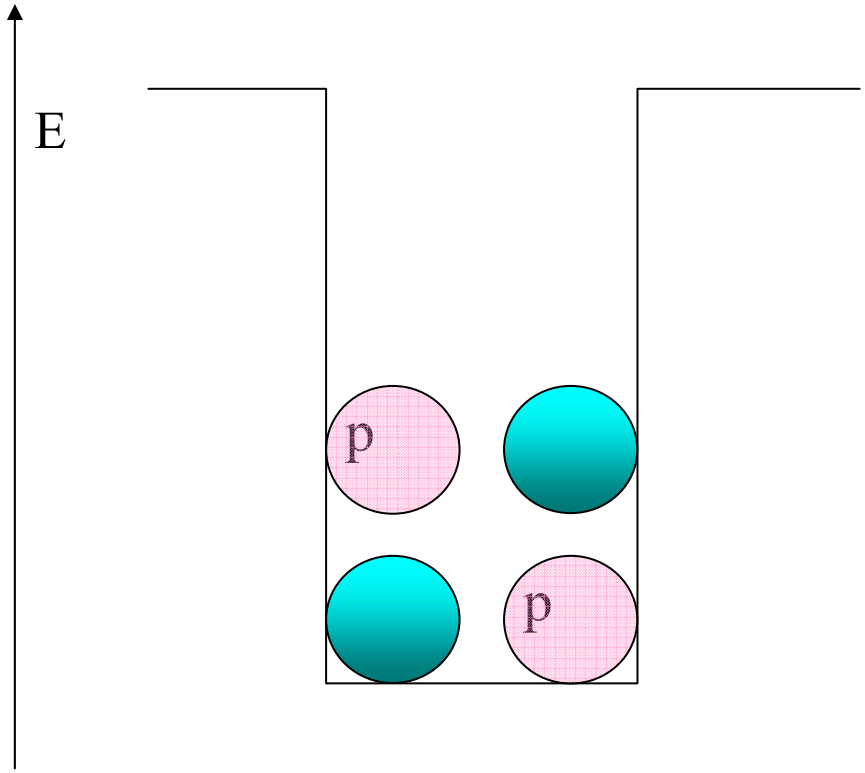
$$m_{top} - m_{bottom} = \frac{m_{bottom}c^2 + m_{bottom}gh}{c^2} - m_{bottom} =$$

$$\frac{m_{bottom}gh}{c^2} = \frac{70Kg \times 9.8m/s^2 \times 550m}{\left(3\times10^8 m/s\right)^2} = 4.2\times10^{-12}\,Kg$$

Nuclear Models

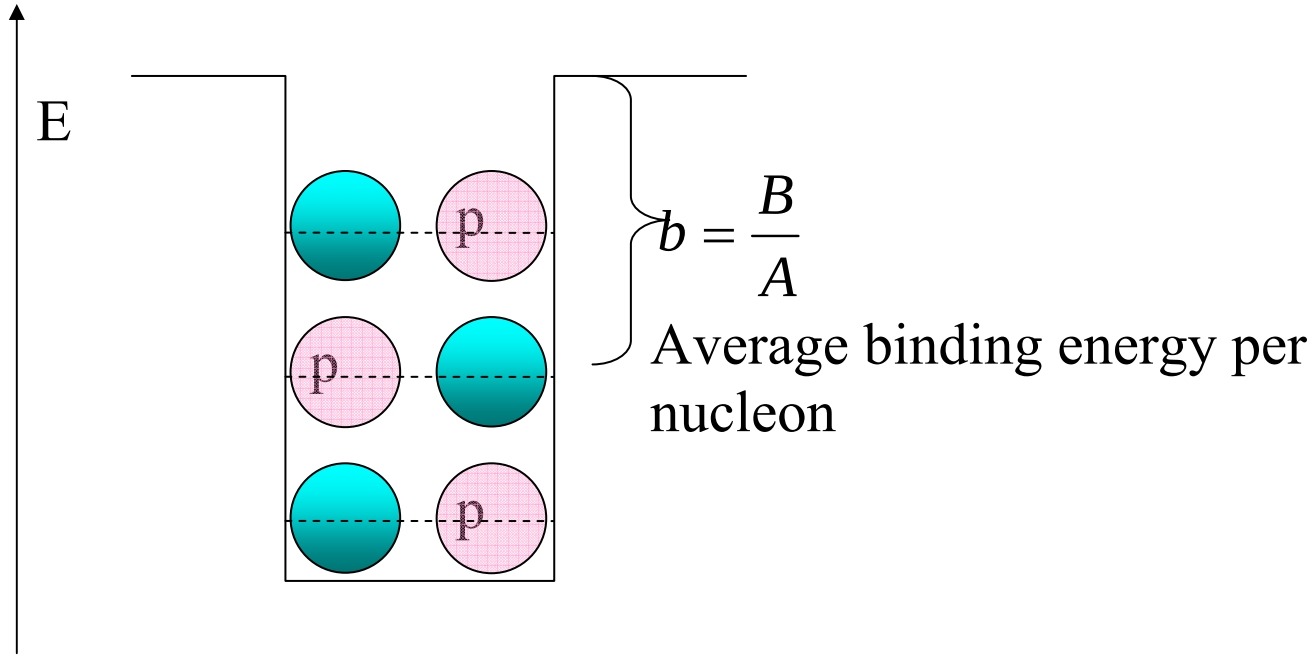
Shell Model - Potential Well

- We can picture the nucleons (protons and neutrons) as "living" in a "potential well" created by the nuclear forces.
- The binding energy is the energy that needs to be communicated to the nucleons to allow all of them to exit the well.



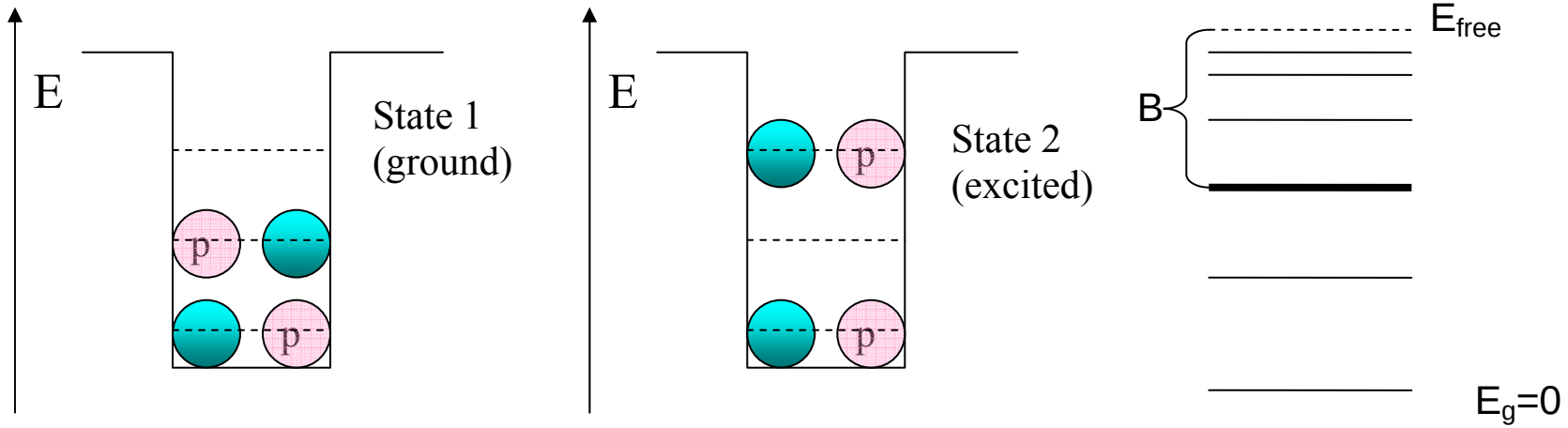
More details on the potential well

- Nucleons can occupy different energy levels in the well, just like electrons can occupy different energy levels in an atom.
- The state of the nucleus is given by the states (energy, spin, parity) of all its nucleons.
- Pauli's exclusion principle applies (No two nucleons can occupy the same state).



More details on the potential well

- Depending on the "arrangement" of nucleons on energy levels inside the well, the nucleus can have different binding energies.
- The lowest energy level of the nucleus (corresponding to the largest binding energy) is called the **ground level**, and the corresponding state is called the **ground state**.
- Higher energy levels are called **excited levels**, and the corresponding states are called **excited states**.

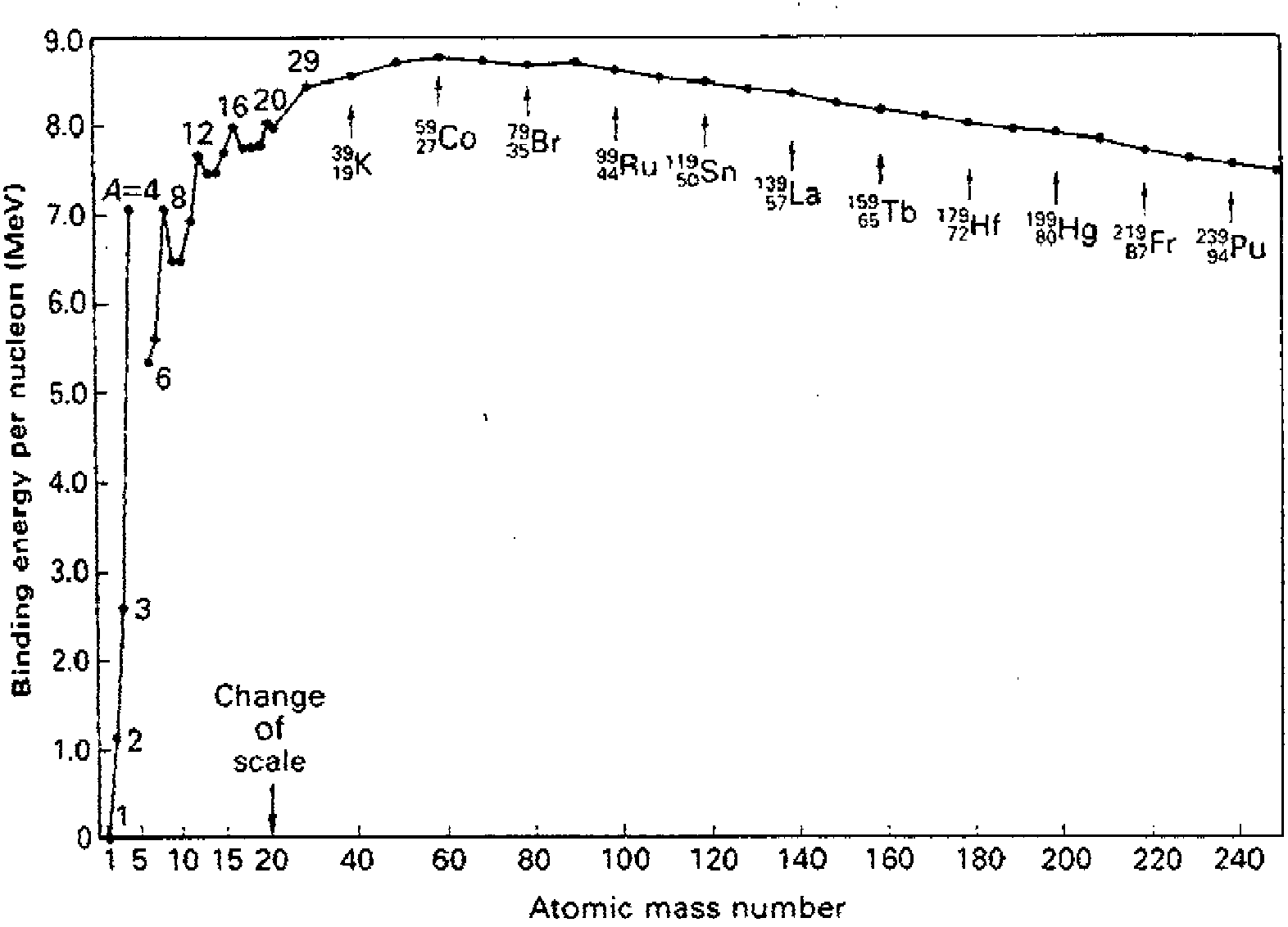


$$E_{excited} > E_{ground}$$

$$B_{ground} > B_{excited}$$

$$M\left({}^{N}_{Z}X^{excited}\right) > M\left({}^{N}_{Z}X^{ground}\right)$$

Binding Energy per Nucleon



Liquid Drop Nuclear Model

- Attempts to express the binding energy as a function of nuclear characteristics.
- Leads to a semiempirical formula.
  - Shape of formula determined from the model
  - Values of constants determined from measurement

$$B = a_v A - a_s A^{\frac{2}{3}} - a_c \frac{Z(Z-1)}{A^{\frac{1}{3}}} - a_A \frac{(A-2Z)^2}{A^2} + \delta(Z,A)$$

Liquid Drop Model – Meaning of Terms

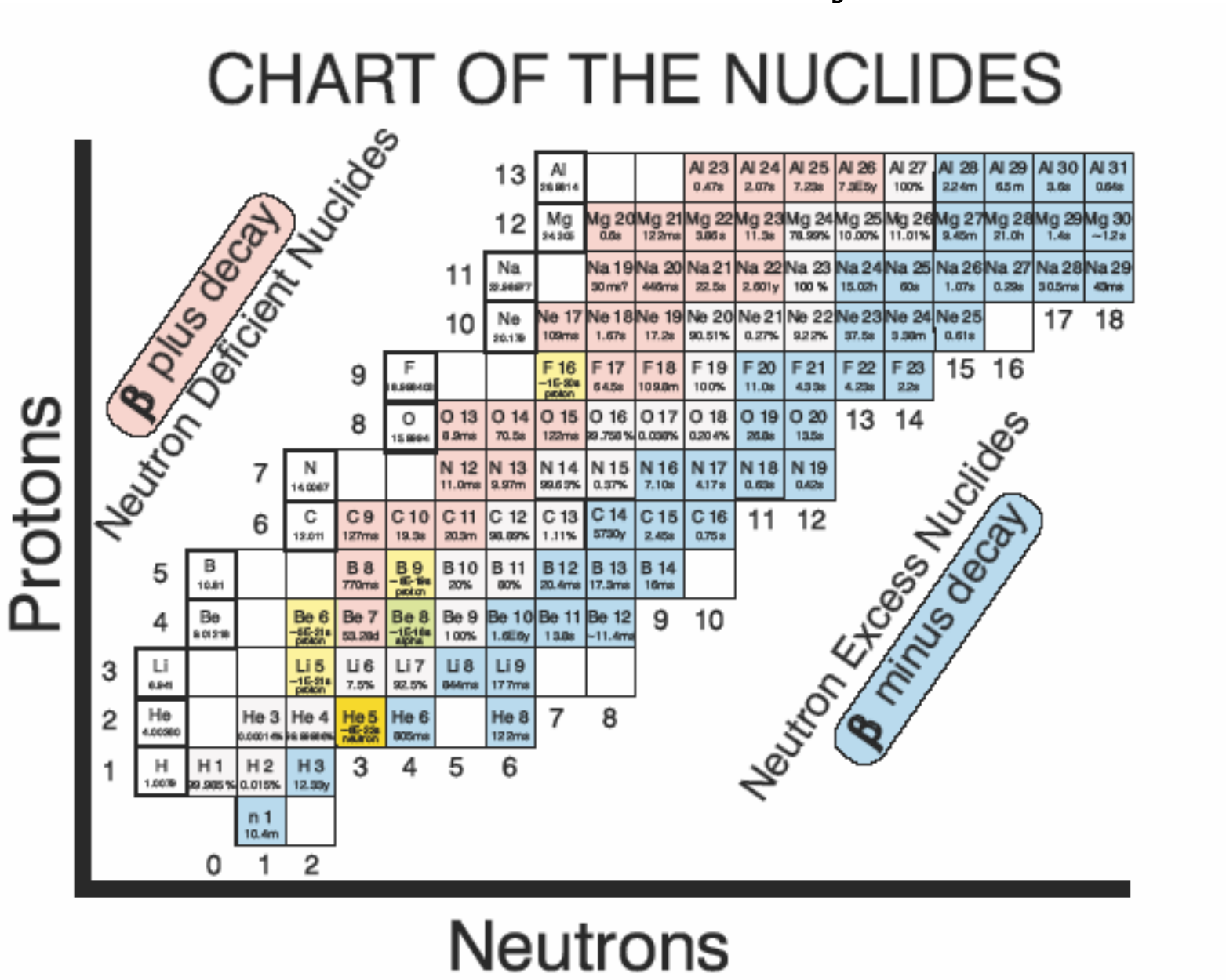
- $\mathbf{a_v}$ – Volume effect – proportional to the "volume" of the nucleus, which can be considered to be roughly proportional to A. This term was introduced because it was observed that the binding energy per nucleon is almost constant.
- $\mathbf{a_s}$ – Surface effect – proportional to the "surface" of the nucleus, roughly proportional to $A^{2/3}$. This negative term was introduced because the nucleons situated close to the surface have fewer neighbors, and hence contribute less to the binding energy.
- $\mathbf{a_c}$ – Coulomb effect – electrostatic repulsion between protons has a potential energy $\frac{Z(Z-1)e^2}{r} \propto \frac{Z(Z-1)}{A^{\frac{1}{3}}}$
- $\mathbf{a_A}$ – Asymmetry effect. It vas observed that nuclei with N=Z are more stable, hence the binding energy is probably smaller if Z and N differ. This term accounts for that effect.
- $\delta(Z,A)$ - Pairing term. Introduced because it was found experimentally that two protons or two neutrons are bound stronger than a proton and a neutron. It is zero for odd A, $-a_p \frac{1}{A^{\frac{1}{2}}}$ for both Z and N odd and $+a_p \frac{1}{A^{\frac{1}{2}}}$ for both Z and N even.

Liquid Drop Model Numerical values of coefficients

a_V	15.7 MeV
a_S	17.8 MeV
a_C	0.71 MeV
a_A	23.6 MeV
a_P	12.0 MeV

Radioactivity

Nuclear Stability



Radioactivity

- Discovered first by Henri Becquerel (1852-1908).
- Becquerel discovered that a mineral containing Uranium would darken a photographic plate even when the latter was wrapped in opaque paper.
- In 1903 Becquerel shared the Physics Nobel Prize with Pierre and Marie Curie, for their discovery and work on radioactivity.

Radioactive Decay

- Some nuclei are **stable**, while others are **unstable**.
- Unstable nuclei **decay**, by emitting a particle and changing into a different nucleus.
- Most common types of decay (others possible too):
  - Alpha ($^{4}_{2}\alpha$), Helium nucleus emission
  - Beta ($^{0}_{-1}\beta$), electron emission
  - Beta plus ($^{0}_{1}\beta$) positron emission
  - Gamma ($^{0}_{0}\gamma$), photon emission (no change in nuclear species)
  - Electron capture (an electron is “captured” rather than emitted)

Radioactive Decay

$$ {}^{A}_{Z}X \rightarrow {}^{A-m}_{Z-n}Y + {}^{m}_{n}P $$

(Parent Nucleus → Daughter Nucleus + Emitted Particle)

- Charge and number of nucleons are conserved.
- For gamma decay, technically the nucleus does not change into a different one. Only its energy state changes.
- Electron capture (still classified as "decay")

$$ {}^{A}_{Z}X + {}^{0}_{-1}e \rightarrow {}^{A}_{Z-1}Y $$

Alternative Notation (no chemical symbol)

- General decay

$$(Z, A) \rightarrow (Z - n, A - m) + (n, m)$$

- Alpha

$$(Z, A) \rightarrow (Z - 2, A - 4) + {}^{4}_{2}\alpha$$

- Beta minus

$$(Z, A) \rightarrow (Z + 1, A) + {}^{0}_{-1}\beta + \tilde{\upsilon}$$

- Electron capture

$$(Z, A) + e^{-} \rightarrow (Z - 1, A)$$

$$(Z, A) + {}^{0}_{-1}\beta \rightarrow (Z - 1, A)$$

Characteristics of Radioactive Decay

- Nuclei decay randomly.
  - It is impossible to predict which nuclei will decay in a given period of time, and which not.
  - It is impossible to predict when a particular nucleus will decay.
- On average, for large initial numbers of nuclei and for short periods of time Δt, the number of nuclei that decay within Δt is proportional to the time Δt, and to the original number of nuclei present at the beginning of the time interval.

Derivation of the Law of Radioactive Decay

- Let N(t) be the number of X-type nuclei present at time t.
- Let Δt be a short time interval.
- According to the second bullet on the previous slide, we have, on average:

$$\Delta N = -N(t) + N(t + \Delta t) = -\lambda \times N(t) \times \Delta t \qquad (1)$$

- λ is called the decay constant, is dependent on nucleus type, and is measured in s^{-1}.

Derivation of the Law of Radioactive Decay

- The previous can be rewritten as:

$$\frac{\Delta N}{\Delta t} = -\lambda N(t) \qquad (2)$$

- which, considering that Δt is small, yields:

$$\frac{dN}{dt} = -\lambda N(t) \qquad (3)$$

Derivation of the Law of Radioactive Decay

Eq. (3) is an ordinary differential equation with constant coefficients. Its solution is of the form:

$$e^{-\lambda t+c} \equiv Ce^{-\lambda t}\ ; \qquad C = e^{c}$$

The multiplicative constant C can be determined from the number of nuclei present at t=0.

$$N(0) \equiv N_0 = C \times e^{-\lambda \times 0} = C$$

It follows that the number of X-type nuclei is given at any time t by:

$$N(t) = N_0 \times e^{-\lambda t}$$

Law of Radioactive Decay

Example

- At t=0, a sample of ^{24}Na weights 1.0 mg. How many beta particles are emitted in an hour? ($\lambda = 1.2836 \times 10^{-5}\ s^{-1}$)
- Solution
  - The number of emitted particles equals the number of decayed nuclei:

$$\Delta N = N_0 - N(t) = N_0 - N_0 \times e^{-\lambda t} = N_0 \times \left(1 - e^{-\lambda t}\right)$$

- The initial number of Na nuclei is:

$$N_0 = \frac{m}{M} = \frac{1.0 \times 10^{-6}\, Kg}{24 amu} = \frac{1.0 \times 10^{-6}}{24} \times \frac{Kg}{amu} =$$

$$\frac{1.0 \times 10^{-6}}{24} \times N_A = \frac{1.0 \times 10^{-6}}{24} \times 6.023 \times 10^{23} = 2.51 \times 10^{16}$$

- Hence the number of emitted particles is:

$$\Delta N = 2.51 \times 10^{16} \times \left(1 - e^{-1.2836 \times 10^{-5} \times 3600}\right) = 1.133 \times 10^{15}$$

Half Life

- Definition
  - The half life, $T_{1/2}$, of a radioactive species is the time after which the initial number of nuclei decreases to one half.
- Expression
  - By definition:

$$N(T_{1/2}) = \frac{N_0}{2}$$

Expression of Half-Life

- The definition of half life is equivalent to:

$$N_0 \times e^{-\lambda \times T_{1/2}} = \frac{N_0}{2}$$

- Dividing be N_0 we obtain:

$$e^{-\lambda \times T_{1/2}} = \frac{1}{2}$$

- By taking the natural logarithm of both sides we get:

$$-\lambda \times T_{1/2} = \ln\left(\frac{1}{2}\right) = -\ln(2)$$

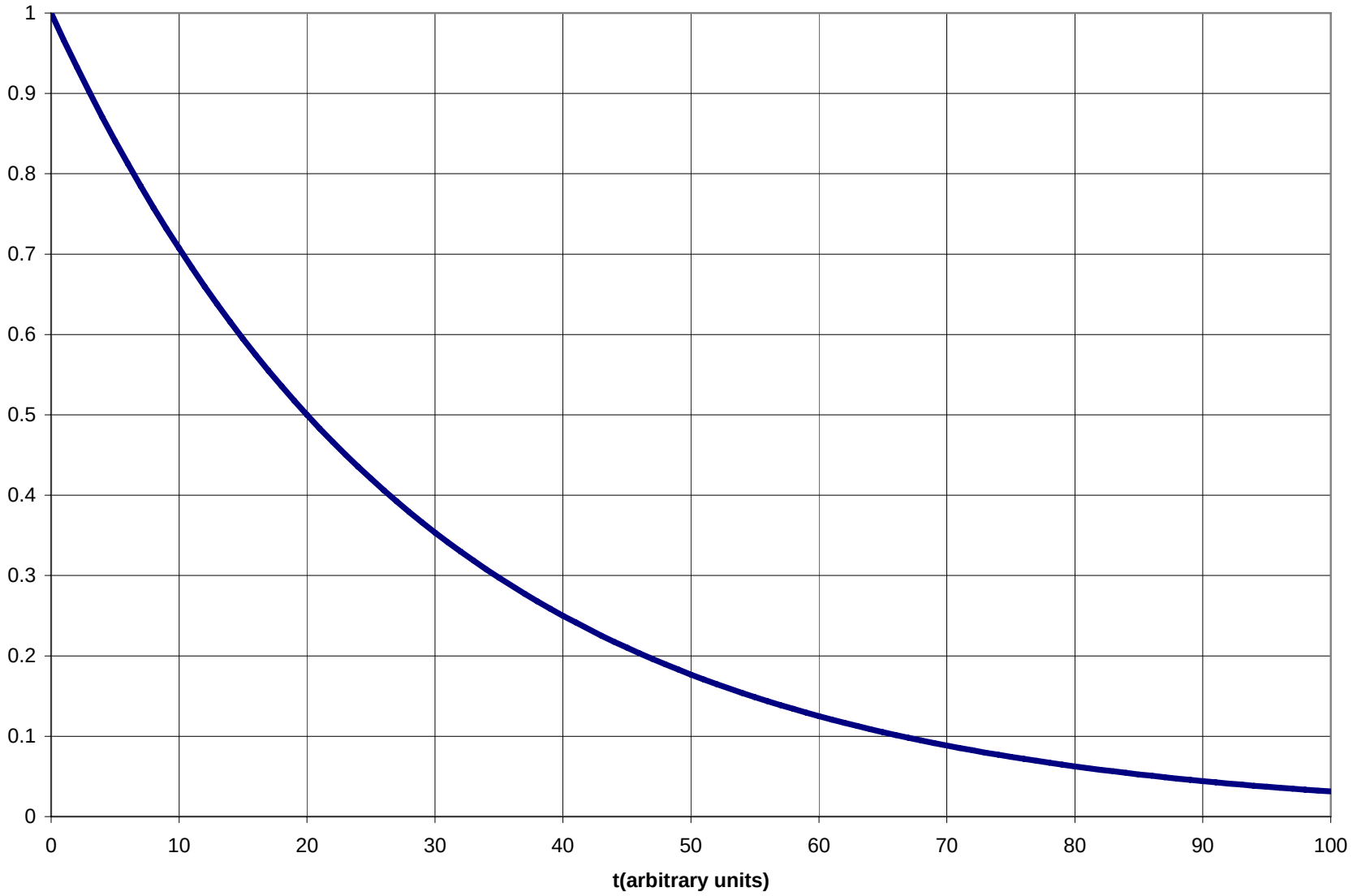
- Finally, we can solve for $T_{1/2}$:

$$T_{1/2} = \frac{\ln(2)}{\lambda}$$

Radioactive Decay and Half Life Important Notes

- Half life can be measured from any moment of time. The number of nuclei left after $T_{1/2}$ elapses will be half of those existent at t_0.
- According to the radioactive decay law, the number of parent nuclei keeps halving every $T_{1/2}$, but never reaches zero. However, it can become negligibly small.
- As the number of remaining nuclei becomes small, deviations from the law of radioactive decay start to appear, as the law of radioactive decay is valid ***on average***.
  We cannot have 2.5 parent nuclei left. What such a number means is that we can have 2 or maybe 3 nuclei left in different experiments, such that the average is 2.5

Exponential Decay



Law of Radioactive Decay – Probabilistic Interpretation

- N(t) out of N_0 nuclei do not decay.
- It cannot be determined *a priori* which nuclei do not decay and which do.
- The ratio $N(t)/N_0$ can be interpreted as the probability of one nucleus **not decaying** after time t.

$$\frac{N(t)}{N_0} = P_{ND} = e^{-\lambda \times t}$$

- Conversely, the probability that a nucleus **does decay** after time *t* is:

$$P_D = 1 - P_{ND} = 1 - e^{-\lambda \times t} \neq e^{-\lambda \times t}$$

Activity

- The rate at which a radioactive sample decays is called **activity.**

$$\Lambda(t) \equiv -\frac{dN(t)}{dt}$$

- Equivalent definition

$$\Lambda(t) = -\lambda N(t) \Leftarrow -\frac{d}{dt} N_0 e^{-\lambda t} = \lambda N_0 e^{-\lambda t} = \lambda N(t)$$

- Units:
  - 1 decay/second = 1 Becquerel (Bq)
  - 1 Curie = 3.7×10^{10} Bq

Average Life of a Nucleus

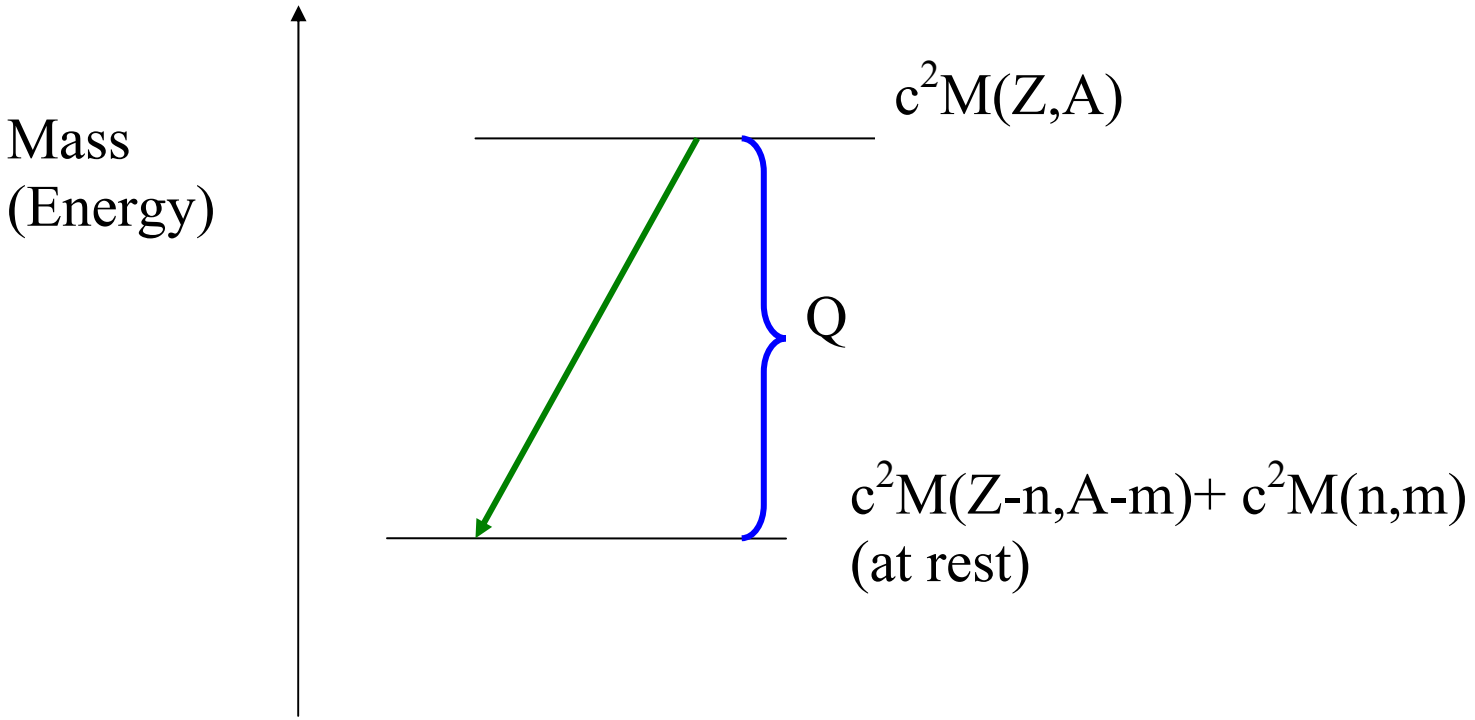
- At t=0 there are N_0 parent nuclei.
- At time *t*, there are N parent nuclei left.
- At time *t*, $\Lambda(t)dt = \lambda N_0 e^{-\lambda t} dt$ decay in *dt*
- These nuclei have "lived" *t* before decaying.
- To get the average life, we need to sum (integrate) over *dt* and divide by the initial number of nuclei.

$$\tau = \frac{\int_0^\infty t\Lambda(t)dt}{N_0} = \frac{\int_0^\infty t\lambda N_0 e^{-\lambda t} dt}{N_0}$$

$$= \frac{N_0\lambda\int_0^\infty te^{-\lambda t} dt}{N_0} = \lambda\int_0^\infty te^{-\lambda t} dt =$$

$$\lambda\left[\left(te^{-\lambda t}\right)\Big|_0^\infty + \int_0^\infty \frac{1}{\lambda}e^{-\lambda t} dt\right] = \int_0^\infty e^{-\lambda t} dt = -\frac{1}{\lambda}\left(e^{-\lambda t}\right)\Big|_0^\infty = \frac{1}{\lambda}$$

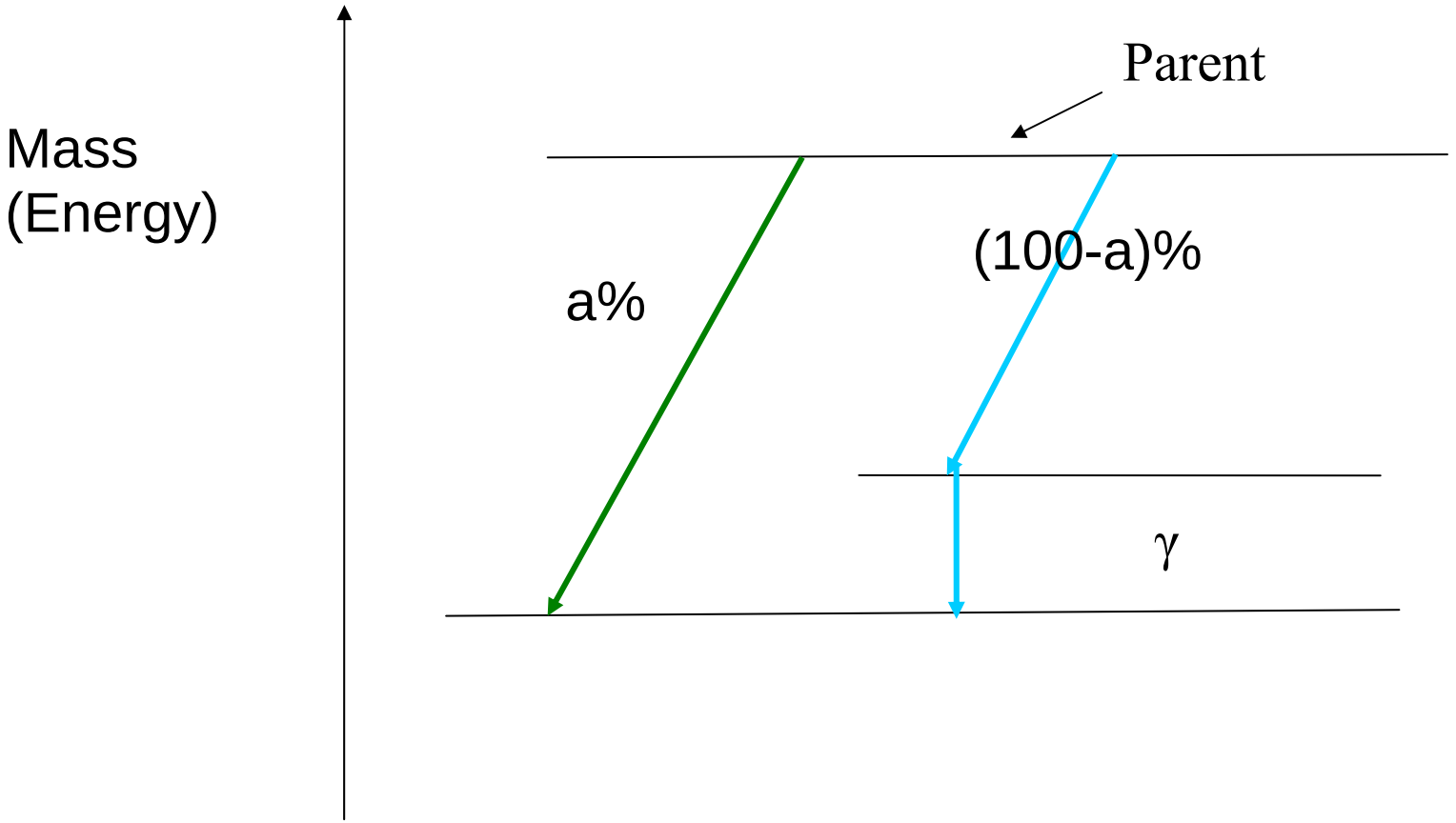
Energy-Level Diagrams for Decay and Decay Scheme



- $Q=[M(Z,A)-M(Z-n,A-m)-M(n,m)]c^2$
- $Q>0$ in order for the decay to be energetically possible
- By convention, the lowest energy on this graph is taken to be zero (Energy is expressed relative to the lowest value.).

Multimodal Decay

- Some nuclei can decay in more than one way



Branching factors

- Fraction of nuclei that decay in a certain mode
- Have to add up to 1. (100%)

- Consider a species of nucleus that can decay by either a reaction 1, or another reaction 2.
- Let *dN* be the total number of nuclei that decay in *dt*. The branching factors are defines as:

$$f_1 = \frac{dN_1}{dN}$$

$$f_2 = \frac{dN_2}{dN}$$

$$f_1 + f_2 = 1$$

Branching factors and derived quantities

- Partial decay constants

$$\lambda_1 = \frac{dN_1}{dN}\frac{1}{N} = \lambda f_1$$

$$\lambda_2 = \frac{dN_2}{dN}\frac{1}{N} = \lambda f_2$$

$$\lambda_1 + \lambda_2 = \lambda$$

- Partial half-lives (What the half life would be if only that decay mode was present).

$$T_1 = \frac{\ln 2}{\lambda_1}$$

$$T_2 = \frac{\ln 2}{\lambda_2}$$

$$\frac{1}{T_1} + \frac{1}{T_2} = \frac{1}{T}$$

Decay Chains
(Radioactive Families)

- Consider a nuclide whose daughter is also unstable and decays.

$$ {}^{A}_{Z}Q \rightarrow {}^{A-m}_{Z-n}R + {}^{m}_{n}P $$

$$ {}^{A-m}_{Z-n}R \rightarrow {}^{A-m-m'}_{Z-n-n'}S + {}^{m'}_{n'}P' $$

- This is called a decay chain.
- Chains can have more than two members. We then talk about radioactive families or series.

Nuclear Reactions

Nuclear Reactions

General Expression

$$ {}^{A_{x1}}_{Z_{1x}}X_1 + {}^{A_{x2}}_{Z_{x2}}X_2 \rightarrow {}^{A_{y1}}_{Z_{y1}}Y_1 + {}^{A_{y2}}_{Z_{y2}}Y_2 $$

Q value

$$ Q = \left[M\left({}^{A_{x1}}_{Z_{1x}}X_1\right) + M\left({}^{A_{x2}}_{Z_{x2}}X_2\right) - M\left({}^{A_{y1}}_{Z_{y1}}Y_1\right) - M\left({}^{A_{y2}}_{Z_{y2}}Y_2\right) \right] c^2 $$

M are rest masses of nuclei/particles

- Q>0 – exothermic reaction (provides energy to the outside)
- Q<0 – endothermic reaction (needs energy from outside in order to proceed)

Conservation Laws

- The following quantities are conserved in a nuclear reaction
  - charge
  - number of nucleons
  - energy
  - momentum

Conservation Laws

- Conservation of charge

$$Z_{x1} + Z_{x2} = Z_{y1} + Z_{y2}$$

- Conservation of number of nucleons

$$A_{x1} + A_{x2} = A_{y1} + A_{y2}$$

Conservation Laws

If additional particles enter or exit the reaction, their charge, number of nucleons, energy and momentum need to be accounted for when writing the conservation laws

Example

$$ {}^{A_{x1}}_{Z_{1x}}X_1 + {}^{A_{x2}}_{Z_{x2}}X_2 \rightarrow {}^{A_{y1}}_{Z_{y1}}Y_1 + {}^{A_{y2}}_{Z_{y2}}Y_2 + e^- $$

Conservation of charge

$$ Z_{x1} + Z_{x2} = Z_{y1} + Z_{y2} - 1 $$

We can represent the electron as ${}^{0}_{-1}e$

Conservation Laws

Conservation of momentum

$$\vec{P}(X_1) + \vec{P}(X_2) = \vec{P}(Y) + \vec{P}(Y_2)$$

Conservation of energy

– Expressed by the definition of the Q value

$$\left[M\left({}^{A_{x1}}_{Z_{1x}}X_1\right) + M\left({}^{A_{x2}}_{Z_{x2}}X_2\right)\right]c^2 = \left[M\left({}^{A_{y1}}_{Z_{y1}}Y\right) + M\left({}^{A_{y2}}_{Z_{y2}}Y_2\right)\right]c^2 + Q$$

– The liberated energy (Q) is found as kinetic energy of the products [including all emitted particles (photons or other)]

Interaction of Radiation with Matter

Atom Density

- Also called ***number density***.
- Is the Number of Atoms per Unit Volume
- Connection with (mass) density
  - n = # of atoms in volume V
  - M = atomic weight of each atom
  - N = Atom density

$$N = \frac{n}{V}$$

$$\rho = \frac{m}{V}$$

$$\rho = \frac{m}{V} = \frac{nM}{V} = M\frac{n}{V} = MN$$

Mechanisms of Interaction for Charged Particles

Heavy Charged Particles e.g. alpha particles

- Interact mostly with electrons (there are usually much more electrons than nuclei) via Coulombic force
- Are much heavier than electrons
- Lose little energy in each individual interaction with any one electron
- Eventually do slow down as a consequence of the many interactions
- Have straight-line trajectories
- Electrons are knocked out of their orbits and atoms become ionized (Hence the name “ionizing radiation”)
- Behave like bowling bowls in a space filled with golf balls

Linear Stopping Power

$$S = -\frac{dE}{dx}$$

- Bethe's formula

$$S = \frac{4\pi e^4 z^2}{m_0 \mathrm{v}^2} NB$$

$$B \equiv Z\left[\ln\frac{m_0 \mathrm{v}^2}{I} - \ln\left(1 - \frac{\mathrm{v}^2}{c^2}\right) - \frac{\mathrm{v}^2}{c^2}\right]$$

- Range (tens of microns)

Fast Light Charged Particles (electrons)

- Interact mostly with electrons (there are usually much more electrons than nuclei) via Coulombic force
- Are much of the same mass as electrons
- Can lose a lot of energy in each individual interaction with any one electron
- Slow down quickly, after only few collisions.
- Have broken-line trajectories
- Can be backscattered
- Atomic electrons are knocked out of their orbits and atoms become ionized (Hence the name “ionizing radiation”)
- When accelerated, incident electrons produce bremsstrahlung (electromagnetic radiation – photons)
- Trajectory of an electron is a broken line (possible backscatter)
- Range (millimeters)

Mechanisms of Interactions for Neutral Particles

Photons

Can have several types of interactions (all depend on energy)

- Photoelectric effect
- Compton Scattering
- Pair production
  - A highly energetic (E>1.02MeV) photon is stopped (by collision with a heavy nucleus) and its energy is converted into an electron and a positron emitted in opposite directions

$$\gamma \rightarrow e^{-} + e^{+}$$

Neutrons

- Interact with nuclei via nuclear forces, since they have no charge, hence they cannot interact electrostatically with electrons
- Possible reactions
  - Elastic scattering
  - Inelastic Scattering
  - radiative capture (absorption)
  - (n, 2n)
  - fission

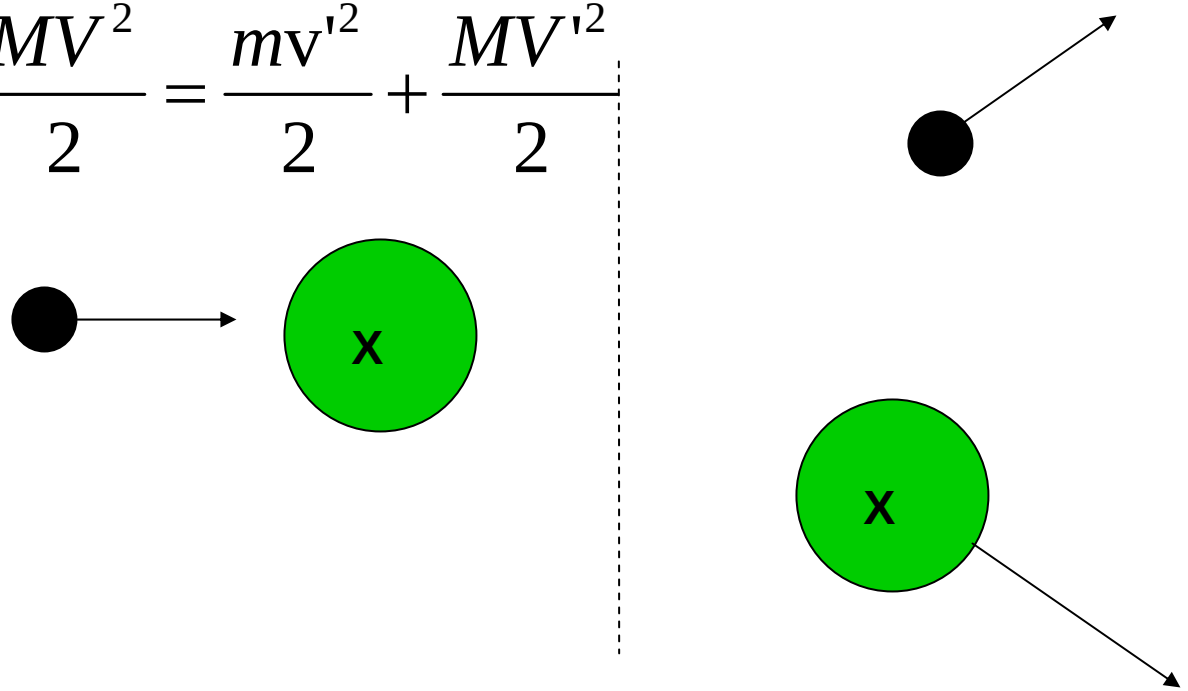
Neutron Elastic Scattering

$$n + {}^{A}_{Z}X \rightarrow n + {}^{A}_{Z}X$$

Kinetic energy is conserved

$$KE_n + KE_X = KE'_n + KE'_X$$

$$\frac{mv^2}{2} + \frac{MV^2}{2} = \frac{mv'^2}{2} + \frac{MV'^2}{2}$$



The incident neutron is slowed down by elastic scattering

Some of its kinetic energy is transferred to the target nucleus

Energy Loss in Elastic Scattering Collisions - Moderation

$$\overline{E}' = \frac{1}{2}\left[1 + \left(\frac{A-1}{A+1}\right)^2\right]E = \frac{1+\alpha}{2}E$$

Scattering of heavy nucleus (^{235}U)- small energy loss (poor moderator)

$$\overline{E}' = \frac{1}{2}\left[1 + \left(\frac{234}{236}\right)^2\right]E = 0.99E$$

Scattering on light nucleus (^{1}H) – large energy loss (good moderator) – Water used as moderator because it contains H.

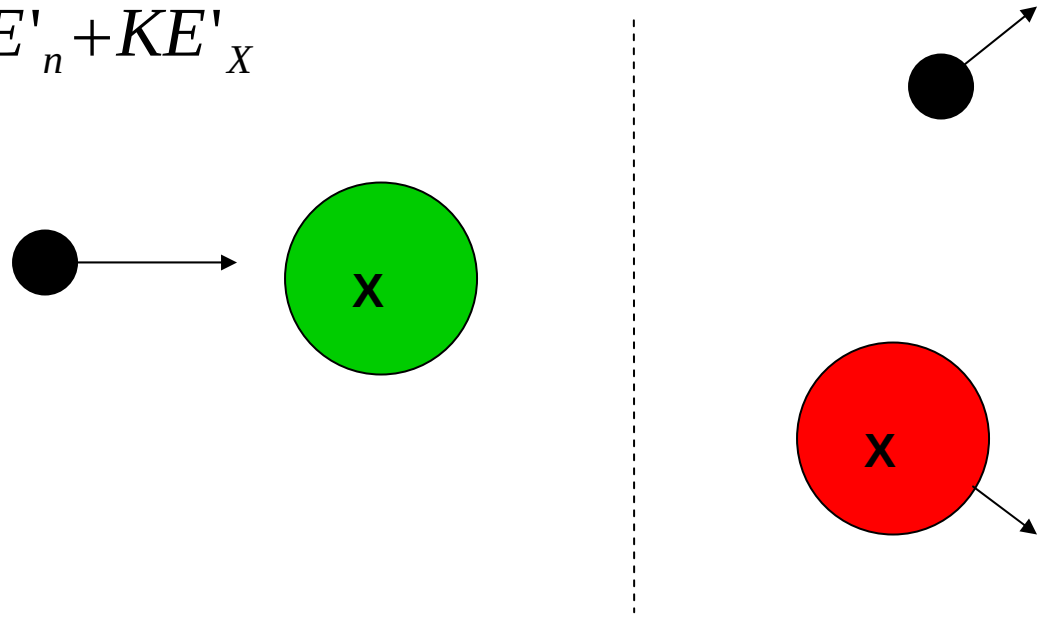
$$\overline{E}' = \frac{1}{2}\left[1 + \left(\frac{0}{2}\right)^2\right]E = 0.5E$$

Inelastic scattering

$$n + {}^{A}_{Z}X \rightarrow n' + {}^{A}_{Z}X$$

Kinetic energy is not conserved any more (total energy is)

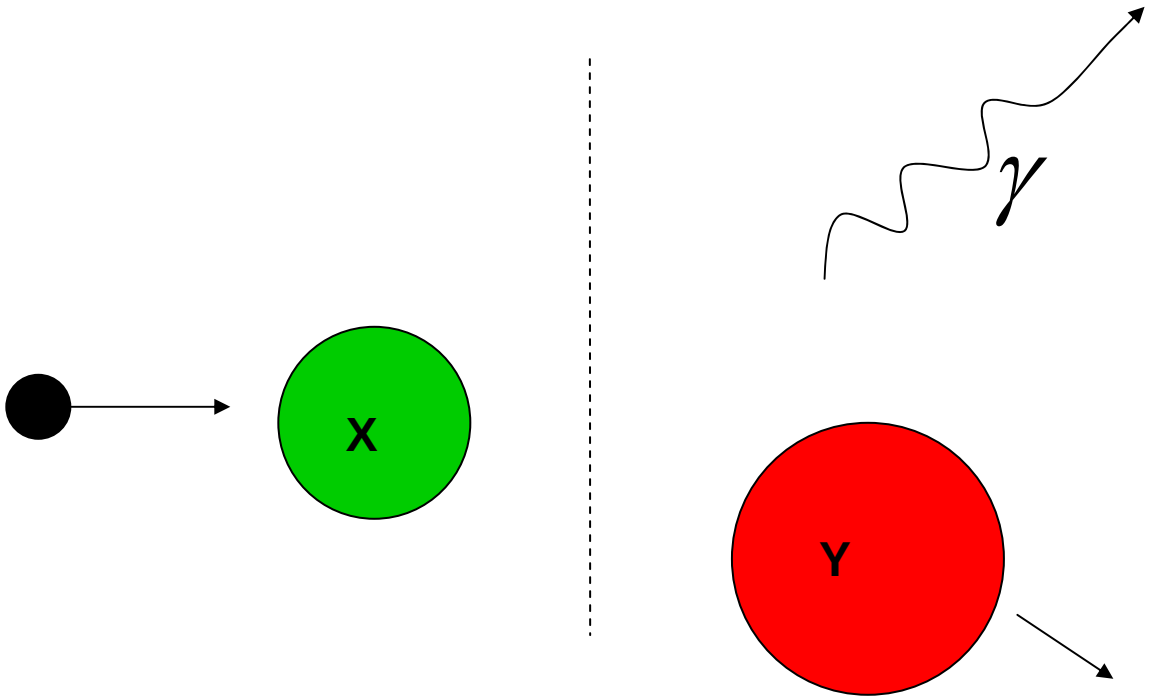
$$KE_n + KE_X > KE'_n + KE'_X$$



The incident neutron is slowed down by inelastic scattering

Radiative Capture

$$n + {}^{A}_{Z}X \rightarrow {}^{A+1}_{Z}Y + \gamma$$



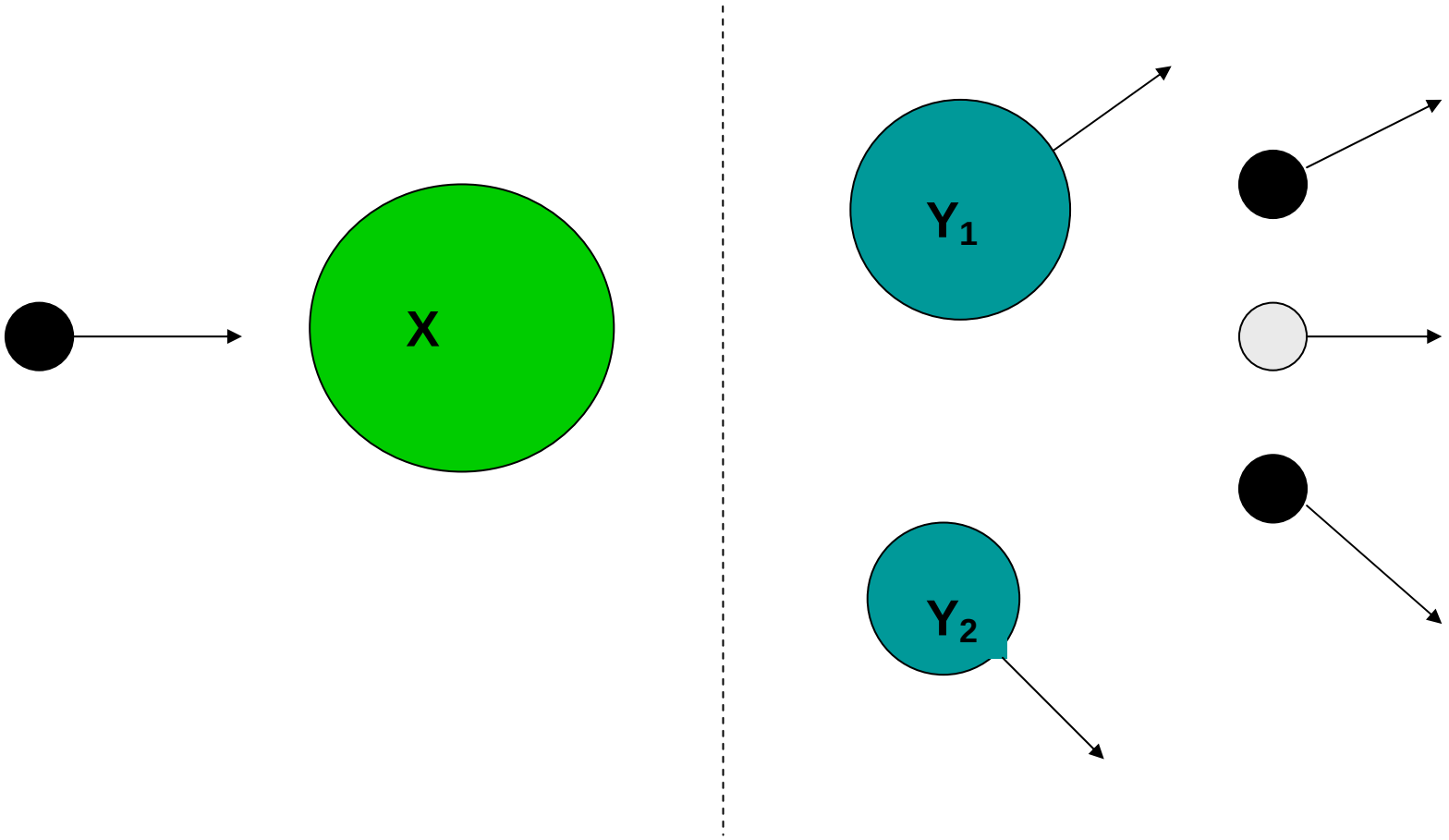
The incident neutron is absorbed (disappears) by radiative capture

Fission

$$n + {}^{A}_{Z}X \rightarrow {}^{A_1}_{Z_1}Y_1 + {}^{A_2}_{Z_2}Y_2 + \nu n + \beta + \gamma + neutrinos + approx.\ 200MeV$$

ν = average number of neutrons ≈ 2.5

(2 to 5 neutrons can be produced)



Fission - Example

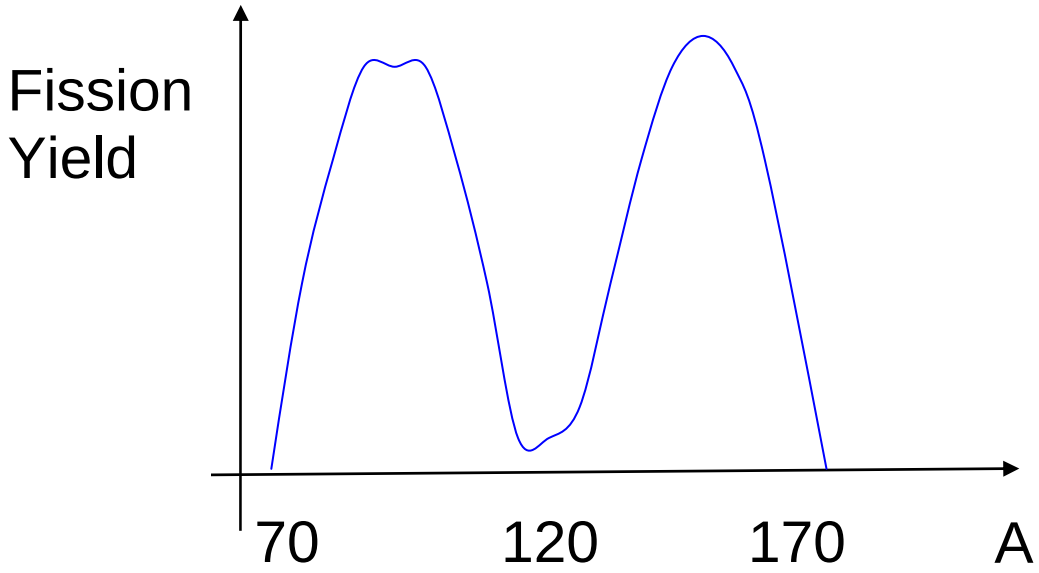
$$ {}_{0}^{1}n + {}_{92}^{235}U \rightarrow X + Y + neutrons $$

- Possible fission reactions

$$ {}_{0}^{1}n + {}_{92}^{235}U \rightarrow {}_{54}^{140}Xe + {}_{38}^{94}Sr + 2n $$

$$ {}_{0}^{1}n + {}_{92}^{235}U \rightarrow {}_{50}^{132}Sn + {}_{42}^{101}Mo + 3n $$

- Distribution of fragments



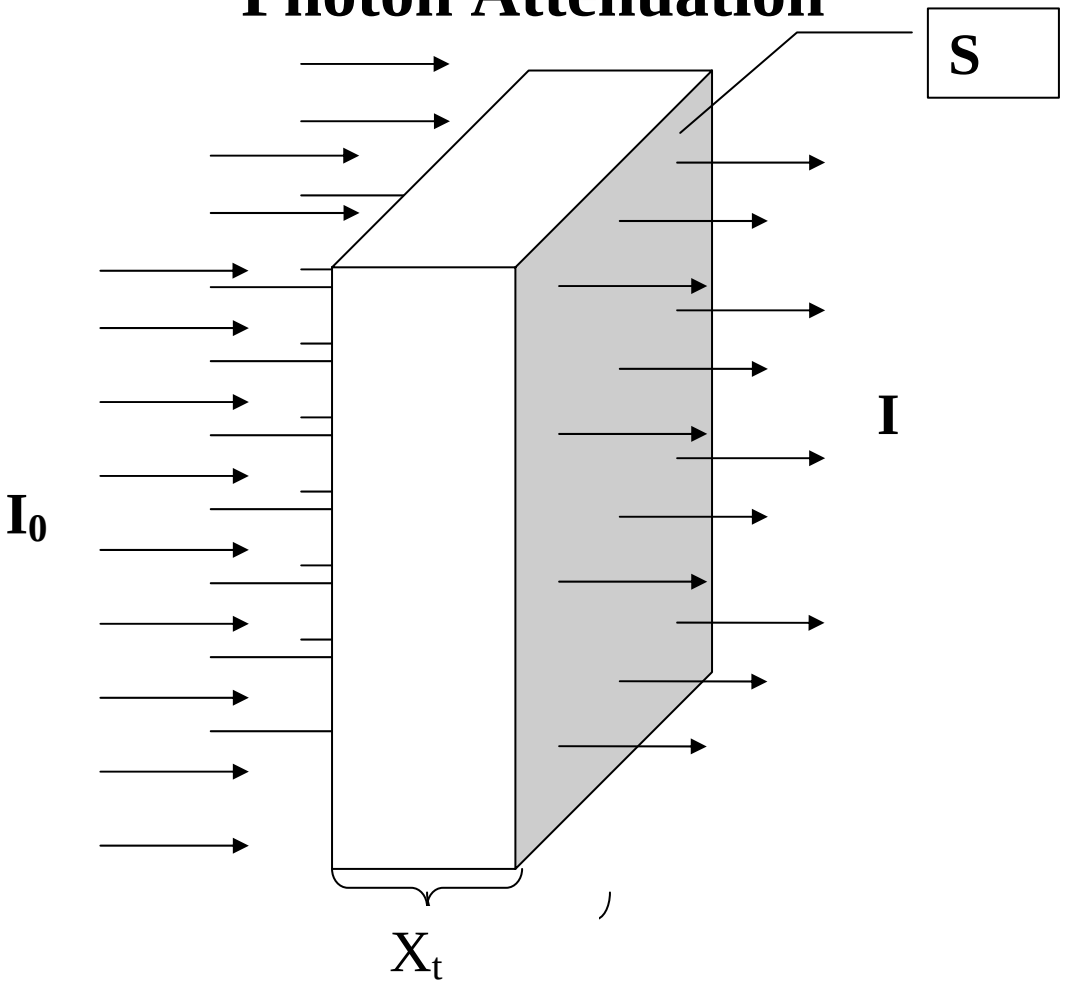
Attenuation of a Photon Beam

Photon Attenuation

Attenuation of a collimated (parallel) beam

- Consider a beam of photons of intensity I_0 that hits a target of thickness x_t, and a collimated detector that measures the intensity of the beam emerging from the target. The fact that the detector is collimated means that only the particles that have not interacted in any way are detected.
- The intensity is defined as the number of photons that pass through a surface S per unit time and per unit area.

Photon Attenuation

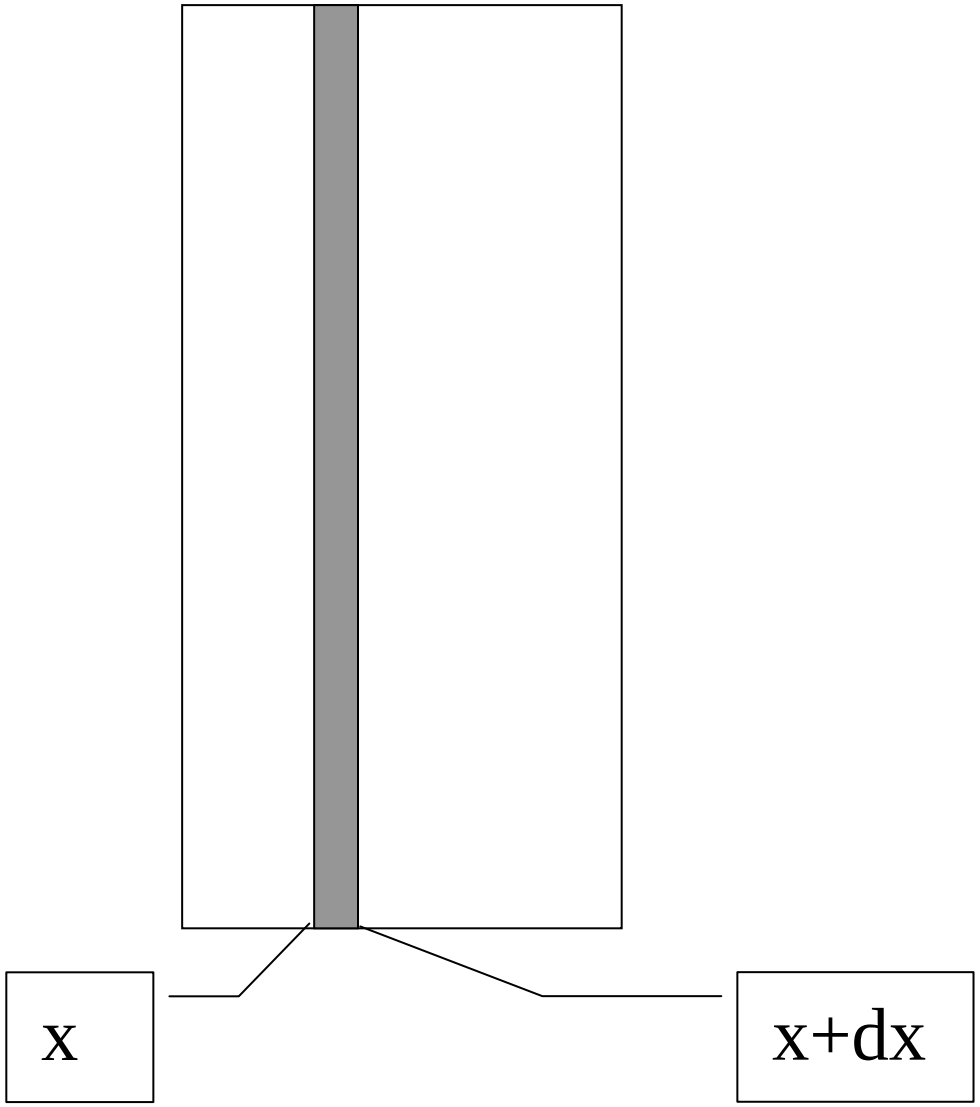


- The atom (number) density of atoms in the slab is N_v. (number of atoms per unit volume)
- The area of the material surface perpendicular to the beam is denoted by S.

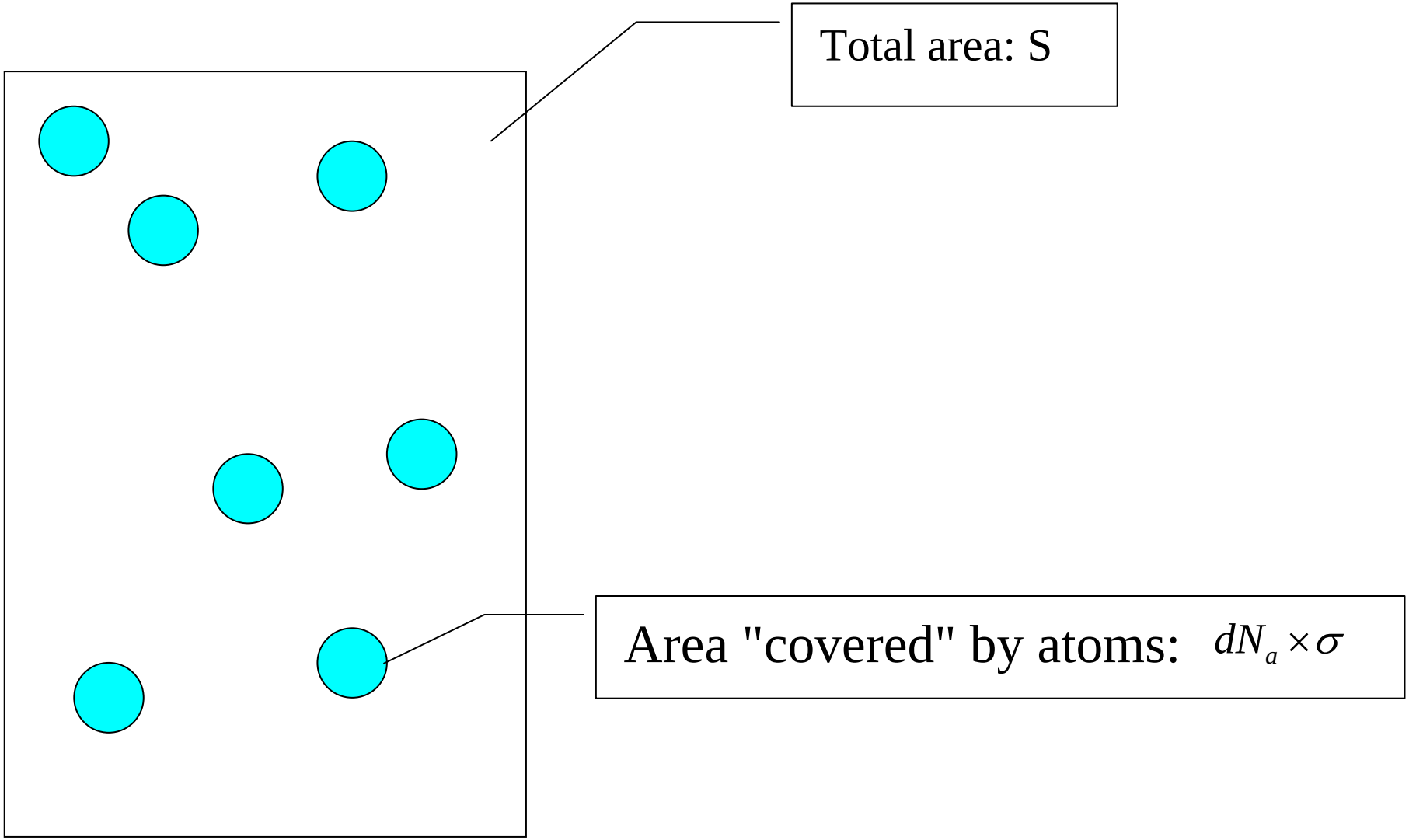
Photon Attenuation

- Consider a thin “slice” of material, of thickness dx situated at depth x in the material.
- Consider each atom can be represented as a hard ball of radius, r_a , and with a corresponding cross-section area $\sigma = \pi r_a^2$
  - Also called “microscopic cross section”
- The number of atoms in the slice is $dN = N_a S dx$
  - where N_a is the atom density
- Consider the photons to be infinitely small (points)

Thin slice of material



View of the dx slice from the photons' perspective



Attenuation of a collimated beam of photons

The probability that a photon "hits" an atom equals the ratio between the area "covered" by atoms and the total area of the slice. Let Np(x) be the total number of photons that enter the slice over a time Δt

$$N_p(x) = I(x)S\Delta t$$

Let N_p(x+dx) be the total number of photons that exit the slice dx over a time Δt

$$N_p(x+dx) = I(x+dx)S\Delta t$$

The probability of a photon interacting with an atom is:

$$P_{coll} = \frac{dn \times \sigma}{S} = \frac{N_a \times S \times dx \times \sigma}{S} = N_a \times \sigma \times dx = \mu \times dx$$

Attenuation coefficient

$$\mu = N_a \times \sigma$$

(units of cm^{-1})

Attenuation of a collimated beam of photons

Number of photons that interact and are therefore removed from the beam

$$dN_p = N_p \times P_{coll} = N_p \times \mu \times dx$$

Setting up the differential equation

- Account for the fact that the number of photons that interact represent the change in the number of photons that exit the slice, with a negative sign

$$-dN_p(x) = N_p(x) \times \mu \times dx$$

- Solution

$$N_p(x) = N_{p_0} e^{-\mu x}$$

- Np_0 is the number of photons entering the material at x=0

Attenuation of a collimated beam of photons

Given that

$$I = \frac{N_p}{S \times \Delta t}$$

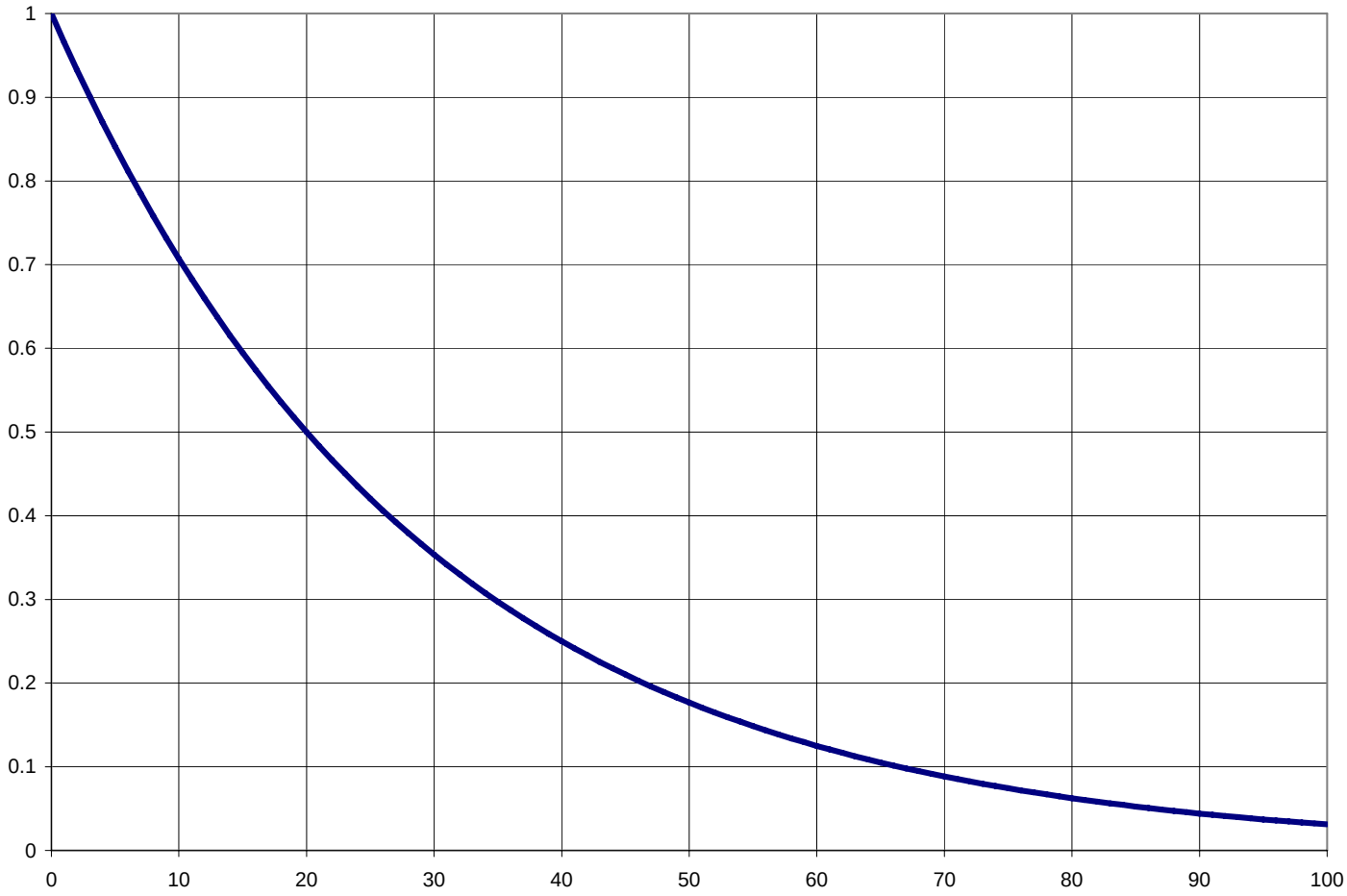
We also have

$$I(x) = \frac{N_p(x)}{S \times \Delta t} = \frac{N_{p_0} e^{-\mu x}}{S \times \Delta t} = I_0 e^{-\mu x}$$

Where Np(x) is the number of photons that “make it” to depth x.

Exponential attenuation (of a collimated beam of photons)

$N_p(x)/Np0$
or
$I(x)/I_0$



Reaction (Collision) Rate

- For a thin slice of thickness dx, the volumetric reaction rate is:

$$R \equiv F = \frac{\text{number of collisions}}{\text{time} \times \text{volume}} =$$

$$= \frac{dN_p}{\Delta t \times S \times dx} = \frac{N_p(x) \times \mu \times dx}{\Delta t \times S \times dx} =$$

$$= \frac{N_p(x)}{\Delta t \times S}\mu = \frac{I(x) \times s \times \Delta t}{\Delta t \times S}\mu = I(x)\mu$$

Attenuation of a Neutron Beam

Neutron Attenuation

- Same reasoning as for photons, but with specific features
  - Instead of the density of atoms previously denoted by Na we talk about the density of nuclei, denoted simply by N. That is because neutrons interact with nuclei and not with atoms as a whole.
- The product $N\sigma$ is called ***macroscopic cross section*** (as opposed to attenuation coefficient) and denoted by Σ (as opposed to μ).

$$\Sigma = N\sigma$$

(units of cm^{-1})

Attenuation:

$$I(x) = I_0 e^{-\Sigma x}$$

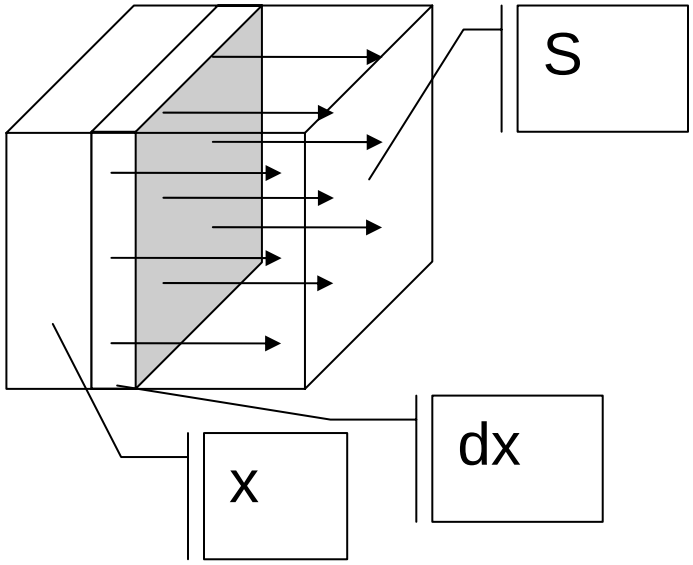
Neutron Reaction (Collision) Rate

$$R \equiv F = I(x)\Sigma$$

Neutron Beam Intensity

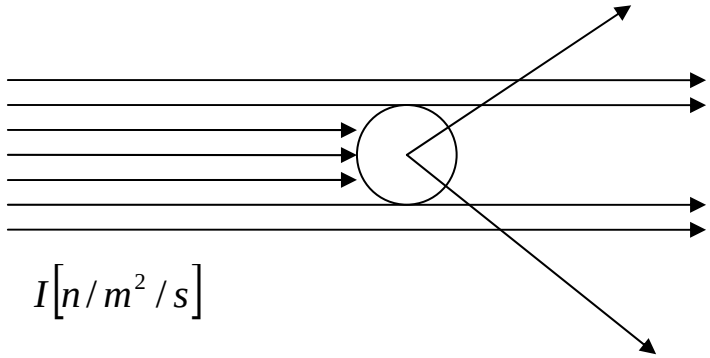
- Let n(x) be the neutron density (neutrons/cm^3)
- Consider monoenergetic neutrons (All have the same speed)
- Let v be the speed of neutrons.
- Consider a thin "slice" of beam of thickness dx, that crosses surface S.
- There are $dN_n = nSdx$ neutrons in this slice.
- It takes the neutrons in the slice time $dt = \frac{dx}{\mathrm{v}}$ to cross surface S.
- The beam intensity is therefore:

$$I = \frac{dN_n}{Sdt} = \frac{nSdx}{S\frac{dx}{\mathrm{v}}} = n\mathrm{v}$$



Microscopic Cross Sections and Reaction Rates for Neutrons

Consider a single nucleus in a parallel beam of monoenergetic neutrons



Assume (for now) that scattering and absorption are the only possible reactions.

Reaction Rates

$$F = R_t = R_s + R_a$$

Probability of a Certain Reaction Type

$$P_s = \frac{R_s}{R_t}$$

$$P_a = \frac{R_a}{R_t}$$

$$P_a + P_s = 1$$

$$P_a + P_s = \frac{R_s + R_a}{R_t} = 1$$

Microscopic Cross sections for Individual Reactions

$$R_t = I\sigma_t$$

$$R_s = R_t P_s = I\sigma_t P_s = I\sigma_s \Rightarrow \sigma_s \equiv \sigma_t P_s$$

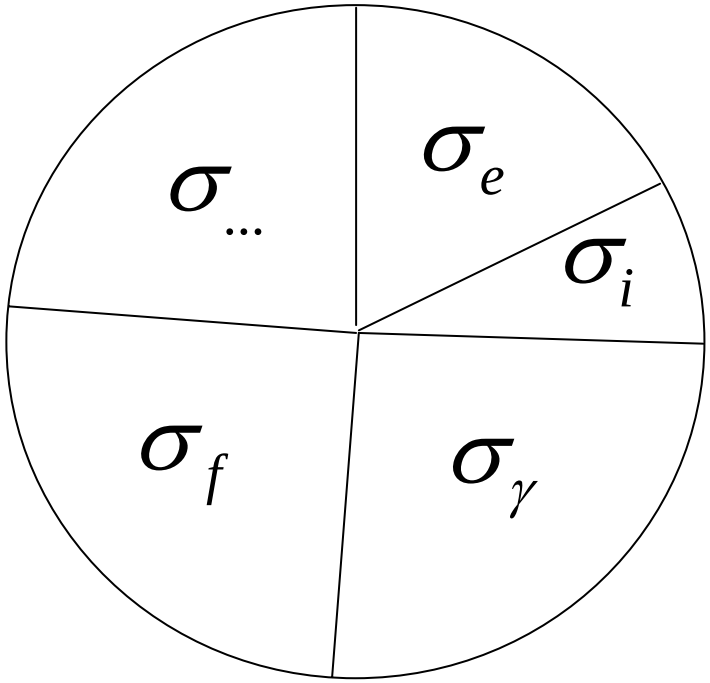
$$R_a = R_t P_a = I\sigma_t P_a = I\sigma_a ; \quad \sigma_a \equiv \sigma_t P_a$$

The sum of individual microscopic cross sections equals the total macroscopic cross section. In our simplified case

$$\sigma_t = \sigma_a + \sigma_s = \sigma_t P_a + \sigma_t P_s = \sigma_t (P_a + P_s) = \sigma_t$$

For the general case:

$$\sigma_t = \sigma_e + \sigma_i + \sigma_\gamma + \sigma_f + \ldots$$



Microscopic Cross Sections as Measures of Probability

We can write:

$$\sigma_t = \frac{R_t}{I}$$

$$\sigma_a = \frac{R_a}{I}$$

$$\sigma_s = \frac{R_s}{I}$$

- The microscopic cross sections can hence be interpreted as the probability of interaction, per unit incident flux.

Energy Dependence of Microscopic Cross Sections

- The microscopic cross sections depend on the energy of the incident neutrons**.** ***The nucleus appears larger or smaller depending on how fast the incoming neutron is moving!***

$$\sigma_t = \sigma_t(E)$$

Where E is the kinetic energy of a neutron

$$\sigma_a = \sigma_a(E)$$

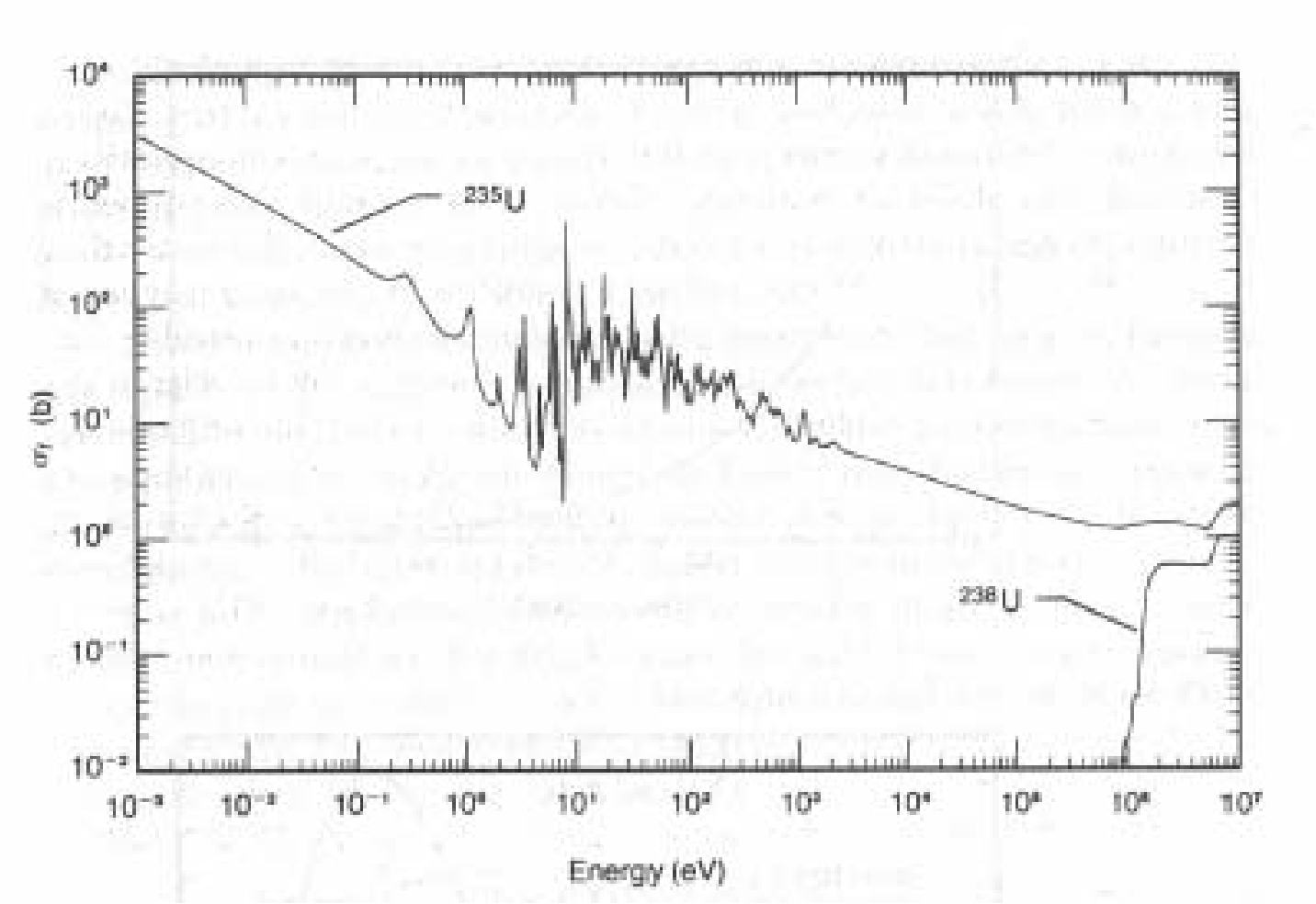
$$\sigma_s = \sigma_s(E)$$

Reaction rate per nucleus

$$R(E) = I\sigma(E)$$

The reaction rate depends on the energy (speed) of the incident neutrons.

Energy Dependence of Microscopic Cross Section (fission)



Volumetric Reaction Rate for a Material (Collision Density)

Consider a small piece of material placed in a beam of monoenergetic neutrons.

$$F = \frac{R_{\text{single}-\text{nucleus}} \times N_{nuclei}}{V} = R_{\text{single}-\text{nucleus}} \times N$$

Where N is the number density of nuclei.

$$F = I\sigma \times N \ = I \times \Sigma$$

Where $\Sigma = \sigma \times N$ is the Macroscopic Cross Section.

We have thus recovered the formula obtained in the previous lecture using the attenuation of a collimated beam.

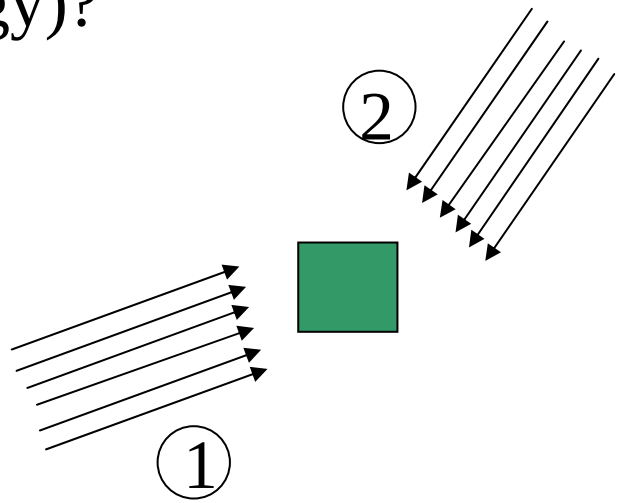
Volumetric Reaction Rate for a Material (Collision Density)

Dependence on the energy of the incident neutrons

$$F(E) = I \times \sigma(E) \times N \ = I \times \Sigma(E)$$

Neutron flux

Q: What happens if we have a small piece of material bombarded by two beams of monoenergetic neutrons (both having the same energy)?



Reaction (collision) Rate

$$F = I_1\sigma \times N \; + I_2\sigma \times N \; = \left(I_1 + I_2\right)\Sigma =$$

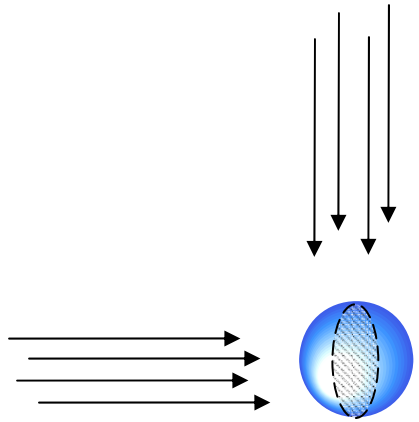
$$= \left(n_1 \mathrm{v} + n_2 \mathrm{v}\right)\Sigma = n\mathrm{v}\Sigma = \Phi\Sigma$$

Neutron flux for monoenergetic neutrons: $\Phi = n\mathrm{v}$

Alternative Interpretation of the Neutron Flux for Monoenergetic Neutrons

Consider a small sphere at the intersection of two beams of same-energy monoenergetic neutrons. The situation is similar to having one nucleus bombarded by two neutron beams.

Cross sectional area of sphere: πr^2



The number of neutrons crossing the sphere per second equals the "reaction rate" for the sphere, due to both beams (which is the sum of the reaction rates due to each beam).

Alternative Interpretation of the Neutron Flux for Monoenergetic Neutrons

$$R_{cross} = R_1 + R_2 = I_1 \times \pi r^2 + I_2 \times \pi r^2 = \left(I_1 + I_2\right) \times \pi r^2 =$$
$$= \left(n_1 \mathrm{v} + n_2 \mathrm{v}\right) \times \pi r^2 = n\mathrm{v} \times \pi r^2 = \Phi \times \pi r^2$$

It follows that:

$$\Phi = \frac{R_{cross}}{\pi r^2}$$

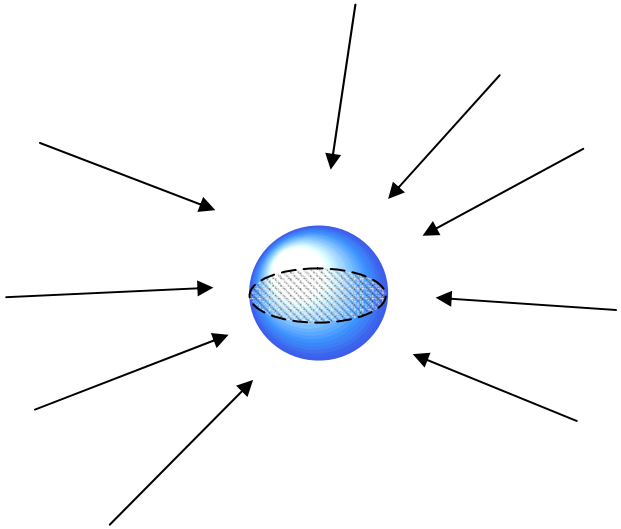
So the flux can also be interpreted as the number of neutrons that cross a sphere per unit time, divided by the cross sectional area of the sphere $\left(\pi r^2\right)$.

Neutron flux for monoenergetic neutrons

For the situation of more than two beams, all of the same energy, the definition of the flux is the same:

$$\Phi = n\mathrm{v}$$

where n is the total neutron density due to all the beams. The flux can still be interpreted as the number of neutrons crossing a small sphere, divided by the cross section area of the sphere.



Macroscopic Cross Sections for Mixtures

Consider a mixture of nuclei with number densities Ni. The volumetric reaction rate density for each nucleus type i is:

$$F_i = \Phi \sigma_i \times N_i = \Phi \Sigma_i$$

The total reaction rate density is:

$$F = \sum_i F_i = \sum_i \Phi \sigma_i \times N_i = \Phi \sum_i \sigma_i \times N_i =$$

$$= \Phi \sum_i \Sigma_i = \Phi \Sigma$$

Macroscopic Cross Sections for Mixtures

The total macroscopic cross section equals the sum of the (partial) macroscopic cross sections for each nucleus species

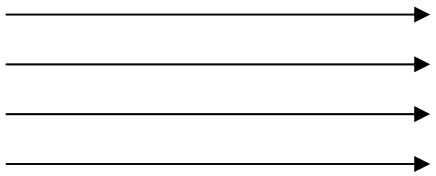
$$\Sigma = \sum_i \Sigma_i = \sum_i N_i \sigma_i$$

$$F = \Phi \Sigma$$

Neutron Intensity, Flux, Current and their Applications

Single Beam

Consider a beam of monoenergetic neutrons



The intensity is given by:

$$I = n\mathrm{v}$$

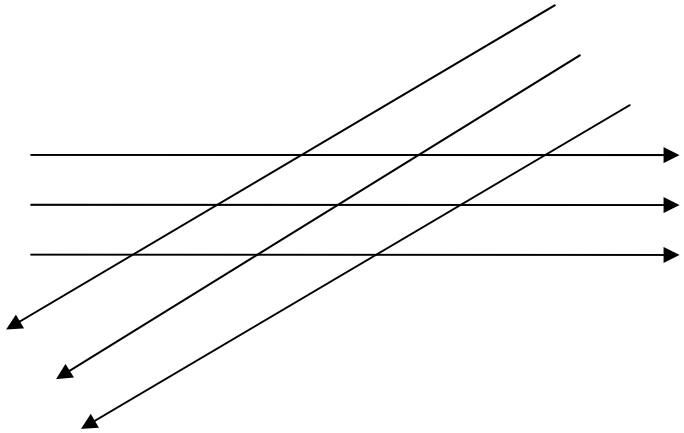
The flux is a scalar quantity given by

$$\Phi = n\mathrm{v}$$

The current is a vectorial quantity given by:

$$\vec{J} = n\vec{\mathrm{v}}$$

Two intersecting beams of different-energy neutrons



Neutron Flux

$$\Phi = n_1 \mathrm{v}_1 + n_2 \mathrm{v}_2 = \Phi_1 + \Phi_2$$

Neutron Current

$$\vec{J} = n_1 \vec{\mathrm{v}}_1 + n_2 \vec{\mathrm{v}}_2 = \vec{J}_1 + \vec{J}_2$$

For many intersecting beams:

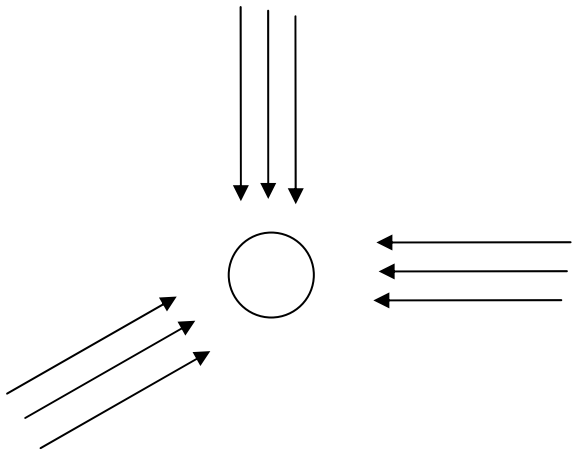
$$\Phi = \sum_i n_i \mathrm{v}_i = \sum_i \Phi_i$$

$$\vec{J} = \sum_i n_i \vec{\mathrm{v}}_i = \sum_i \vec{J}_i$$

Usefulness of Neutron Flux

Consider a small sample of material placed at the intersection of several beams of neutrons.

The total collision density in the sample is equal to the sum of the collision densities due to the neutrons in each beam.



$$F = \sum_i F_i$$

We rewrite the expression for the total collision density

$$F = \sum_i F_i = \sum_i \Sigma \times \Phi_i =$$

$$= \Sigma \sum_i \Phi_i = \Sigma \Phi$$

Where $$\Phi = \sum_i \Phi_i$$

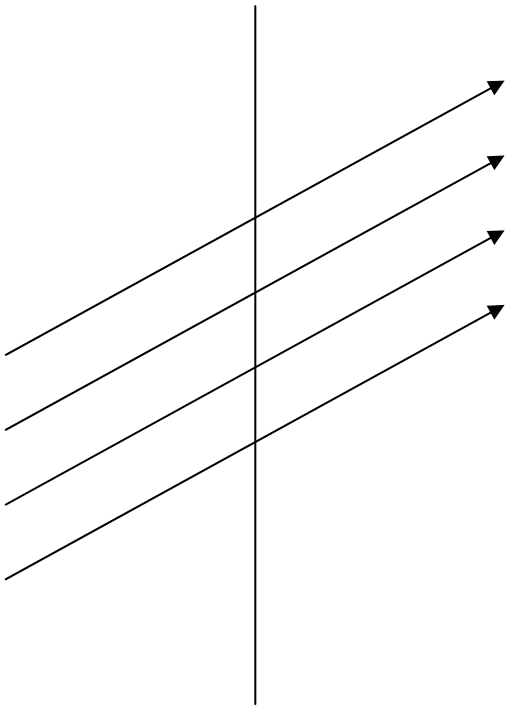
So:

$$F = \Sigma \Phi$$

Regardless of how many beams we have (one or more).

Usefulness of Neutron Current

Consider a monoenergetic neutron beam that intersects a plane surface.



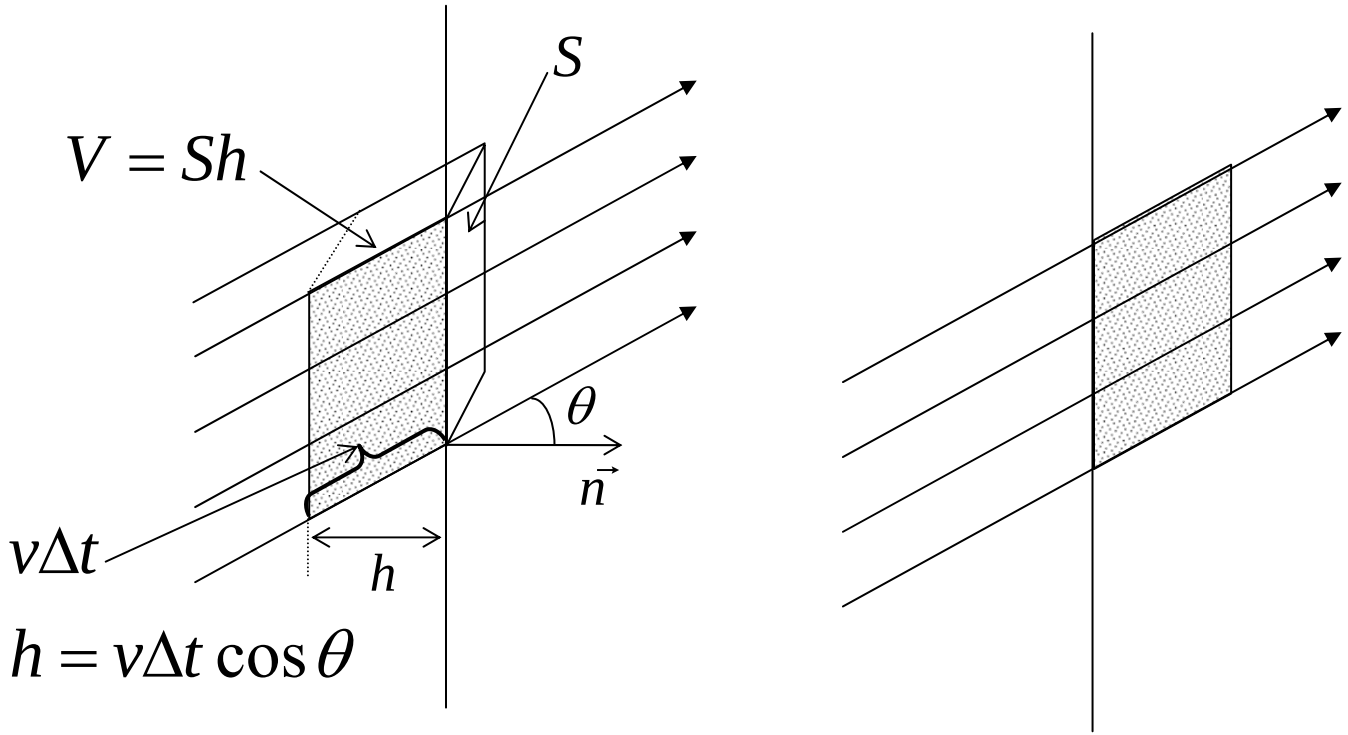
We want to determine the rate at which neutrons cross this surface. Per unit area.

$$R = \frac{\Delta N}{S \Delta t}$$

Where ΔN is the number of neutrons crossing the plate at time Δt through surface area *S*.

Before
(t=0)

After (Δt elapsed)
(t=Δt)



$\vec{n}$ is the unit vector normal to *S*. $\vec{h} = h\vec{n}$

$$V = Sv\Delta t \cos\theta$$

The rate at which neutrons cross the surface in Δt is given by the neutrons in the marked region.

$$R = \frac{\Delta N}{S\Delta t} = \frac{n\Delta V}{S\Delta t} = \frac{nS\mathrm{v}\Delta t\cos\theta}{S\Delta t} =$$

$$= n\mathrm{v}\cos\theta = n\vec{\mathrm{v}}\vec{n} = (n\vec{\mathrm{v}})\vec{n} = \vec{J}\vec{n}$$

Multiple Beams

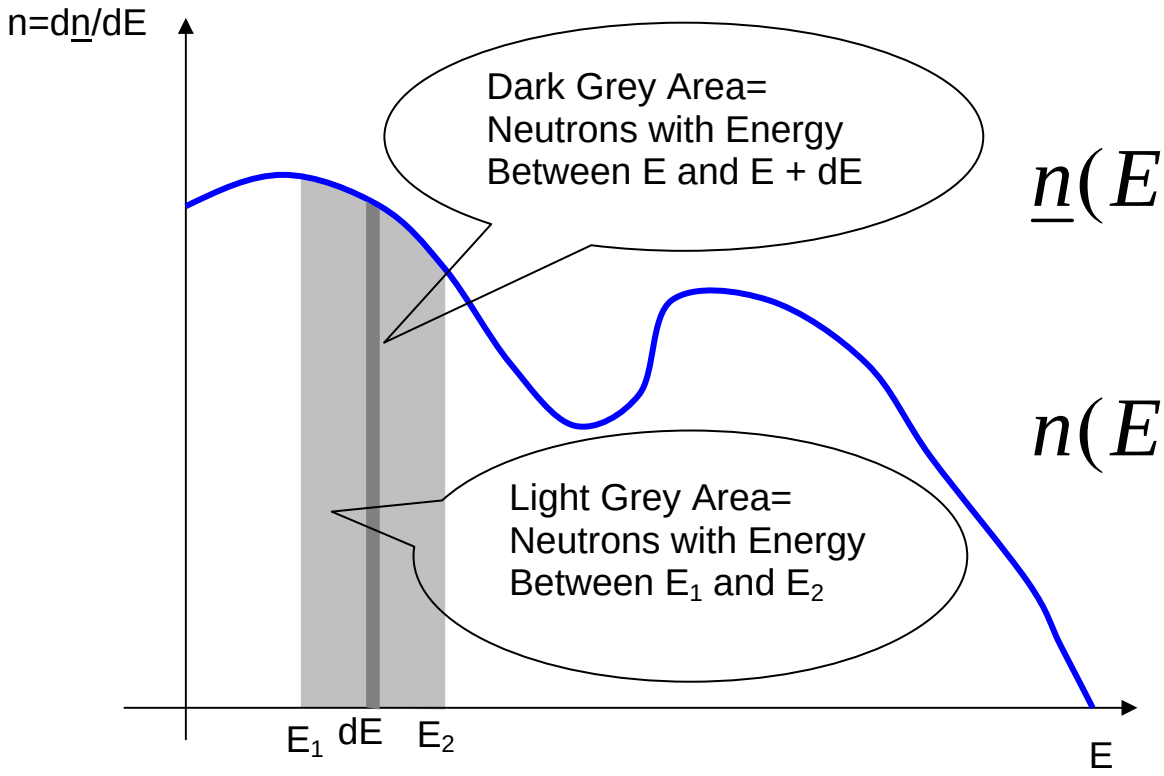
The number of neutrons crossing the surface per unit time per unit area is the sum of the neutrons in each beam that cross the surface per unit time per unit area.

$$R = \sum_i R_i = \sum_i \vec{J}_i \vec{n} =$$

$$= \left(\sum_i \vec{J}_i \right) \vec{n} = \vec{J} \vec{n}$$

Polyenergetic Neutrons

Consider now a parallel beam that has neutrons of different energies (speeds).



$$\underline{n}(E) = \int_0^E n(E')dE'$$

$$n(E) = \frac{d\,\underline{n}(E)}{dE}$$

(volumetric) density of neutrons with energy less or equal to E.

$$\underline{n}(E)$$

Neutron density spectrum

$$n(E) = \frac{d\,\underline{n}(E)}{dE}$$

(volumetric) density of neutrons with energy between E and E+dE.

$$d\,\underline{n} = n(E)dE$$

Beam intensity for neutrons with energy between E and E+dE

$$d\underline{I}(E) = d\underline{n}(E) \times \mathrm{v(E)} = n(E)\mathrm{v}(E)dE$$

The above is the same as eq. 3.36 in the textbook but the textbook does not use the underline.

Energy-dependent beam intensity (Beam Intensity Spectrum)

$$I(E) = \frac{d\underline{I}(E)}{dE} = n(E)\mathrm{v}(E)$$

Energy-dependent Flux (Flux spectrum)

$$\Phi(E) = n(E)\mathrm{v}(E)$$

Energy Dependent Current (Current Spectrum)

$$\vec{J}(E) = n(E)\vec{\mathrm{v}}(E)$$

Where n (number of neutrons with energy between E and $E+dE$, divided by dE) per unit volume.
Total Reaction Rate for Reaction x

$$R_x = \int n(E)\mathrm{v}(E)\Sigma_x(E)dE = \int_0^\infty \Phi(E)\Sigma_x(E)dE$$

Subscript x can stand for total collisions, or just absorption, or elastic scattering, etc.

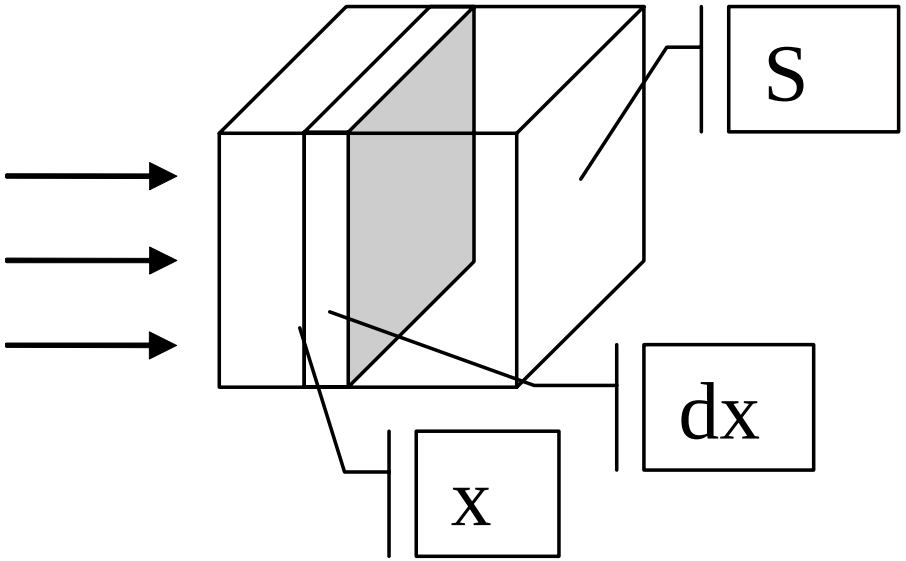
Attenuation of a Neutron Beam from a Neutron Balance Perspective

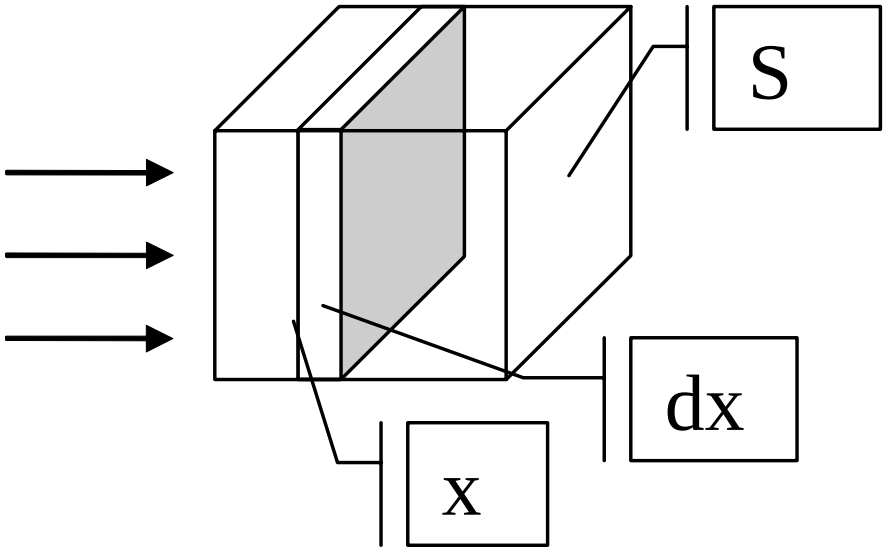
Neutron Attenuation Revisited

Parallel beam of monoenergetic neutrons

For such a beam

$$I \equiv \Phi \equiv J$$





Neutron balance in the volume of thickness dx

$$\underbrace{J(x)S}_{\text{neutrons entering the volume}} - \underbrace{J(x+dx)S}_{\text{neutrons exiting the volume}} = \underbrace{\Sigma(x)\Phi(x)Sdx}_{\text{neutrons colliding (reacting) in the volume}}$$

The neutron balance equation can be rewritten:

$$I(x)S - I(x+dx)S = \Sigma(x)I(x)Sdx$$

Dividing by Sdx on both sides we obtain

$$\frac{I(x) - I(x+dx)}{dx} = \Sigma(x)I(x)$$

Equivalent to:

$$-\frac{dI(x)}{dx} = \Sigma(x)I(x) \Leftrightarrow \frac{dI(x)}{dx} = -\Sigma(x)I(x)$$

If the macroscopic cross section is constant, then:

$$\frac{dI(x)}{dx} = -\Sigma I(x)$$

Which can be integrated to obtain:

$$I(x) = I(0)e^{-\Sigma x}$$

Exactly what we obtained before using a different kind of reasoning.

Moral: If assumptions are right and reasoning correct, the results are the same regardless of the method used.

Mean Free Path

Neutrons that react (collide) between x and x+dx have had a "free path" of length x.

To find the mean free path, we need to average over all the neutrons that interact from $x=0$ to $x=\infty$.

$$\lambda = \frac{\int_0^\infty xI(x)\Sigma dx}{\int_0^\infty I(x)\Sigma dx} = \frac{\int_0^\infty xI(0)e^{-\Sigma x}\Sigma dx}{\int_0^\infty I(0)e^{-\Sigma x}\Sigma dx} =$$

$$\frac{I(0)\Sigma\int_0^\infty xe^{-\Sigma x}dx}{I(0)\Sigma\int_0^\infty e^{-\Sigma x}dx} = \frac{\int_0^\infty xe^{-\Sigma x}dx}{\int_0^\infty e^{-\Sigma x}dx}$$

The numerator is integrated by parts to give

$$\int_0^\infty xe^{-\Sigma x}dx = \int_0^\infty x\left(-\frac{e^{-\Sigma x}}{\Sigma}\right)' dx =$$

$$= x\left(-\frac{e^{-\Sigma x}}{\Sigma}\right)\Bigg|_0^\infty - \int_0^\infty \left(-\frac{e^{-\Sigma x}}{\Sigma}\right)dx =$$

$$= \left(-\frac{e^{-\Sigma x}}{\Sigma^2}\right)\Bigg|_0^\infty = \frac{1}{\Sigma^2}$$

(Since $\lim_{x\to\infty}(xe^{-\alpha x}) = 0$)

The denominator integrates as:

$$\int_0^{\infty} e^{-\Sigma x} dx = \left(-\frac{e^{-\Sigma x}}{\Sigma} \right) \Bigg|_0^{\infty} = \frac{1}{\Sigma}$$

It follows that:

$$\lambda = \frac{\frac{1}{\overline{\Sigma^2}}}{\frac{1}{\overline{\Sigma}}} = \frac{1}{\Sigma}$$

Fission

Fission

$$ {}_{0}^{1}n + {}_{Z_X}^{A_X}X \rightarrow {}_{Z_A}^{A_A}A + {}_{Z_B}^{A_B}B + \tilde{\nu}\,{}_{0}^{1}n + \tilde{\mu}\,{}_{-1}^{0}e + \gamma $$

$$ \tilde{\nu} = 2, 3, 4, 5 $$

on average $\nu \cong 2.5$

A & B = Fission Products (Fission Fragments)

Conservation Laws

Number of nucleons

$$A_X + A_n = A_A + A_B + \tilde{\nu}$$

$$A_X + 1 = A_A + A_B + \tilde{\nu}$$

Charge

$$Z_X = Z_A + Z_B - \tilde{\mu}$$

Energy

$$c^2\left[M(n)+M(X)\right]=c^2\left[M(A)+M(B)\right]+$$
$$+\tilde{\nu}c^2M(n)+c^2\tilde{\mu}M(e)+E_\gamma$$

M is the relativistic mass

Using the rest mass and kinetic energy E, we have:

$$
\begin{aligned}
& c^2\left[M_0(n)+M_0(X)\right]+E_n = \\
& = c^2\left[M_0(A)+M_0(B)\right]+ \\
& +\tilde{\nu}c^2M_0(n)+c^2\tilde{\mu}M_0(e)+ \\
& +E_\gamma+E_A+E_B+E_{\tilde{\nu}n}+E_{\tilde{\mu}e}
\end{aligned}
$$

M_0 is the rest mass

The above can be rewritten using the Q:

$$c^2\left[M_0(n)+M_0(X)\right]+E_n=$$
$$=c^2\left[M_0(A)+M_0(B)\right]+$$
$$+\tilde{\nu}c^2M_0(n)+c^2\tilde{\mu}M_0(e)+Q+E_n$$

For fission, Q is approximately 200 MeV

Distribution of Energy From Fission

Carrier	Energy (MeV)
Fission Fragments	168
Beta	8
Gamma	14
neutrinos	12
neutrons	5
Total	**207**

Most energy is taken by fission fragments and deposited locally.

Fission Mechanism (simplified)

In reality, fission occurs through a compound nucleus which, in turn, can decay very rapidly in several different ways.

$$ {}_{0}^{1}n + {}_{Z_X}^{A_X}X \rightarrow {}_{Z_X}^{A_X+1}X' $$

(fast)

$$ {}_{Z_X}^{A_X+1}X' \begin{cases} \rightarrow A' + B' + \gamma & (\text{mode 1 - prompt } \gamma) \\ \rightarrow A' + B' + \tilde{\nu}_p n & (\text{mode 2 - prompt n}) \end{cases} $$

$$ \nu_p = 2 - 3 $$

Both A' and B' can be stable or further decay in several possible modes:

$$
A' \begin{cases}
\xrightarrow{\lambda} A & (A' \text{ was stable}) \\
\xrightarrow{\lambda} A + \gamma & (\text{delayed } \gamma) \\
\xrightarrow{\lambda} A + {}^{0}_{-1}\beta_{HE} & (\text{mode 3}) \\
\xrightarrow{\lambda} A'' + {}^{0}_{-1}\beta_{LE} & (\text{mode 4}) \\
\quad \downarrow \text{ (fast)} & \\
\quad A + n & (\text{delayed neutron})
\end{cases}
$$

If A' decays according to mode 4, it is called a *precursor*. There are six possible types of precursor, and six possible values for λ

A" is then called an *emitter*.

We cannot predict in advance which nuclei will be precursors, but we can predict, on the average how many will do so. This number is equal to the number of delayed neutrons emitted, called the *delayed neutron yield*.

$$\nu_d = \frac{\#\ of\ \ delayed\ \ neutrons}{\#\ of\ \ \ fissions}$$

We cannot predict how many prompt neutrons will be emitted in each reaction either. But we can predict how many will be produced on the average. This is called the prompt neutron yield.

$$\nu_p = \frac{\#\ of\ \ \ prompt\ \ neutrons}{\#\ of\ \ \ fissions}$$

On the average, the fission reaction can be written:

$$n + X \rightarrow A + B + \nu_p n_p + \nu_d n_d + \mu e + \gamma$$

The *total neutron yield* is defined as:

$$\nu = \nu_d + \nu_p \quad \cong 2.5$$

The delayed neutron fraction is:

$$\beta = \frac{\nu_d}{\nu}$$

Delayed Neutrons

Are emitted by emitters which result from the beta decay of precursors.

There are 6 precursor (delayed neutron) groups, based on their half-life.

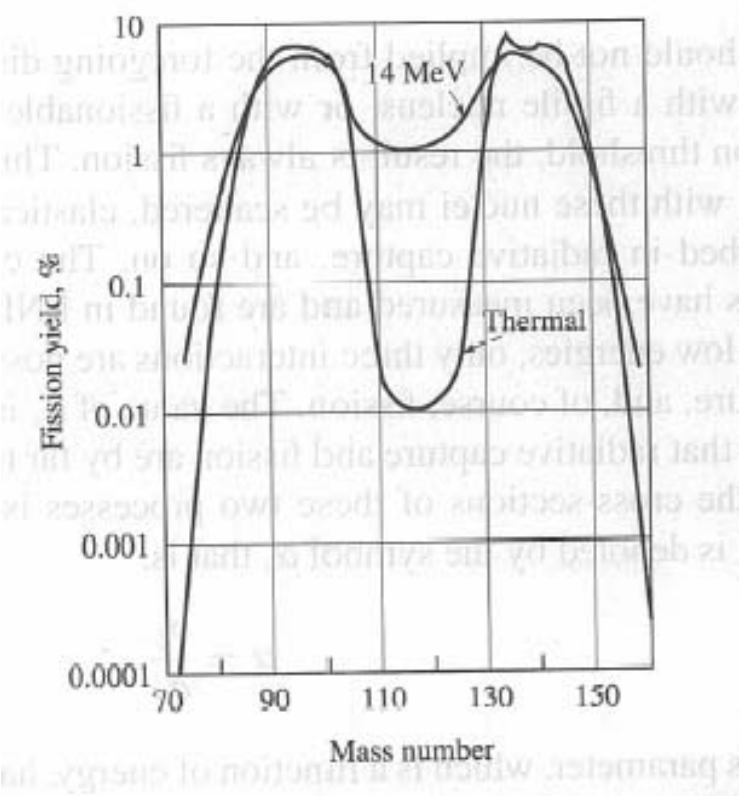
$$\nu_d = \nu_{d1} + \nu_{d2} + \nu_{d3} + \nu_{d4} + \nu_{d5} + \nu_{d6}$$

TABLE 3.5 DELAYED NEUTRON DATA FOR THERMAL FISSION IN $^{235}U^{x}$

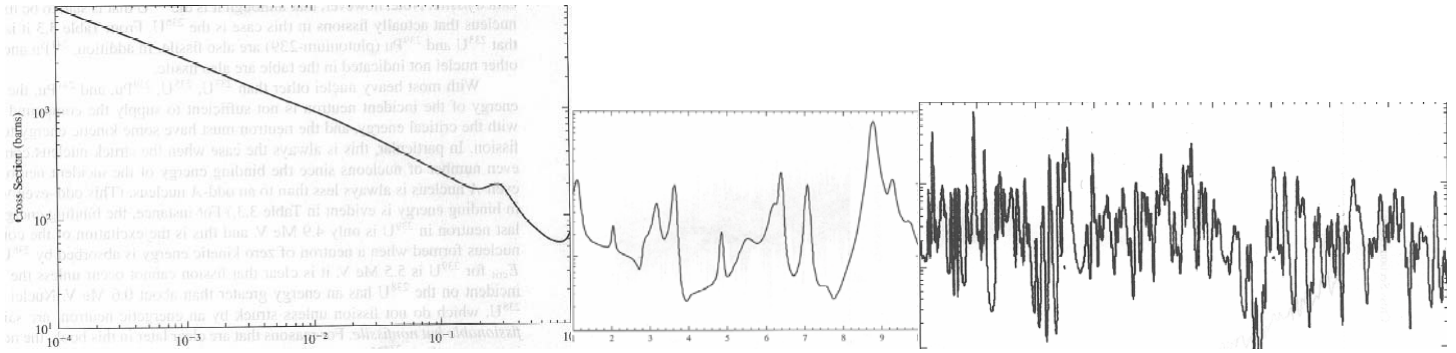
Group	Half-Life (sec)	Decay Constant (l_i, sec^{-1})	Energy (ke V)	Yield, Neutrons per Fission ν_{d_i}	Fraction (β_i)
1	55.72	0.0124	250	0.00052	0.000215
2	22.72	0.0305	560	0.00346	0.001424
3	6.22	0.111	405	0.00310	0.001274
4	2.30	0.301	450	0.00624	0.002568
5	0.610	1.14	—	0.00182	0.000748
6	0.230	3.01	—	0.00066	0.000273
				Total yield: 0.0158	
				Total delayed fraction (β): 0.0065	

Fission Products (Heavy Nuclei)

Mass is distributed asymmetrically.



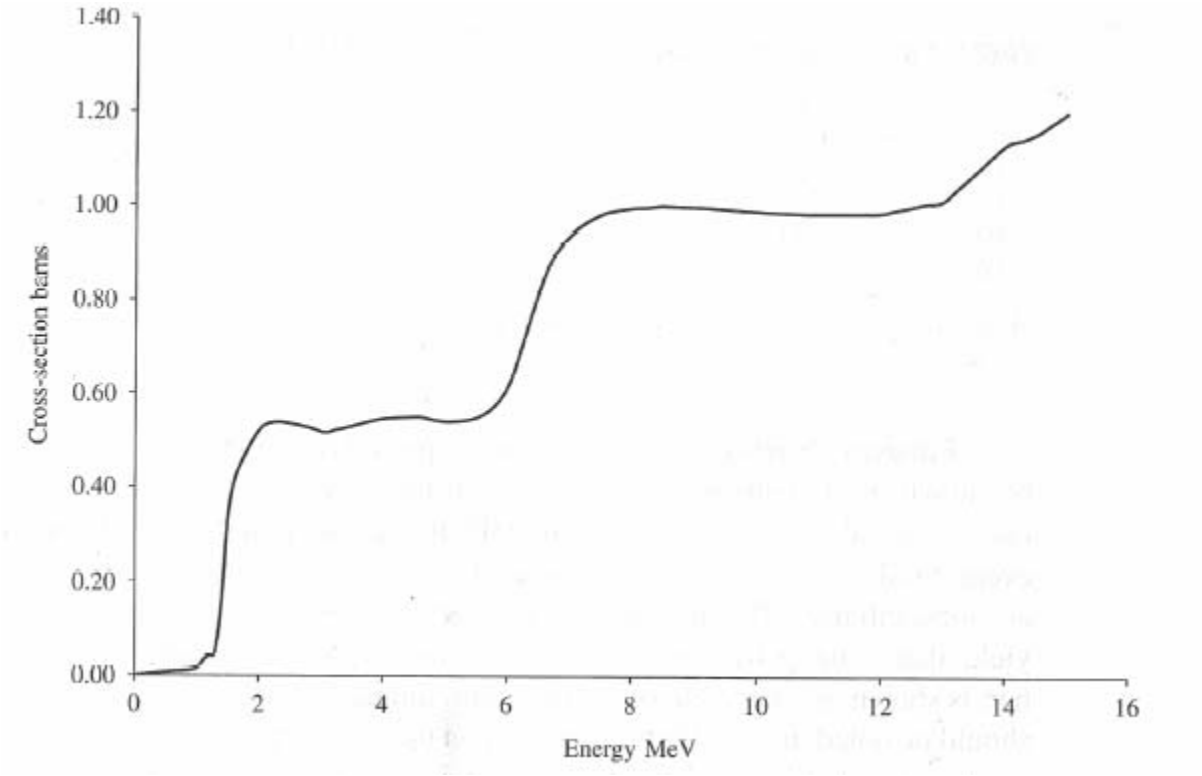
Energy Dependence of Fission Cross Section for ^{235}U



^{235}U is *fissile*, i.e. undergoes fission with near-zero energy neutrons with high probability.

$$\sigma_{f \quad \textit{lower energies}} \propto E^{-\frac{1}{2}} \propto \frac{1}{v}$$

Energy Dependence of Fission Cross Section for ^{238}U



^{238}U is *fissionable*, but not fissile, i.e. it can undergo fission, but with higher energy neutrons and with low probability.

Energy Spectrum of Fission Neutrons

Energy Spectrum

$$\chi(E) \equiv \frac{n(E)}{n_t} = \frac{n(E)}{\int_0^\infty n(E)dE}$$

It follows that:

$$\int_0^\infty \chi(E)dE = \int_0^\infty \frac{n(E)}{\int_0^\infty n(E)dE} dE = \frac{1}{\int_0^\infty n(E)dE} \int_0^\infty n(E)dE = 1$$

Energy Spectrum of Fission Neutrons

Prompt-neutron spectrum (E_{avg}=2MeV)

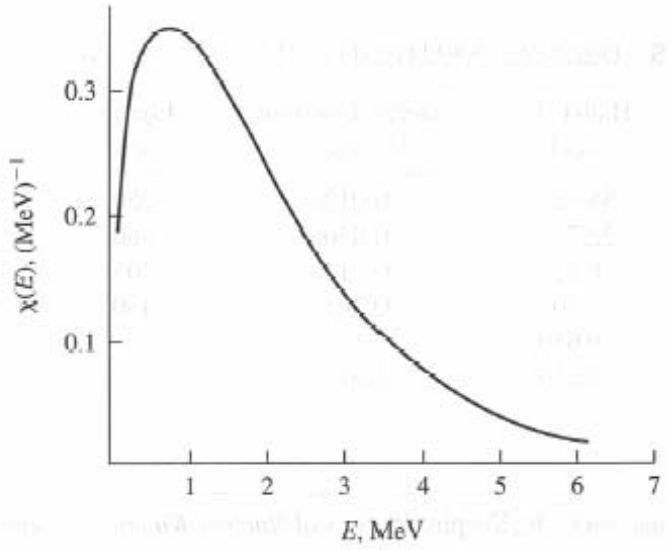


Figure 3.14 The prompt neutron spectrum.

Delayed-neutron energies are slightly lower.

Important Facts

- Fission neutron energies are much higher than thermal energies (0.025 eV), so they are not appropriate for efficient fission in fissile materials.
- To achieve fission efficiently, the neutrons need to be slowed down (their energy needs to be reduced). This process is called *moderation*. It is achieved by elastic collision with light nuclei (usually Hydrogen or Deuterium)
- Reactors that use thermal neutrons for fission are called *Thermal Reactors*.
- Special reactor designs can be conceived, where fast neutrons are used for fission. These are called *Fast Reactors*.

Fission-Related Parameters

Capture-to-fission ratio

$$\alpha = \frac{\sigma_\gamma}{\sigma_f}$$

Number of neutrons released per absorbed neutron.

$$\eta = \nu \frac{\sigma_f}{\sigma_a}$$

For mixtures of fissile **and** non-fissile elements:

$$\eta = \frac{1}{\Sigma_a} \sum_i \nu_i \Sigma_{fi}$$

Nuclear Reactors – The Basics

Nuclear Reactors

- Can be of two Types:
  - **Thermal -** fissions induced by thermal (E<1eV) neutrons in fissile nuclei
  - **Fast -** fissions induced by fast (E≅1MeV) in fissile/fissionable nuclei

Thermal Reactor Components

- ***Fuel*** **-** consists of nuclei that fission liberating energy
- ***Moderator*** **-** slows down fast neutrons resulting from fission to thermal energies so they can fission fuel nuclei
- ***Coolant*** **-** removes the heat

The three can be:

- mixed together $\rightarrow$ Homogeneous Reactor
- separated $\rightarrow$ Heterogeneous Reactor

Most reactors are **heterogeneous**.

Power Reactors

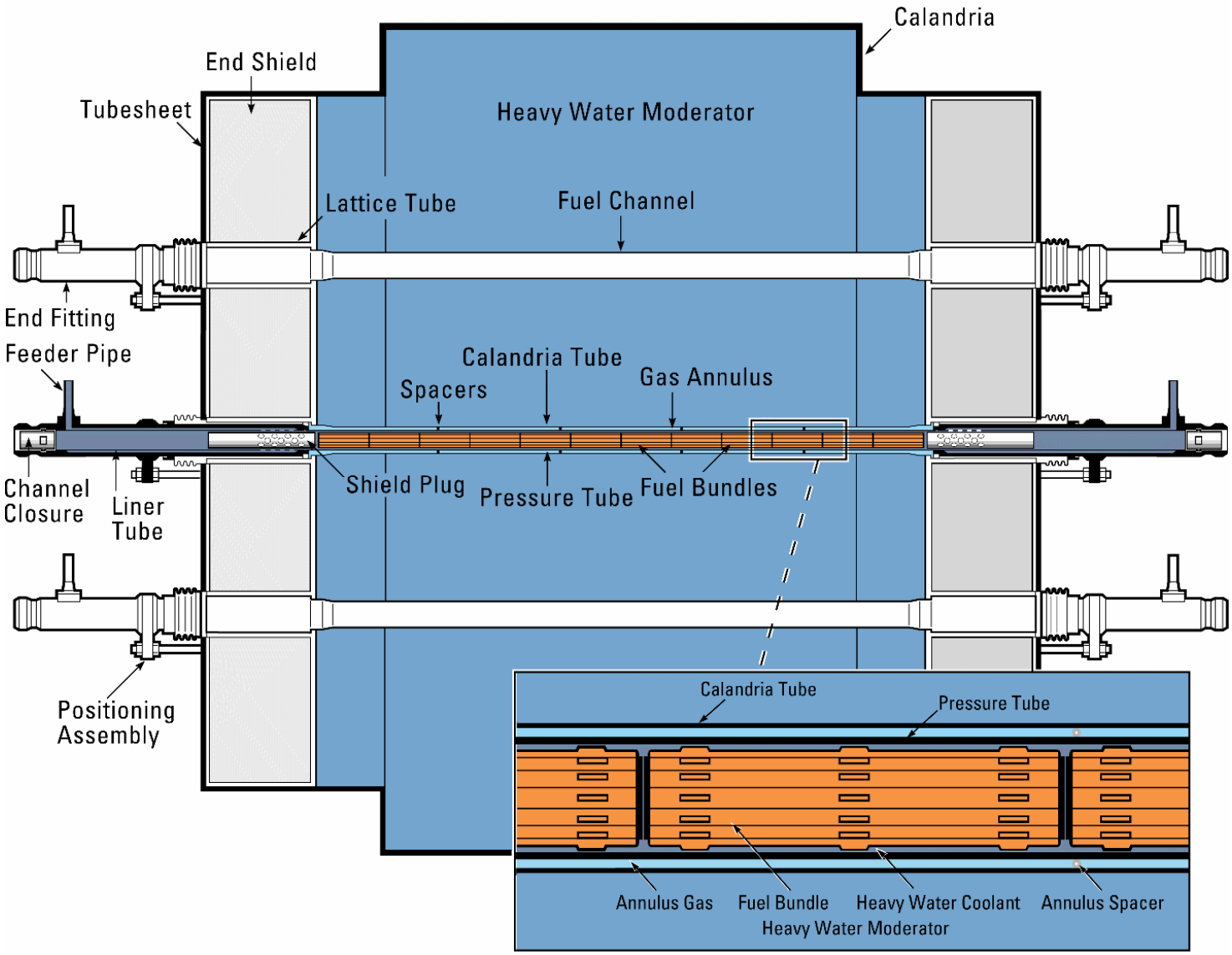
- **Pressurized Water Reactors**
- ***Pressurized Heavy-Water Reactors (CANDU)***
- Gas-Cooled Reactors
- Other

CANDU Reactors

- Heterogeneous

- *Fuel:* Natural Uranium Oxide
  - (UO_2 0.7% ^{235}U, 99.3% ^{238}U)
- *Coolant:* Heavy Water (D_2O)
- *Moderator:* Heavy Water (D_2O)

CANDU Reactor Schematic



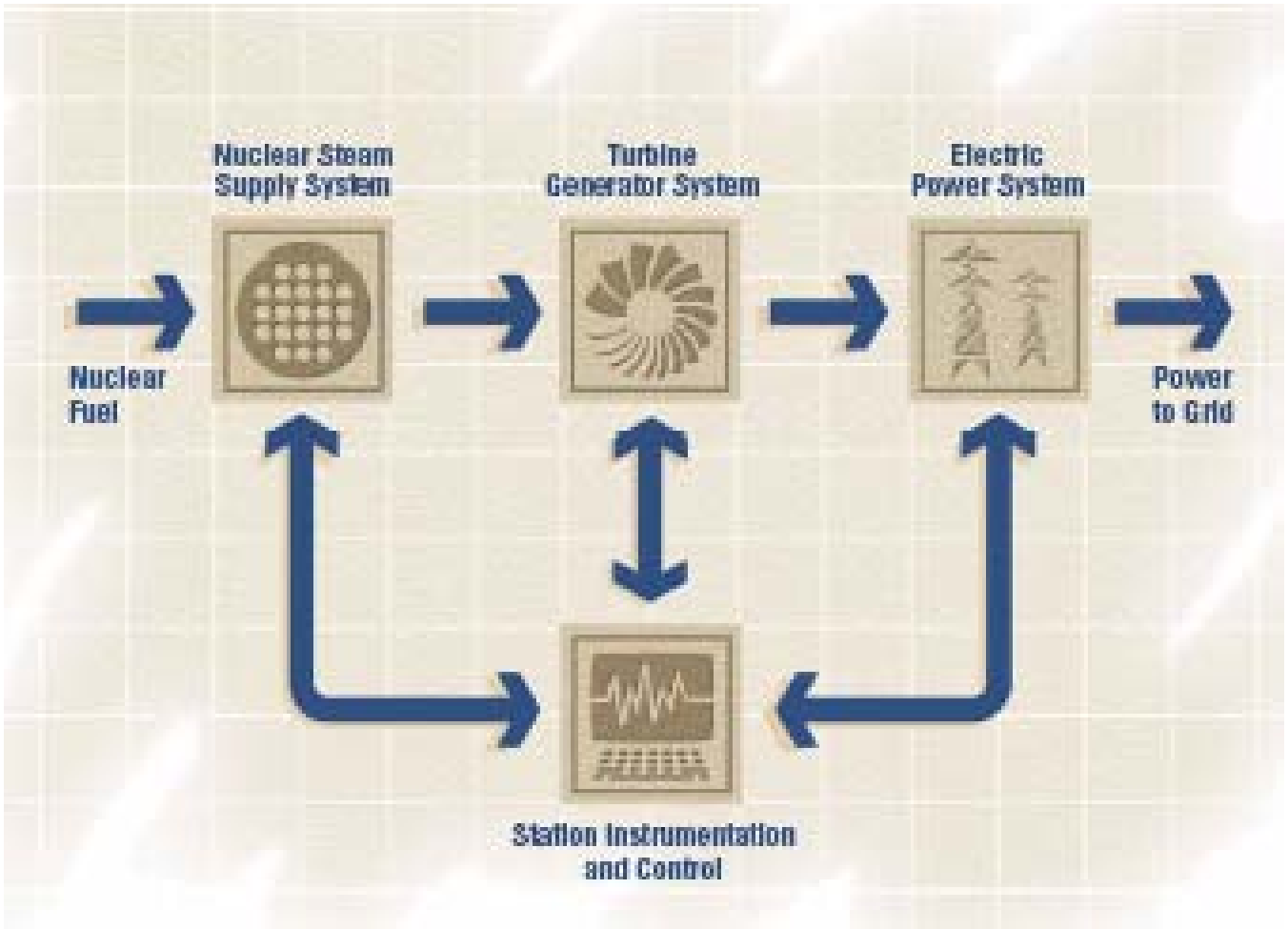
CANDU Reactor - How it Works

- Fissions take place in the fuel
- Most energy from fissions is taken up by fission fragments which stop in less that one micron.
- In stopping, the fission fragments' kinetic energy becomes heat, which raises the fuel temperature.
- The fuel is cooled by the coolant, which takes the heat from the fuel to the steam generators.
- Neutrons are also produced from fission.
- Fission neutrons are slowed-down by elastic collisions in the moderator and, to a smaller extent, in coolant.
- Once they become thermal, neutrons can induce new fissions, keeping the chain reaction going.

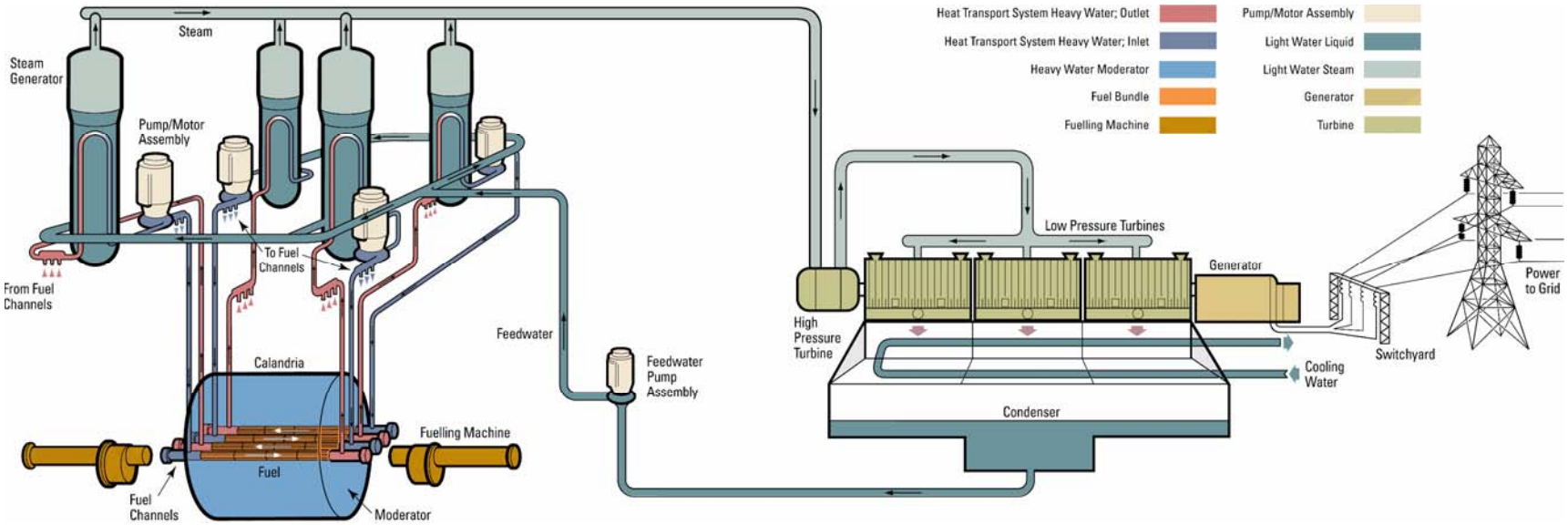
CANDU Reactor - How it Works (cont.)

- Part of the neutrons get absorbed by radiative capture or "leak" out of the reactor. These do not induce fissions.
- On the average, only one neutron per each fission succeeds in inducing a new fission, so there is a uniform rate of fissions and not an avalanche of fissions.

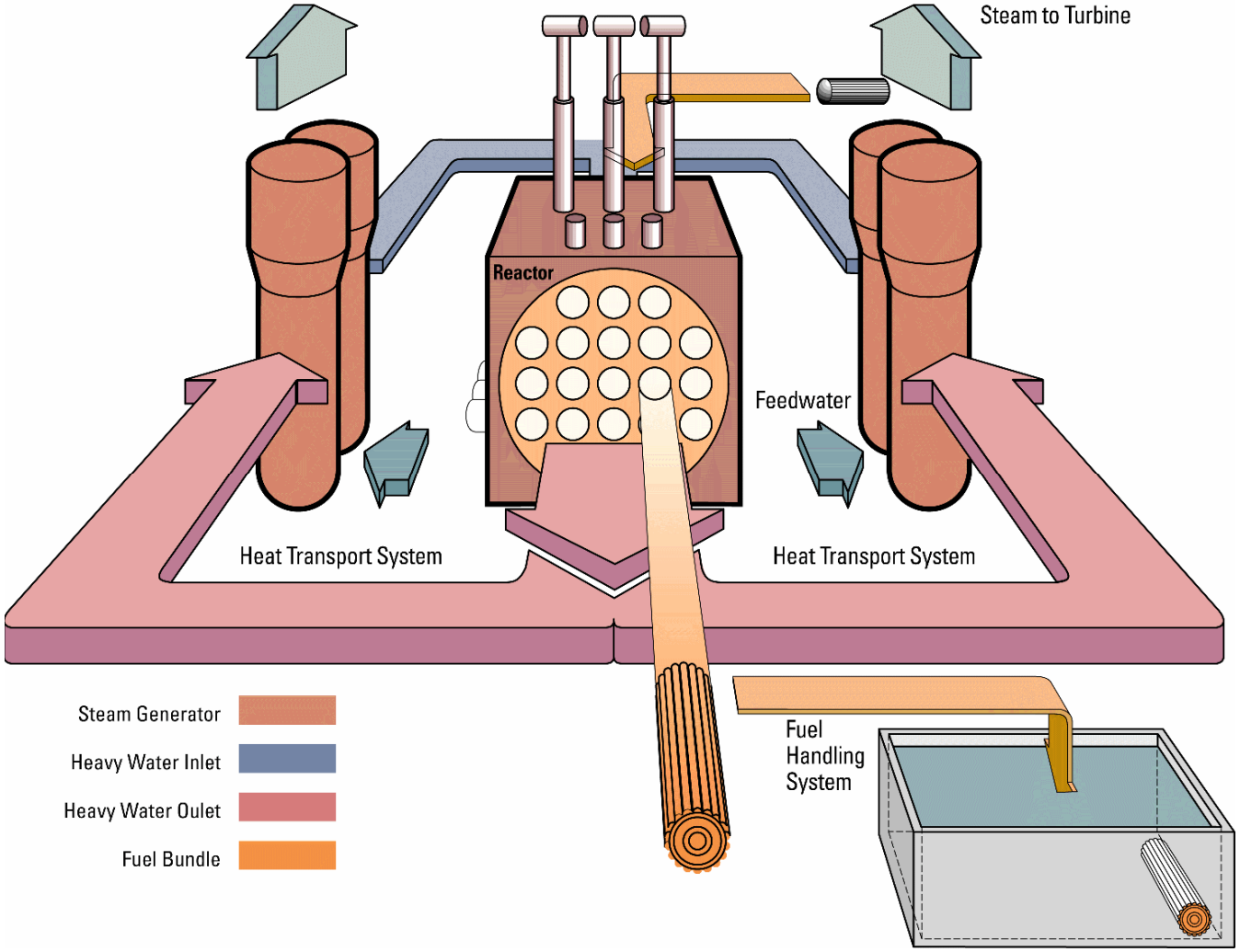
The CANDU Power Plant



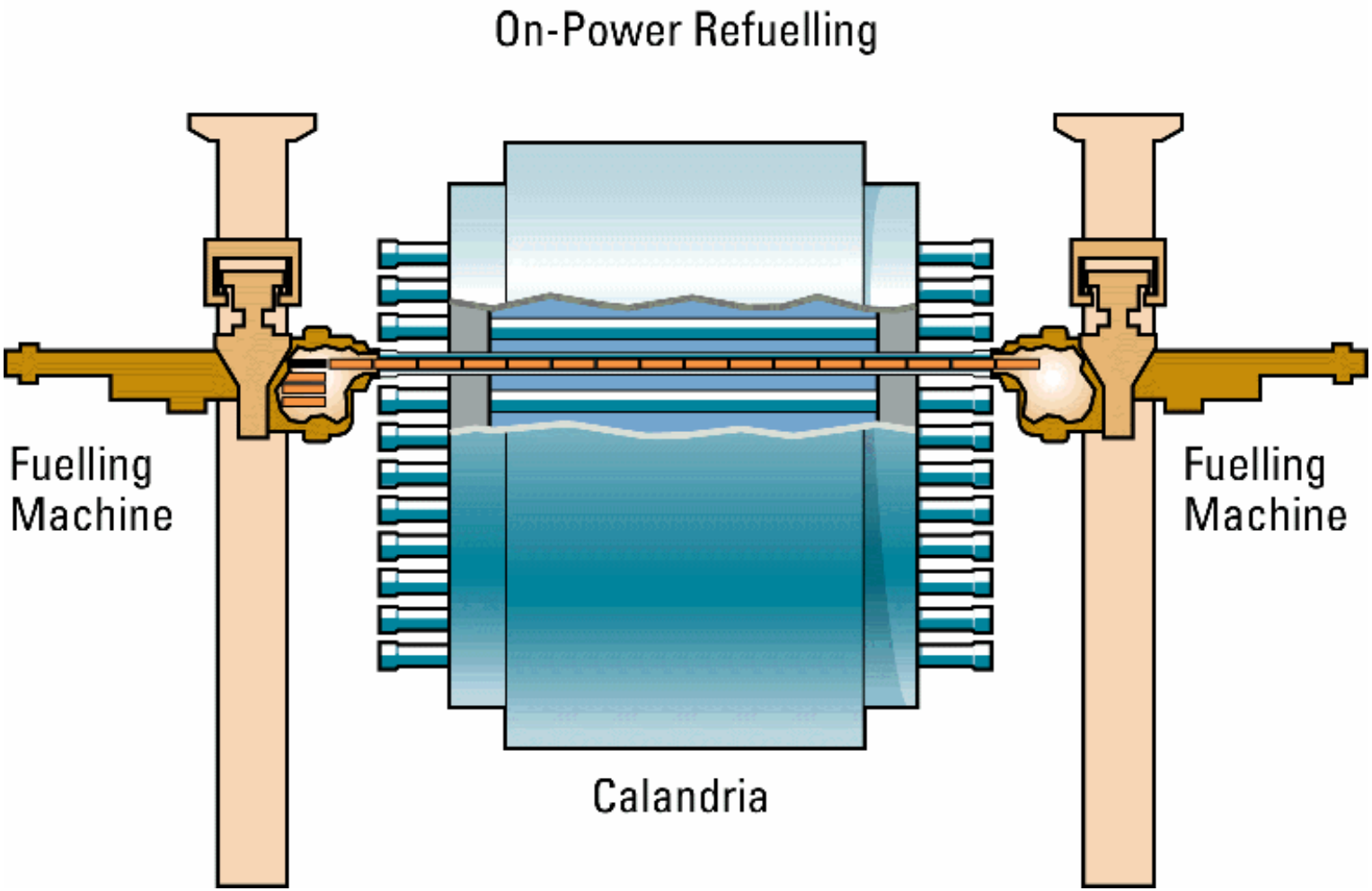
The CANDU Power Plant



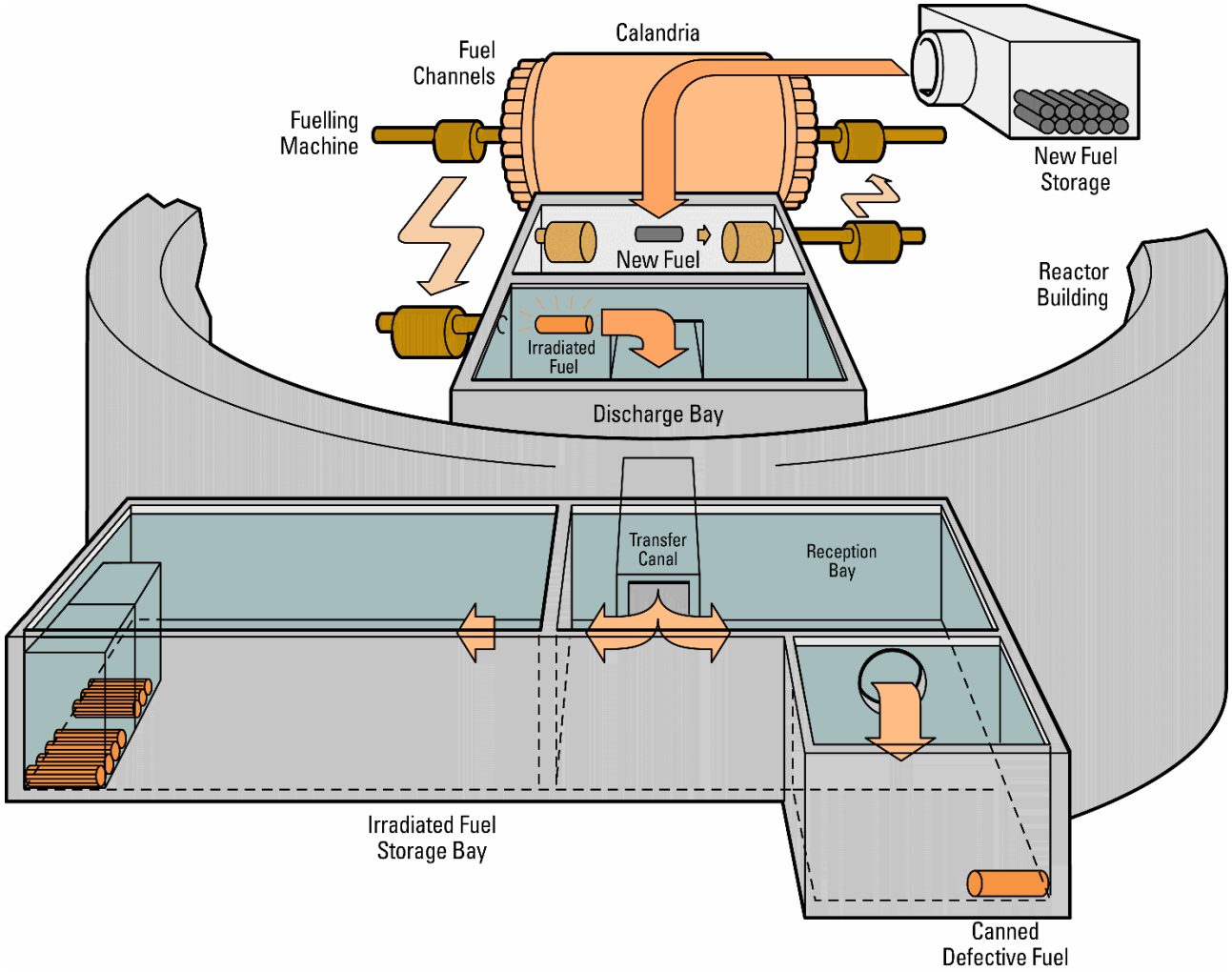
The Nuclear Steam Supply System



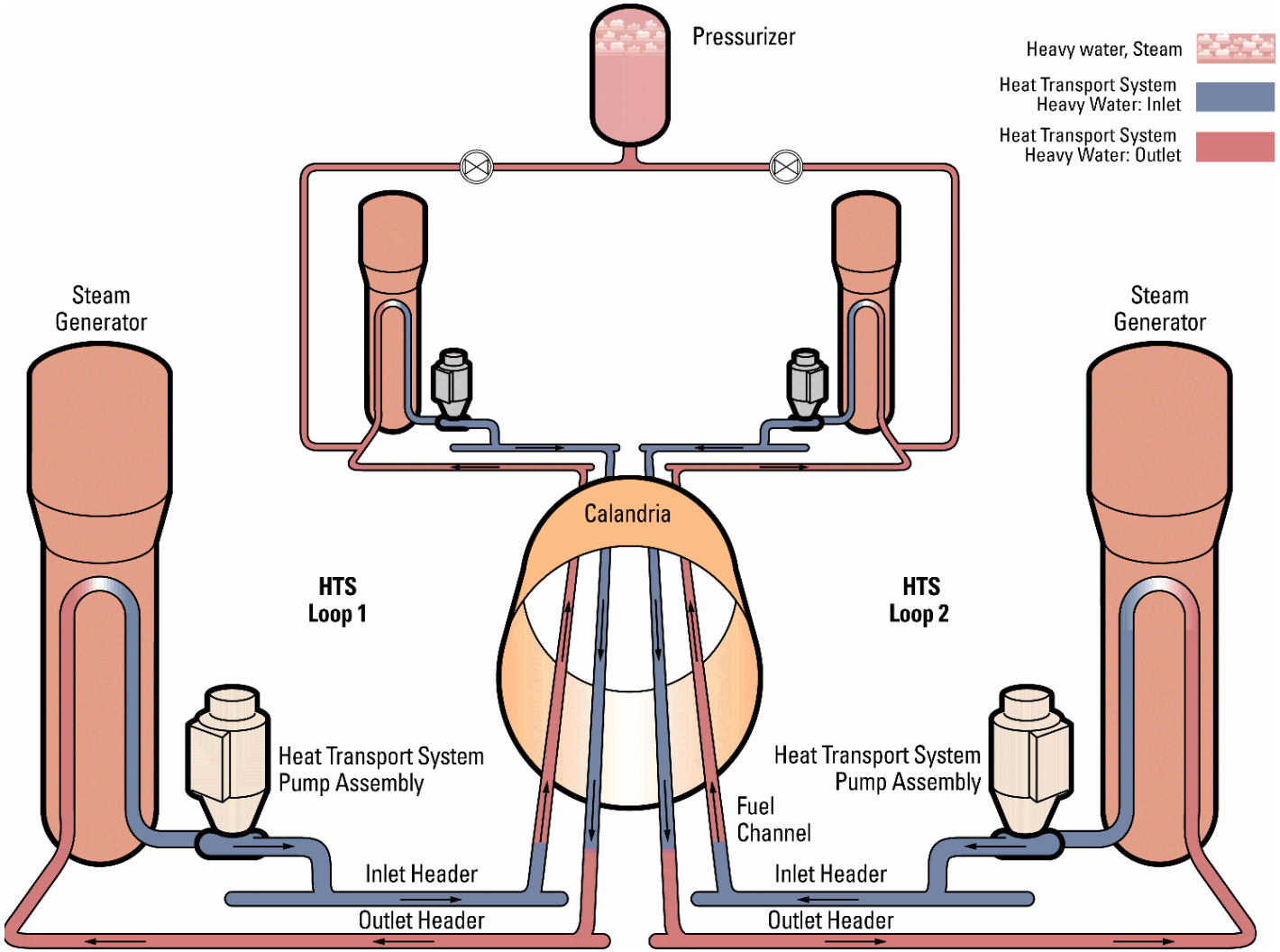
CANDU Reactor Fuelling



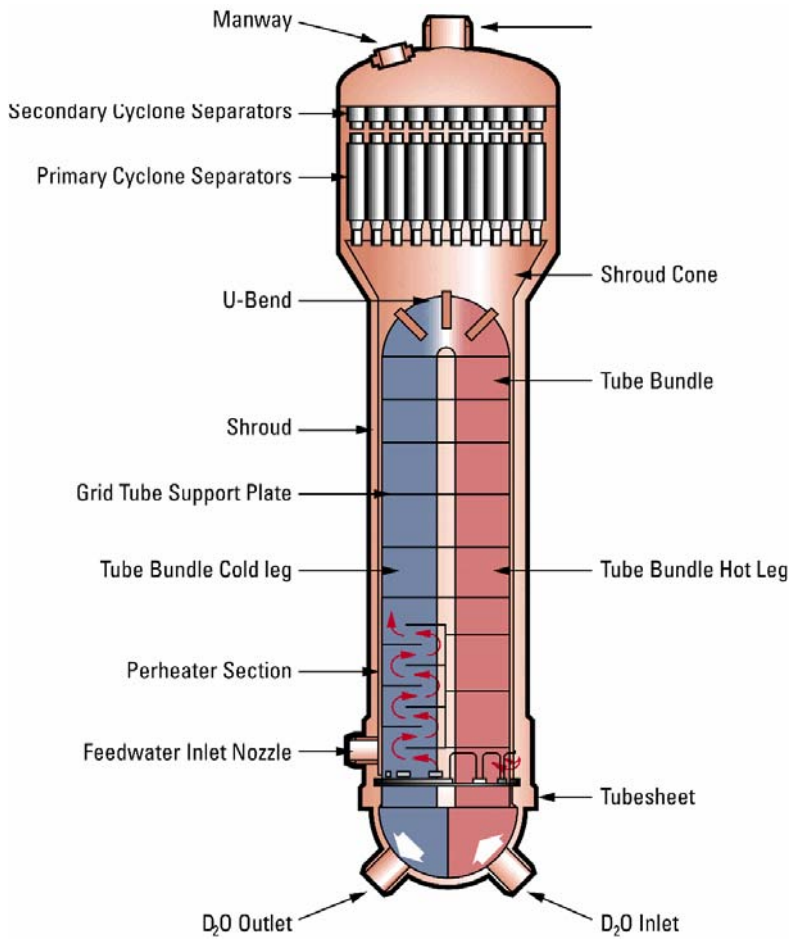
Fuel Handling



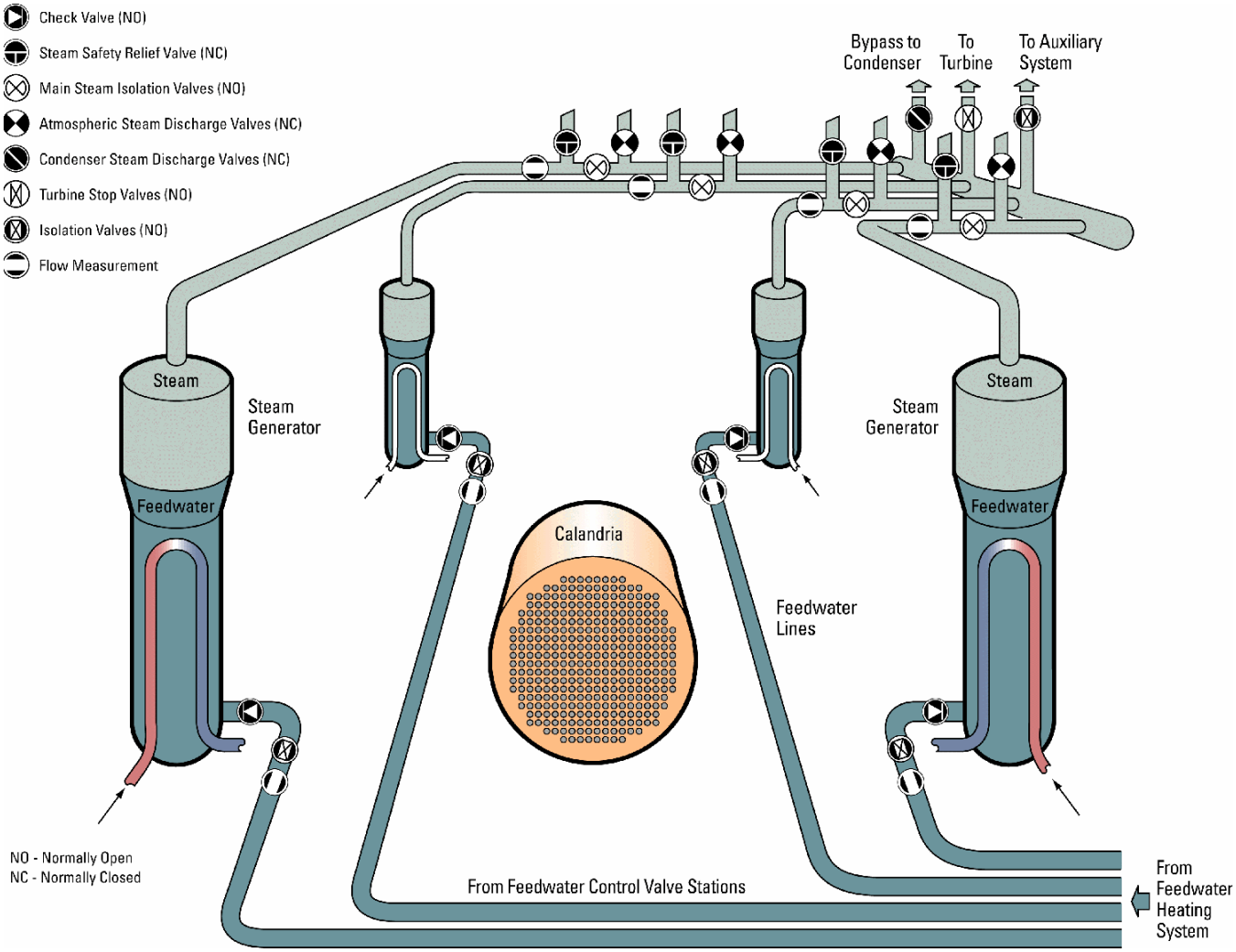
Primary Heat Transport System



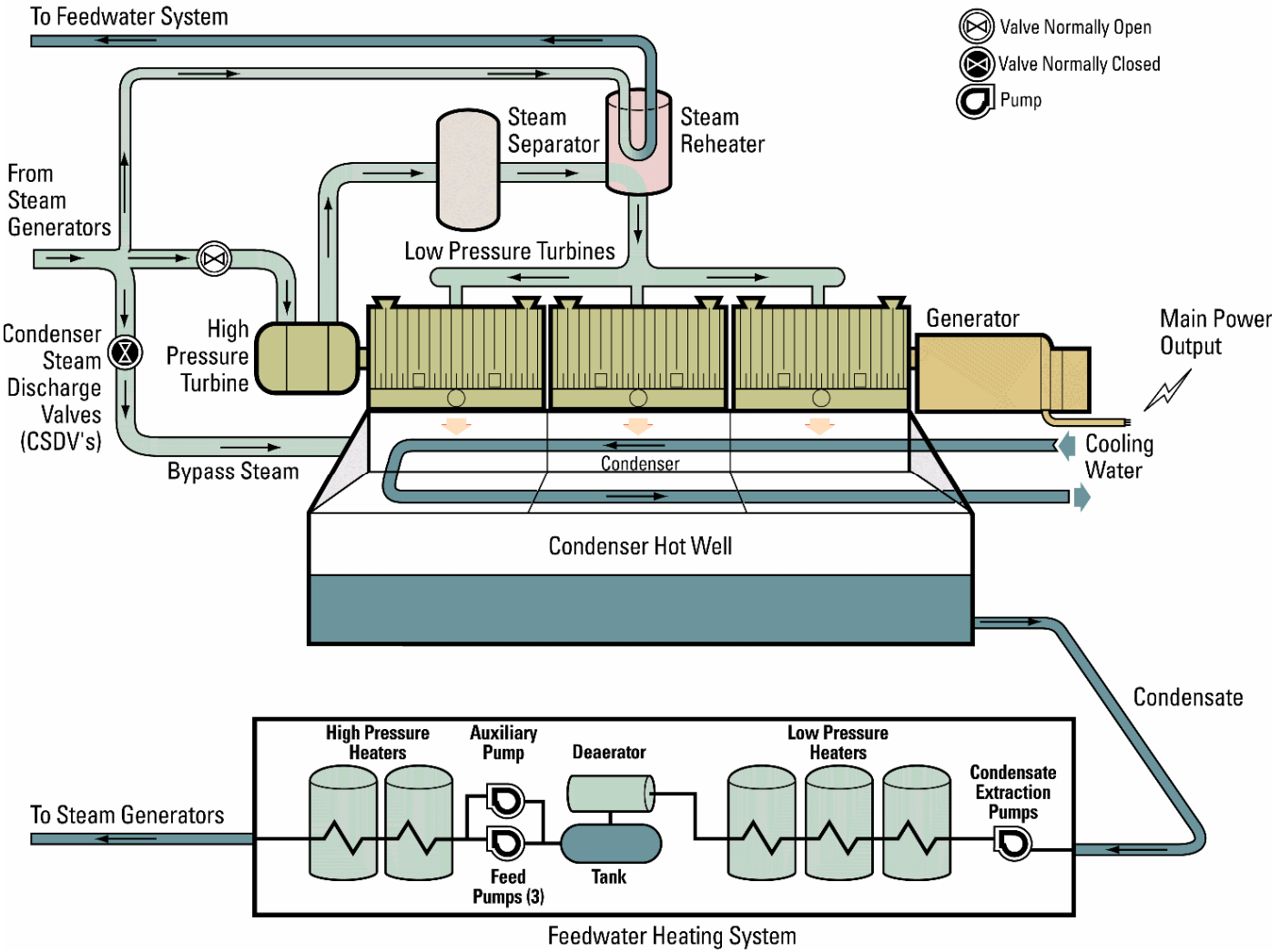
Steam Generator



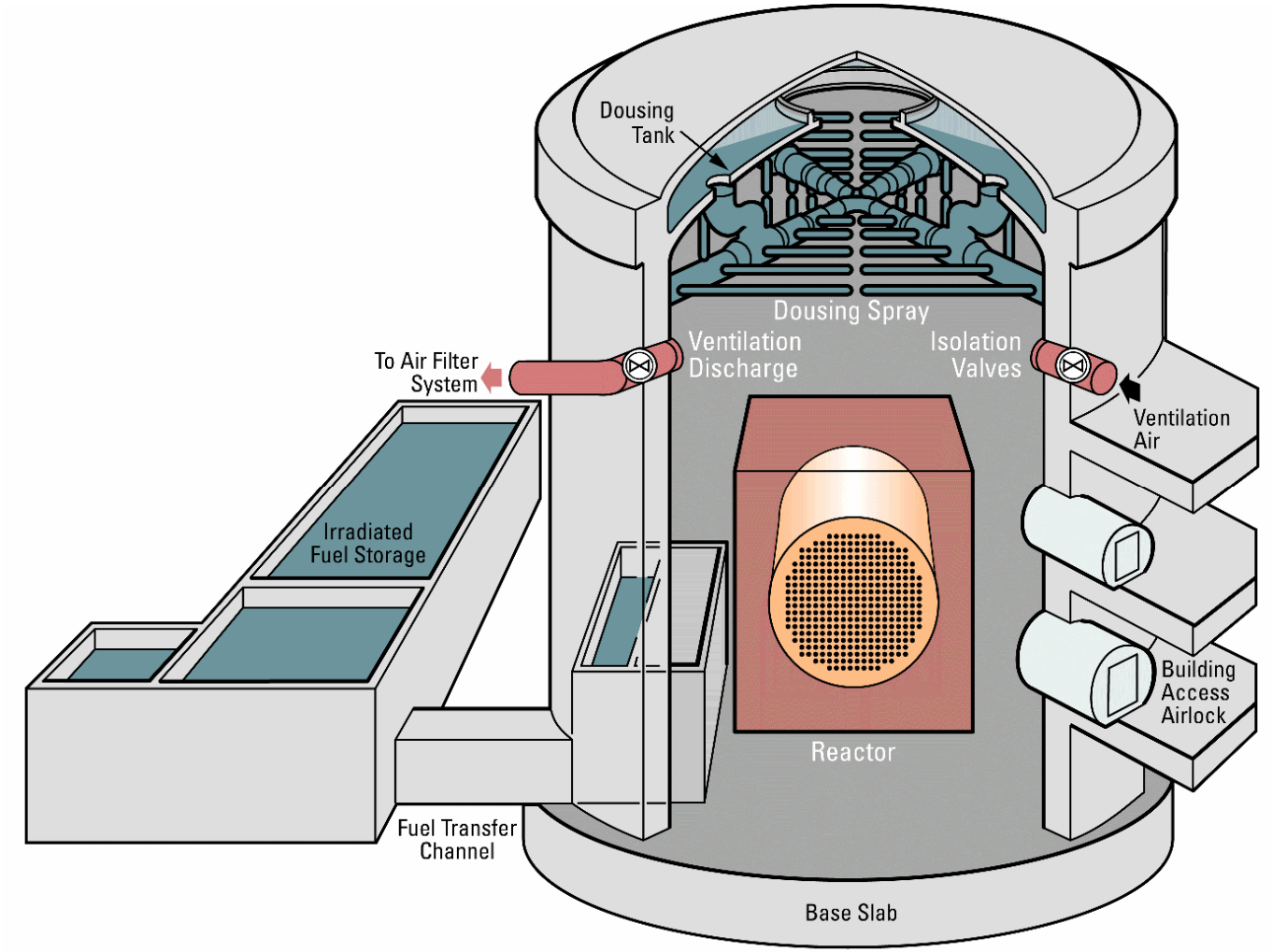
Secondary Heat Transport System



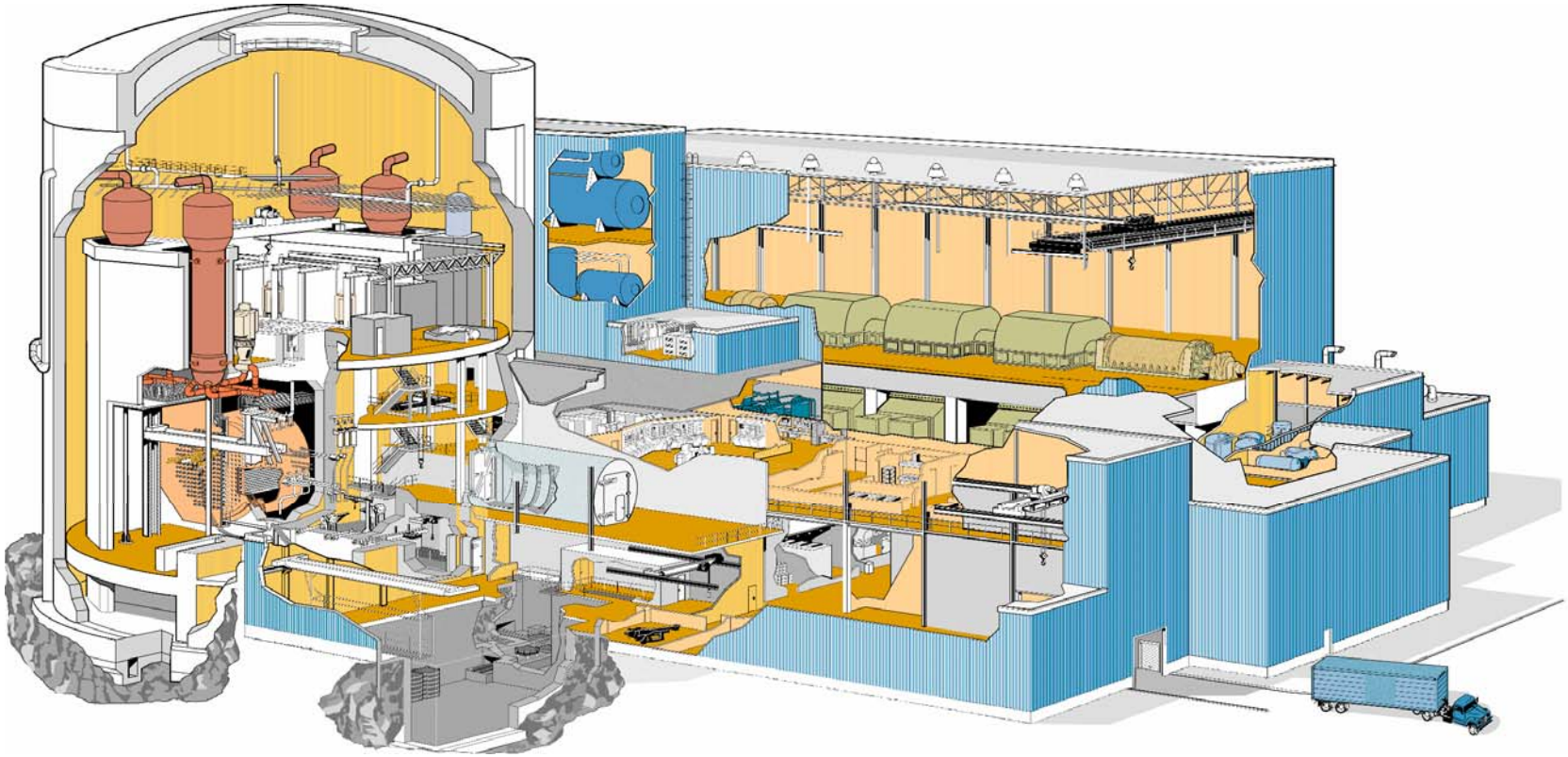
Turbine and Generator



Containment Building



Plant Layout



Neutron Diffusion and Moderation

Nomenclature

General Nomenclature

Consider a quantity, say the number of collisions N_{coll}:

Rate

We call *rate*, the ratio between the amount of that quantity that is found or produced between time t and time t+dt and dt. (i.e. the collision rate is the ratio between the number of collisions that occur between t and t+dt divided by dt):

$$R_{coll} = \frac{dN_{coll}}{dt}$$

Spectrum

We call (energy) *spectrum* the ratio between the amount of that quantity that is found or produced between energy E and E+dE and dE (i.e. the collision spectrum is the ratio between the number of collisions suffered by neutrons with energies between E and E+de and dE):

$$F(E) = \frac{dN_{coll}}{dE}$$

Density

We call (volumetric) *density*, the ratio between the total quantity dQ existing or produced in volume dV and dV (i.e. the collision density is the ratio between the number of collisions suffered by neutrons in volume dV and dV)

$$F = \frac{dN_{coll}}{dV}$$

We can have names that imply double ratios, e.g.

Collision *density spectrum*. - the ratio between the number of collisions suffered by neutrons in dV with energies between E and E+dE and dVdE

$$F(E) = \frac{dN_{coll}}{dVdE}$$

Collision *density rate*:

$$F = \frac{dN_{coll}}{dVdt}$$

Oftentimes, when talking about double ratios people omit to name one of them, so you must pay attention to the context.

For example, one will often refer to the collision rate or collision density, when, in fact, meaning collision density rate.

The same letter is sometimes used to denote different quantities.

Always look at the context.

Recapitulation of Important Concepts

Recapitulation of Important Concepts

Volumetric total reaction (collision) rate density for monoenergetic neutrons

$$F = \Sigma_t \Phi$$

or

$$F = \Sigma_t n \mathrm{v}$$

Reaction rate for neutrons with energies between E and E+dE:

$$dF = \Sigma(E) \times n(E)dE \times \mathrm{v(E)}$$

(Total) Reaction rate for neutrons of all energies:

$$F = \int_0^\infty \Sigma_t(E)n(E)\mathrm{v(E)dE} = \int_0^\infty \Sigma_t(E)\Phi(E)dE$$

where

$$\Phi(E) = n(E)\mathrm{v(E)}$$

Reaction Rates for Individual Reactions

Scattering reaction rate density:

$$F_s = \int_0^\infty \Sigma_s(E)\Phi(E)dE$$

Absorption reaction rate density (number of neutrons absorbed per cm^3 per s):

$$F_a = \int_0^\infty \Sigma_a(E)\Phi(E)dE$$

Neutron Diffusion

Fick’s Law

Fick's Law (Diffusion Law)

- Will accept it without proof.
- Valid far from interfaces.
- Valid for materials with relatively low absorption.

Gives the neutron current as a function of the neutron flux

Assume monoenergetic neutrons
Assume the flux only varies along the x axis:

$$J_x = -D\frac{d\Phi(x)}{dx}$$

D = Diffusion Coefficient

In three dimensions (and monoenergetic neutrons):

$$\vec{J} = -D\,grad\Phi = -D\nabla\Phi$$

Definition of gradient:

$$\nabla f(x, y, z) = \begin{bmatrix} \frac{\partial f}{dx} \\ \frac{\partial f}{dy} \\ \frac{\partial f}{dz} \end{bmatrix}$$

Number of particles crossing a surface of orientation $\vec{n}$ per unit time per unit area (normal current):

$$J_n = \vec{J} \cdot \vec{n}$$

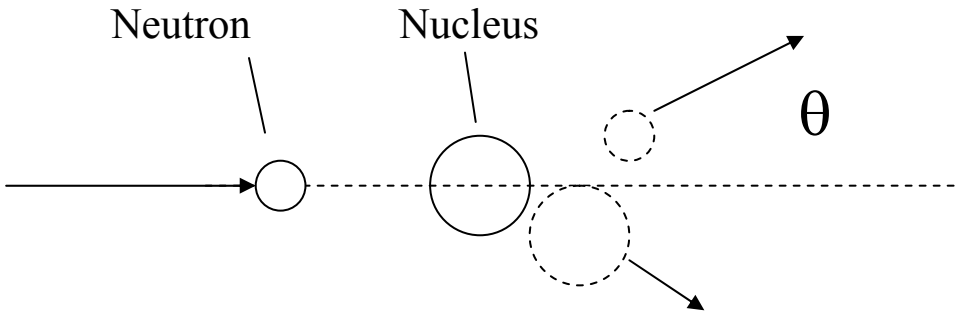
$$D = \frac{\lambda_{tr}}{3}, \quad (\lambda = \frac{1}{\Sigma})$$

Transport mean free path

$$\lambda_{tr} = \frac{1}{\Sigma_{tr}} = \frac{1}{\Sigma_s (1 - \overline{\mu})}$$

Average of the cosine of the scattering angle

$$\bar{\mu} = \overline{\cos\theta}$$



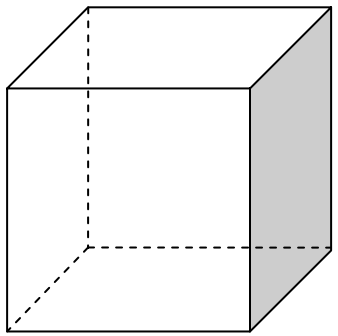
$$\bar{\mu} = \frac{2}{3A}$$

Diffusion Equation

Neutron Balance Equation (equation of Continuity) for Monoenergetic Neutrons

Expresses the conservation of neutrons

$$\left[\text{Rate of change in the number of neutrons in a small volume dV}\right] =$$
$$= \left[\text{Rate of neutron production in volume dV}\right] -$$
$$- \left[\text{Rate of neutron absorption in volume dV}\right] -$$
$$\left[\text{Rate of neutron leakage from dV}\right]$$

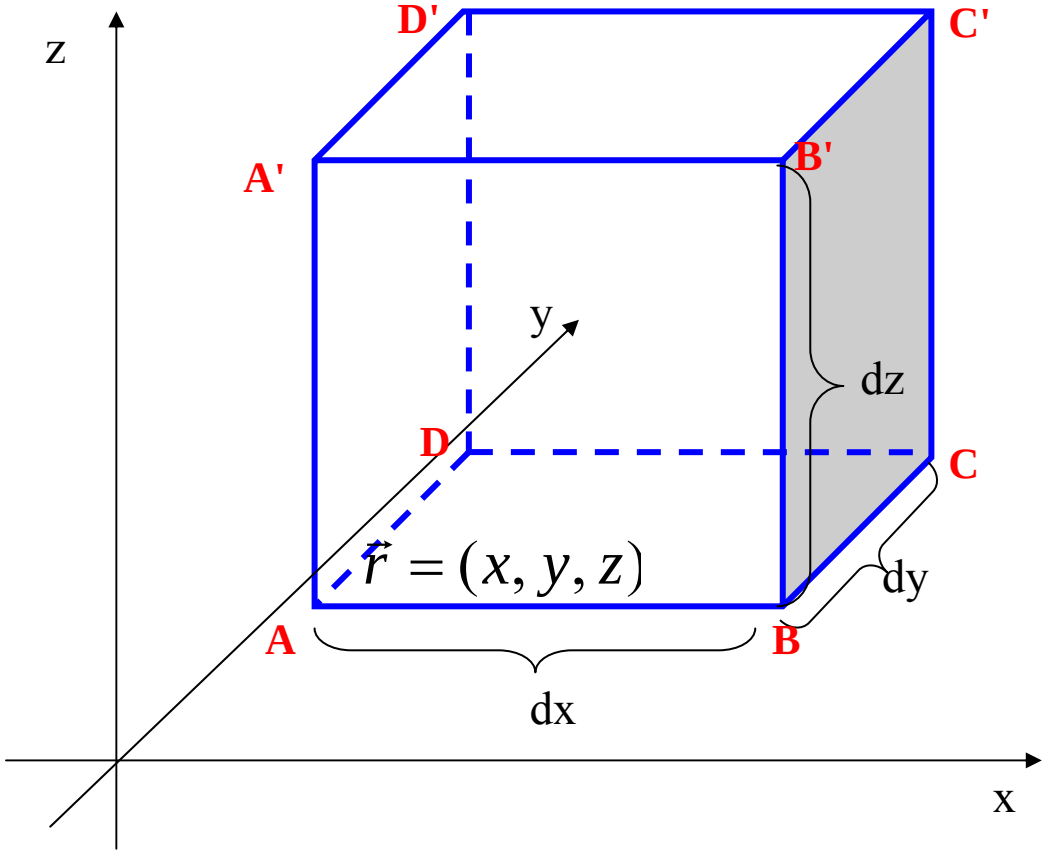


$$dV = dxdydz$$

Will follow a derivation slightly different from the one in the textbook.

You are welcome to use the derivation in the book. Brush up on your vector calculus and Gauss' formula if you want to follow the derivation in the book.

Infinitesimal (i.e. very small) Volume



$$dV = dxdydz$$

dx, dy, dz small-enough that:

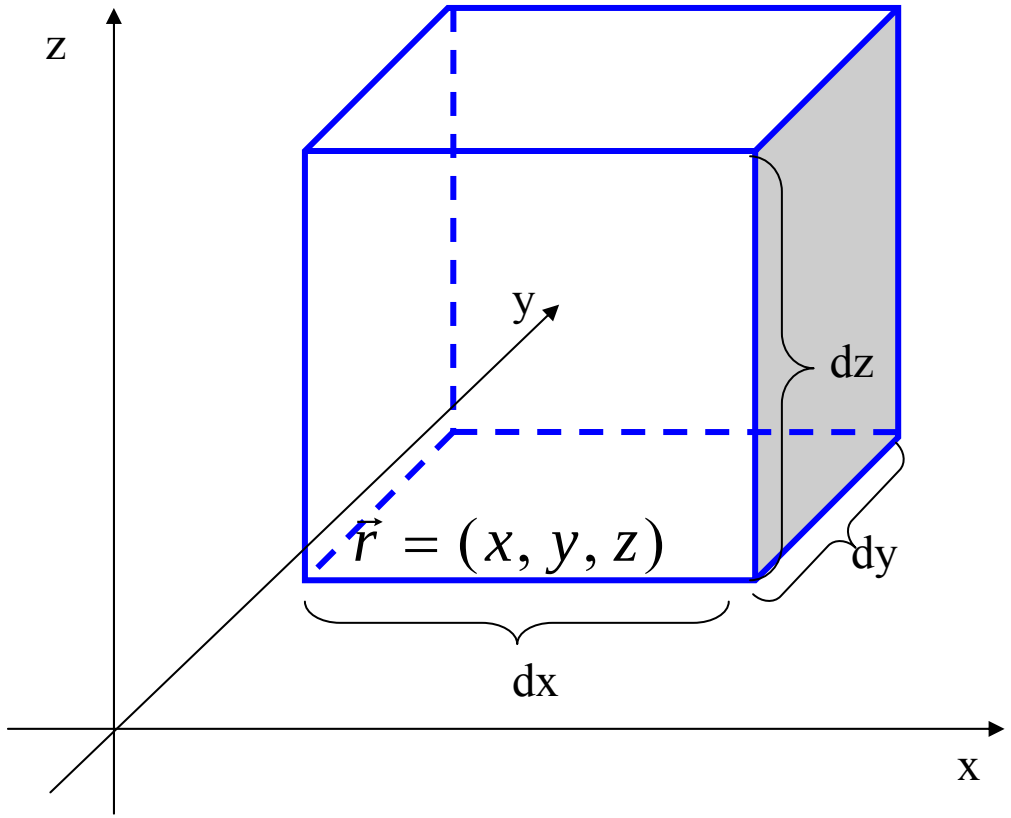
$$\Phi(x, y, z) \cong \Phi(x + dx, y, z)$$

$$\vec{J}(x, y, z) \cong \vec{J}(x + dx, y, z)$$

and similarly for y and z

Production

The number of neutrons being produced per unit time in volume dV. (Neutron Source)



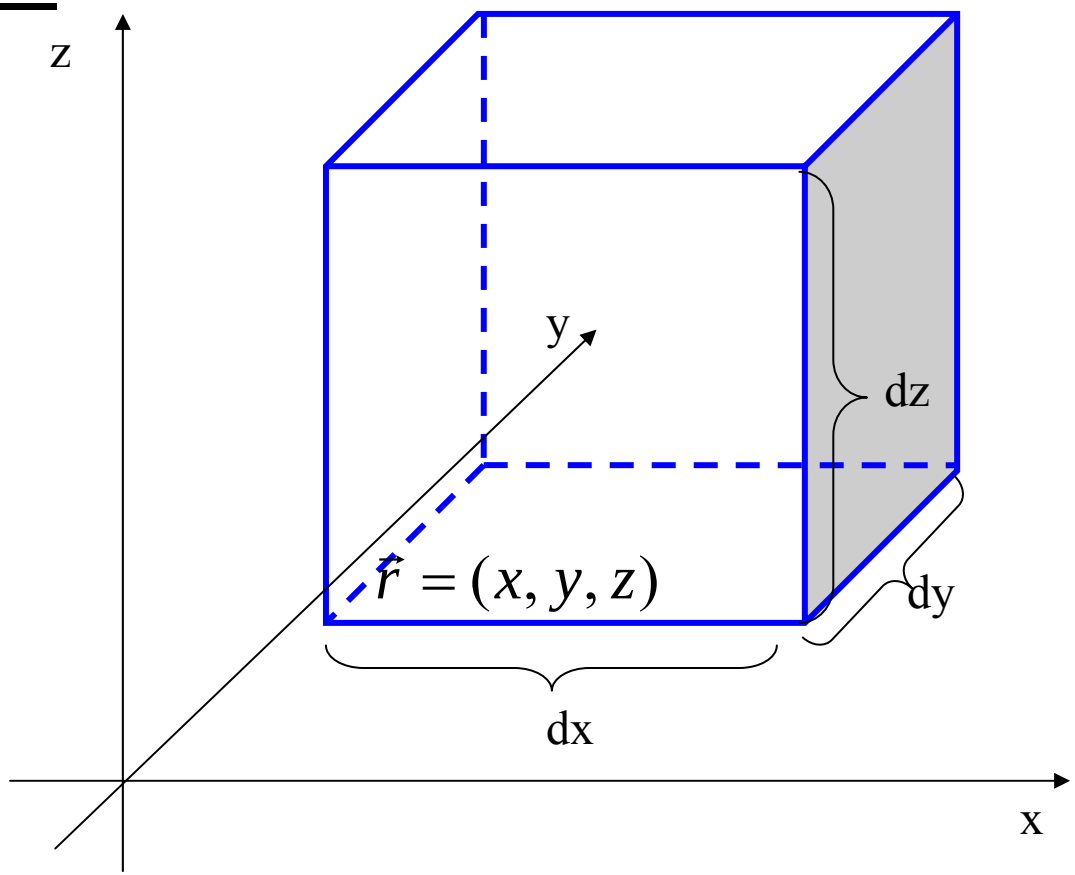
Production Rate:

SdV

$s(cm^{-3}s^{-1})$

Production Rate $= SdV = Sdxdydz$

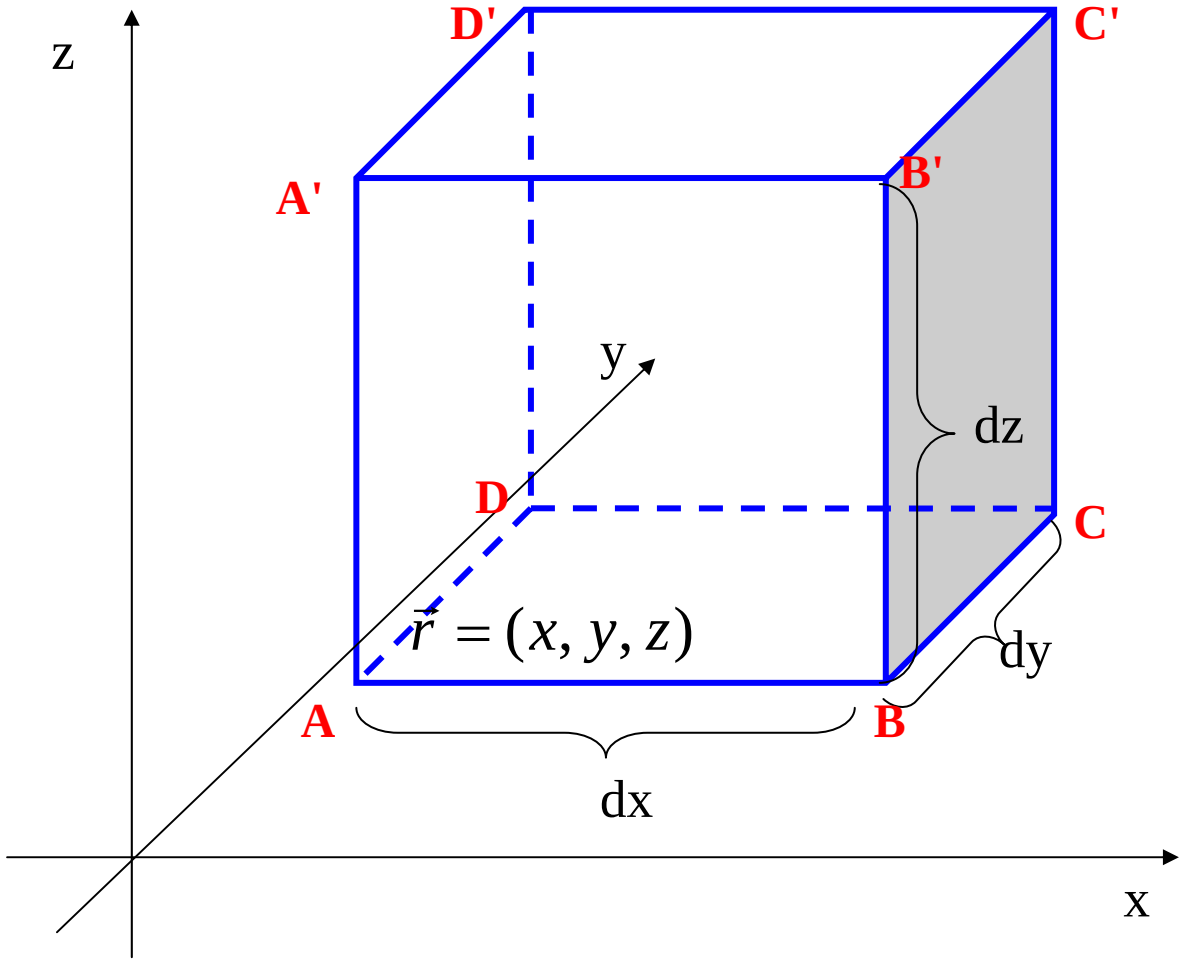
Absorption



Assume, dx, dy, dz are small enough that the flux Φ varies negligibly inside our volume

$$R_a = \Sigma_a \Phi dV = \Sigma_a \Phi dx dy dz$$

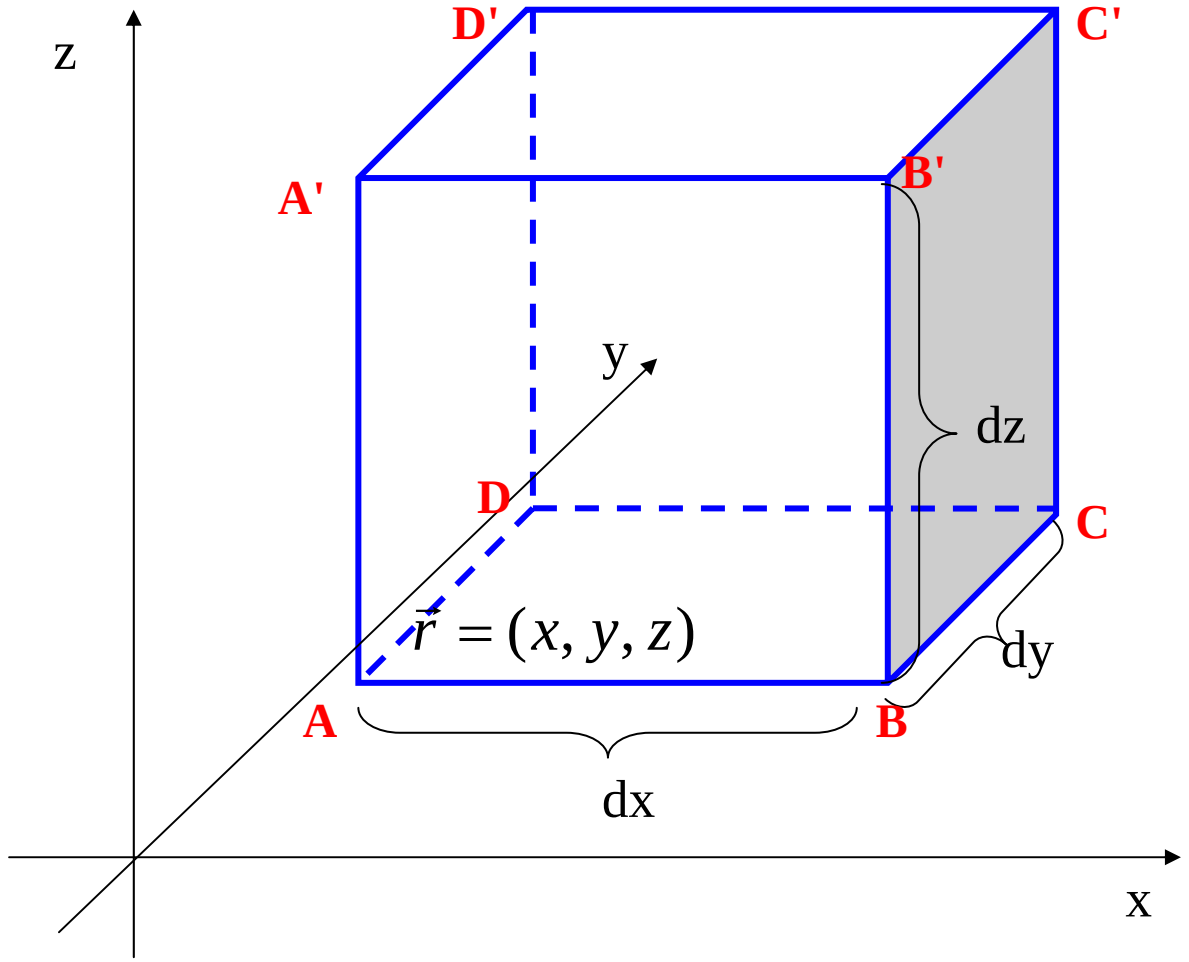
Leakage Through Face BCC'B'



$$LK_{x+} = LK_{BCC'B'} \cong \vec{J}\left(x + dx, y + \frac{dy}{2}, z + \frac{dz}{2} \right) \cdot \hat{u}_x dydz$$

$$LK_{x+} = LK_{BCC'B'} \cong J_x\left(x + dx, y, z\right)dydz$$

Leakage Through Face ADD'A'



$$LK_{x-} = LK_{ADD'A'} \cong \vec{J}\left(x, y + \frac{dy}{2}, z + \frac{dz}{2}\right) \cdot \left(-\hat{u}_x\right) dydz$$

$$LK_{x-} = LK_{ADD'A'} \cong -J_x\left(x, y, z\right) dydz$$

Net Leakage Along X Axis

$$LK_x = LK_{x+} + LK_{x-} = LK_{BCC'B'} + LK_{ADD'A'} \cong$$
$$\cong J_x(x+dx, y, z) - J_x(x, y, z)dydz$$

Let's remember that:

$$\frac{\partial J_x}{\partial x}(x, y, z) = \frac{J_x(x+dx, y, z) - J_x(x, y, z)dydz}{dx}$$

Hence:

$$J_x(x+dx, y, z) - J_x(x, y, z)dydz = \frac{\partial J_x}{\partial x}(x, y, z)dx$$

$$LK_x \cong J_x(x+dx, y, z) - J_x(x, y, z)dydz =$$
$$= \frac{\partial J_x}{\partial x}(x, y, z)dxdydz$$

Total Leakage out of dV

$$LK = LK_x + LK_y + LK_z =$$

$$= \frac{\partial J_x}{\partial x}(x,y,z)dxdydz + \frac{\partial J_y}{\partial y}(x,y,z)dxdydz + \frac{\partial J_z}{\partial z}(x,y,z)dxdydz =$$

$$= \left(\frac{\partial J_x}{\partial x} + \frac{\partial J_y}{\partial y} + \frac{\partial J_z}{\partial z} \right)\Bigg|_{x,y,z} dxdydz = \left(div\vec{J}\right)dxdydz = \left(\nabla \cdot \vec{J}\right)dxdydz$$

Definition of divergence for a vector function $\vec{f}(x,y,z)$:

$$div\vec{f} \equiv \nabla \cdot \vec{f} = \left(\frac{\partial f_x}{dx} + \frac{\partial f_y}{dy} + \frac{\partial f_z}{dz} \right)$$

Rate of Change of Number of Neutrons in dV

$$R_{change} = \frac{\#neutrons(t+dt) - \#neutrons(t)}{dt} = \frac{n(t+dt)dV - n(t)dV}{dt} =$$

$$= \frac{n(t+dt) - n(t)}{dt} dV = \frac{\partial n}{\partial t} dxdydz$$

Neutron Balance Equation for dV

$$\frac{\partial n}{\partial t} dxdydz = sdxdydz - \Sigma_a \Phi dxdydz - \nabla \cdot \vec{J} dxdydz$$

Dividing by the volume $dV = dxdydz$ we obtain:

$$\frac{\partial n}{\partial t} = s - \Sigma_a \Phi - \nabla \cdot \vec{J}$$

Valid regardless of whether Fick's law holds true or not

Neutron Balance in the Diffusion Approximation

Assume Fick's Law to be true:

$$\vec{J} = -D\nabla\Phi$$

Substitute into the neutron balance eq:

$$\frac{\partial n}{\partial t} = -\nabla \cdot \left(-D\nabla\Phi\right) - \Sigma_a \Phi + s$$

This is the *time-dependent diffusion equation for monoenergetic neutrons.*

It is important because by solving it we find the flux and the flux allows us to calculate all reaction rates, including fission rate - which is really what we are after, by using $R = \Sigma\Phi$.

If the diffusion coefficient is constant:

$$\frac{\partial n}{\partial t} = D\nabla \cdot \left(\nabla \Phi\right) - \Sigma_a \Phi + s$$

Remember the definition of the Laplacian:

$$\Delta f(x, y, z) \equiv \nabla^2 f = \nabla \cdot \left(\nabla f\right) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

The diffusion eq. can then be rewritten:

$$\frac{\partial n}{\partial t} = D\nabla^2 \Phi - \Sigma_a \Phi + s$$

If we keep in mind that

$$\phi = n\mathrm{v} \Rightarrow n = \frac{\Phi}{\mathrm{v}} \Rightarrow \frac{\partial n}{\partial t} = \frac{\partial}{\partial t}\left(\frac{\Phi}{V}\right) = \frac{1}{V}\frac{d\Phi}{dt}$$

We obtain:

$$\frac{1}{\mathrm{v}}\frac{\partial \phi}{\partial t} = D\nabla^2\Phi - \Sigma_a\Phi + s$$

Steady-State Situation (no time dependence)

$$D\nabla^2\Phi - \Sigma_a\Phi + s = 0$$

Steady-State diffusion equation for monoenergetic neutrons and constant D

Dividing by D:

$$\nabla^2\Phi - \frac{\Sigma_a}{D}\Phi + \frac{s}{D} = 0$$

Introducing notation (Diffusion Length):

$$L^2 = \frac{D}{\Sigma_a}$$

$$-\nabla^2\Phi + \frac{1}{L^2}\Phi = \frac{s}{D}$$

Interface Conditions for the Diffusion equation:

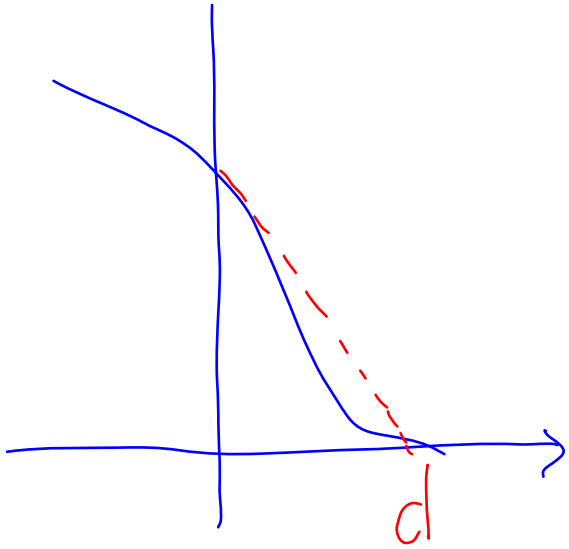
Continuity of flux: $\Phi_A = \Phi_B$

Continuity of normal component of current $J_{A\perp} = J_{B\perp}$

Vacuum Interface

$$\Phi(d) = 0$$

Extrapolation distance $d = 0.71\lambda_{tr}$



$$d = 2.13D$$

$$\lambda_{tr} = \frac{1}{\Sigma_{tr}}; \quad \Sigma_{tr} = \Sigma_s(1-\bar{\mu}); \quad D = \frac{1}{3\Sigma_{tr}} \Rightarrow \Sigma_{tr}\,\frac{1}{3D}$$

The Concept of Infinite Homogeneous Medium

Medium is the same at any point
Hence, there is no reason why the flux would be different an any particular point

$$\Phi(x, y, z) = \Phi = const$$

The current is given by Fick's Law

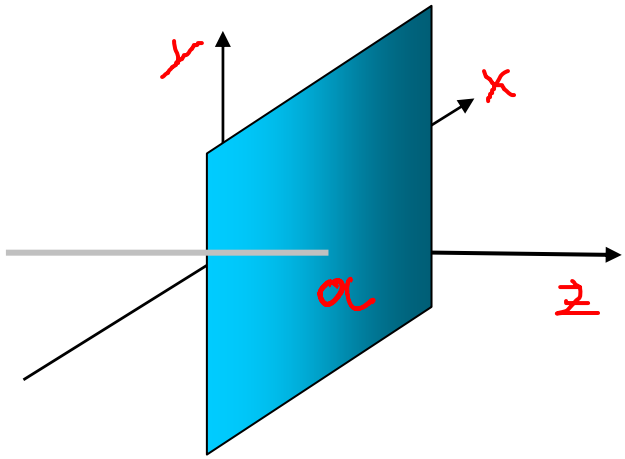
$$\vec{J} = \nabla\Phi = \begin{bmatrix} \frac{\partial\Phi}{\partial x} \\ \frac{\partial\Phi}{\partial y} \\ \frac{\partial\Phi}{\partial z} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0$$

The current is zero in an infinite homogeneous medium

$$\Sigma_a\Phi = S; \quad \nabla^2\Phi = 0; \quad \Phi = \frac{S}{\Sigma_a}$$

The Concept of Homogeneous Half Space

$$x \in (-\infty, \infty)$$
$$y \in (-\infty, \infty)$$
$$z \in (a, \infty)$$



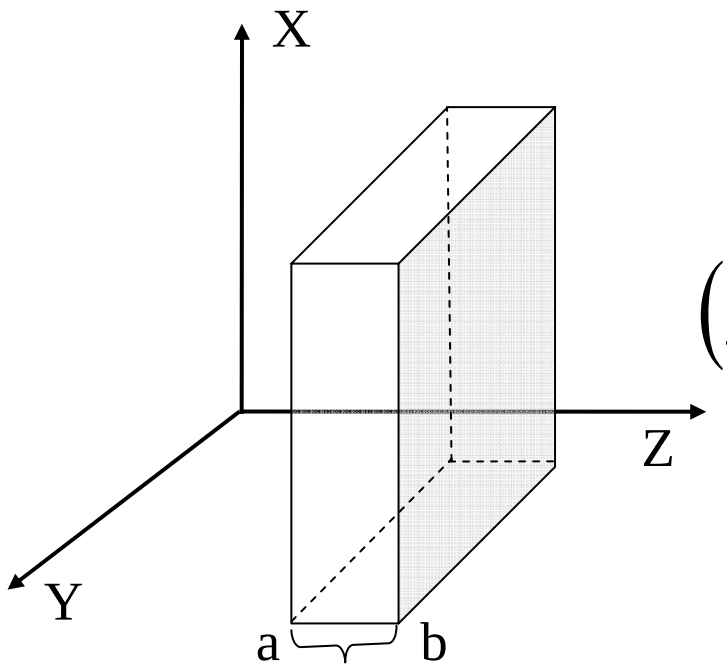
In such a configuration, since for the same z all points are identical, there is no variation in the flux with x or y

$$\Phi(x, y, z) = \Phi(z); \quad \nabla^2\Phi = \left(0 + 0 + \frac{\partial^2 \Phi}{\partial z^2} \right)$$

$$D\frac{\partial^2 \Phi(z)}{\partial z^2} - \Sigma_a \Phi(z) + S(z) = 0$$

The Concept of Infinite Homogeneous Slab

- finite in Z, but infinite in X and Y directions



$$(x,y,z)\;\Big|\;\begin{array}{l} x\in(-\infty;\infty) \\ y\in(-\infty;\infty) \\ z\in(a;b) \end{array}$$

Because there is no change in the material properties in either X or Y direction,

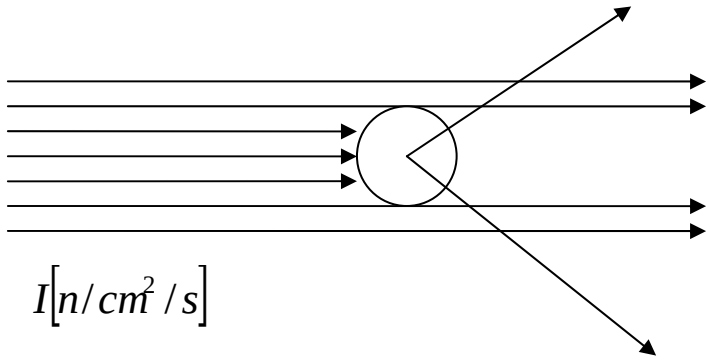
$$\Phi(x,y,z)=\Phi(z)$$

Energy-Dependent Diffusion

Differential Microscopic Scattering Cross Sections

Beam of monoenergetic neutrons

$$I = n\mathrm{v}(E)$$



Scattering rate:

$$R_s = I\sigma_s(E)$$

Equivalently, we can write (using only macroscopic quantities that can be measured):

$$\sigma_s(E) = \frac{R_s}{I}$$

By scattering, neutrons lose energy.

Let $dR_s(E')$ be the rate at which neutrons are scattered in energy range E', E'+dE'

We have:

$$\int_0^\infty dR_s(E') = R_s$$

Definition of the differential scattering microscopic cross section

$$\sigma_s(E \to E') \equiv \frac{dR_s(E')}{I dE'}$$

Equivalently, we can write:

$$\sigma_s(E \to E') \equiv \frac{dR_s(E')}{I dE'} \frac{R_s}{R_s} = \frac{R_s}{I} \frac{dR_s(E')}{R_s dE'} = \sigma_s(E) \frac{dR_s(E')}{R_s dE'}$$

Scattering Kernel

$$k(E \to E') = \frac{dR_s(E')}{R_s dE'} = \frac{dP(E, E')}{dE'}$$

$$\sigma_s(E \to E') = \sigma_s k(E \to E')$$

The scattering kernel can be interpreted as the probability density function for a neutron of energy E to be scattered such that its final energy is between E' and E'+dE'.

The differential and total scattering cross section satisfy:

$$\sigma_s(E) = \int_0^\infty \sigma_s(E \to E')dE'$$

Differential Macroscopic Scattering Cross Sections

$$\Sigma_s(E \to E') = N\sigma_s(E \to E')$$

or, using the scattering kernel:

$$\Sigma_s(E \to E') = N\sigma_s(E)k(E \to E') =$$
$$= \Sigma_s(E)k(E \to E')$$

Volumetric reaction rate at which neutrons scatter within energy range (E, E+dE)

$$R_s(E \to E') = I\Sigma_s(E \to E')$$

Energy-Dependent Neutron Balance Equation

Balance Equation for Neutrons with Energy Between E and E+dE

[rate of change of number of neutrons in volume dV with energy within range (E, E+dE)] =
[rate of production in volume dV of neutrons with energy within range (E, E+dE)] +
[rate of scattering of neutrons in dV into energy range (E, E+dE)] -
[rate of absorption in dV of neutrons with energy in range (E, E+dE)] -
-[rate of scattering of neutrons in dV outside of energy range (E, E+dE)] -
[rate of leakage out of dV of neutrons with energy within range (E, E+dE)]

[rate of change of number of neutrons in volume dV with energy within range (E, E+dE)]

$$R_{change}(E) = \frac{n(E,t+dt)dEdV - n(E,t)dEdV}{dt} =$$

$$= \frac{n(E,t+dt) - n(E,t)}{dt} dV = \frac{\partial n(E)}{\partial t} dEdV$$

[rate of production in volume dV of neutrons with energy within range (E, E+dE)]

$$R_p(E) = s(E)dEdV$$

s(E) = number of neutrons produced inside dV with energies between E and E+dE, divided by dEdV.

[rate of scattering of neutrons in dV into energy range (E, E+dE)]

Rate at which neutrons with energy within $(E';E'+dE)$ scatter such that their energy is within $(E;E+dE)$

$$R_s(E' \rightarrow E) = \Phi(E')dE' \times \Sigma_s(E' \rightarrow E)dEdV$$

Rate at which all neutrons scatter such that their energy is within $(E;E+dE)$

$$R_s(\rightarrow E) = \left[\int_0^{\infty} \Phi(E') \times \Sigma_s(E' \rightarrow E)dE'\right]dEdV$$

[rate of absorption in dV of neutrons with energy in range (E, E+dE)]

$$R_a(E) = \Phi(E)dE \times \Sigma_a(E)dV = \Phi(E) \times \Sigma_a(E)dEdV$$

[rate of scattering of neutrons in dV outside of energy range (E, E+dE)]

$$R_s(E \rightarrow) = \Phi(E)dE \times \Sigma_s(E)dV = \Phi(E) \times \Sigma_s(E)dEdV$$

Note that:

$$\Sigma_s(E) = \int_0^{\infty} \Sigma_s(E \rightarrow E')dE'$$

[rate of leakage out of dV of neutrons with energy within range (E, E+dE)]

$$LK(E) = \nabla \cdot \vec{J}(E) dE dV$$

Balance Equation for Neutrons with Energy Between E and E+dE

$$R_{change}(E) = R_p(E) + R_s(\rightarrow E) - R_a(E) - R_s(E \rightarrow) - LK(E)$$

$$\frac{\partial n(E)}{\partial t} dEdV = s(E)dEdV + \left[\int_0^\infty \Phi(E') \times \Sigma_s(E' \rightarrow E)dE'\right] dEdV -$$

$$- \Phi(E) \times \Sigma_a(E)dEdV - \Phi(E) \times \Sigma_s(E)dEdV - \nabla \cdot \vec{J}(E)dEdV$$

Dividing by dEdV we obtain the energy-dependent neutron balance equation (continuity equation):

$$\frac{\partial n(E)}{\partial t} = s(E) + \int_0^\infty \Phi(E') \times \Sigma_s(E' \rightarrow E)dE' -$$

$$- \Phi(E) \times \Sigma_a(E) - \Phi(E) \times \Sigma_s(E) - \nabla \cdot \vec{J}(E)$$

We can show the dependence on time explicitly:

$$\frac{\partial n(E,t)}{\partial t} = s(E,t) + \int_0^\infty \Phi(E',t) \times \Sigma_s(E' \to E) dE' -$$

$$- \Phi(E,t) \times \Sigma_a(E) - \Phi(E,t) \times \Sigma_s(E) - \nabla \cdot \vec{J}(E,t)$$

Definition of energy-dependent flux:

$$\Phi(E) = n(E)\mathrm{v(E)} \Rightarrow n(E) = \frac{\Phi(E)}{\mathrm{v(E)}}$$

Substituting the expression for the energy-dependent neutron density, we obtain:

$$\frac{1}{\mathrm{v}(E)} \frac{\partial \Phi(E,t)}{\partial t} = s(E,t) + \int_0^\infty \Phi(E',t) \times \Sigma_s(E' \to E) dE' -$$

$$- \Phi(E,t) \times \Sigma_a(E) - \Phi(E,t) \times \Sigma_s(E) - \nabla \cdot \vec{J}(E,t)$$

Energy-Dependent Steady-State Neutron Balance Equation

$$0 = s(E,t) + \int_0^\infty \Phi(E',t) \times \Sigma_s(E' \rightarrow E) dE' -$$

$$- \Phi(E,t) \times \Sigma_a(E) - \Phi(E,t) \times \Sigma_s(E) - \nabla \cdot \vec{J}(E,t)$$

Diffusion Approximation (use Fick's Law)

$$\vec{J}(E) = D(E) \nabla \Phi(E)$$

$$0 = s(E,t) + \int_0^\infty \Phi(E',t) \times \Sigma_s(E' \rightarrow E) dE' -$$

$$- \Phi(E,t) \times \Sigma_a(E) - \Phi(E,t) \times \Sigma_s(E) + \nabla \cdot \left(D(E) \nabla \Phi(E) \right)$$

For position-independent diffusion coefficient:

$$0 = s(E,t) + \int_0^{\infty} \Phi(E',t) \times \Sigma_s(E' \rightarrow E) dE' -$$

$$- \Phi(E,t) \times \Sigma_a(E) - \Phi(E,t) \times \Sigma_s(E) + D(E) \nabla^2 \Phi(E)$$

Multigroup Formalism

Approximate treatment of the energy-dependent diffusion equation.

Energy Groups

Divide the energy domain $(0, E_{\max})$ into intervals called *groups*

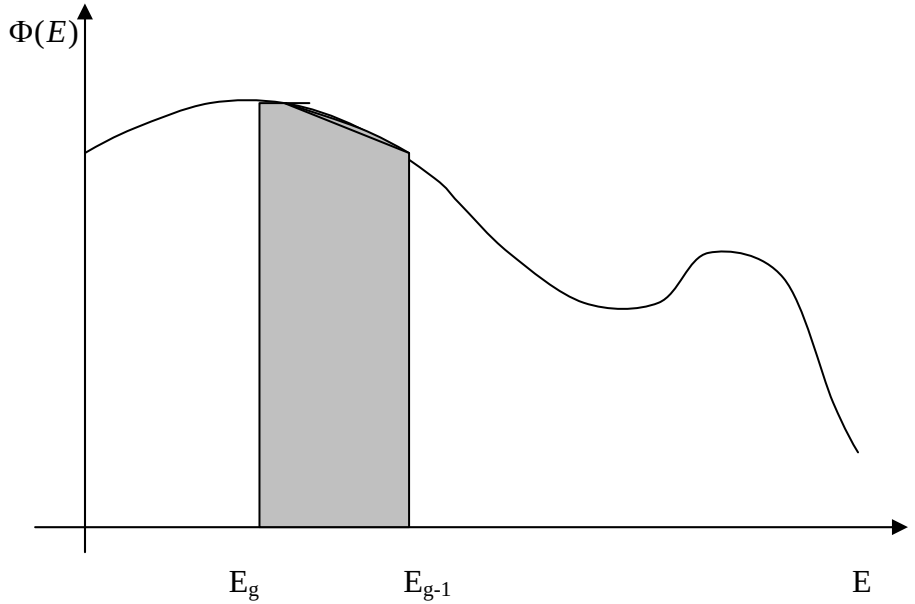
$$0 = E_G < E_{G-1} ..... < E_2 < E_1 < E_0 = E_{\max}$$

Group neutron density

$$n_g \equiv \int_{E_g}^{E_{g-1}} n(E)dE$$

(Energy) Group Flux

$$\Phi_g \equiv \int_{E_g}^{E_{g-1}} \Phi(E)dE$$



Can depend on parameters such as position and/or time

$$\Phi_g(\vec{r}) \equiv \int_{E_g}^{E_{g-1}} \Phi(\vec{r}, E)dE$$

Group Current

$$\vec{J}_g \equiv \int_{E_g}^{E_{g-1}} \vec{J}(E)dE$$

Can depend on parameters such as position and/or time

$$\vec{J}_g(\vec{r}) \equiv \int_{E_g}^{E_{g-1}} \vec{J}(\vec{r},E)dE$$

Group Source

$$s_g = \int_{E_g}^{E_{g-1}} s(E)dE$$

Group Reaction Rates

Reaction Rate for a single Nucleus

$$R_g^{\text{single nucleus}} \equiv \int_{E_g}^{E_{g-1}} R(E)dE = \int_{E_g}^{E_{g-1}} \Phi(E)\sigma(E)dE$$

Reaction Rate Density for a Material

$$R_g \equiv \int_{E_g}^{E_{g-1}} R(E)dE = \int_{E_g}^{E_{g-1}} \Phi(E)\Sigma(E)dE$$

Can depend on parameters such as position and/or time

$$R_g(\vec{r}) \equiv \int_{E_g}^{E_{g-1}} R(\vec{r},E)dE = \int_{E_g}^{E_{g-1}} \Phi(\vec{r},E)\Sigma(\vec{r},E)dE$$

Group Cross Sections

Microscopic Group Cross Sections

$$\sigma_g \equiv \frac{\int_{E_g}^{E_{g-1}} \Phi(E)\sigma(E)dE}{\int_{E_g}^{E_{g-1}} \Phi(E)dE} = \frac{R_g^{\text{single nucleus}}}{\Phi_g}$$

Macroscopic Group Cross Sections

$$\Sigma_g \equiv \frac{\int_{E_g}^{E_{g-1}} \Phi(E)\Sigma(E)dE}{\int_{E_g}^{E_{g-1}} \Phi(E)dE} = \frac{R_g}{\Phi_g}$$

Inter-Group Scattering (Transfer) Cross Sections

Microscopic

$$\sigma_{g\to g'} = \frac{\int_{E_g}^{E_{g-1}} \left[\int_{E_{g'}}^{E_{g'-1}} \Phi(E)\sigma_s(E \to E')dE' \right] dE}{\int_{E_g}^{E_{g-1}} \Phi(E)dE}$$

Macroscopic

$$\Sigma_{g\to g'} = \frac{\int_{E_g}^{E_{g-1}} \left[\int_{E_{g'}}^{E_{g'-1}} \Phi(E)\Sigma_s(E \to E')dE' \right] dE}{\int_{E_g}^{E_{g-1}} \Phi(E)dE} \qquad g \neq g'$$

Intra-Group Scattering Cross Section

Microscopic

$$\sigma_{g\to g} = \frac{\int_{E_g}^{E_{g-1}} \left[\int_{E_g}^{E_{g-1}} \Phi(E)\sigma_s(E \to E')dE' \right] dE}{\int_{E_g}^{E_{g-1}} \Phi(E)dE}$$

Macroscopic

$$\Sigma_{g\to g} = \frac{\int_{E_g}^{E_{g-1}} \left[\int_{E_g}^{E_{g-1}} \Phi(E)\Sigma_s(E \to E')dE' \right] dE}{\int_{E_g}^{E_{g-1}} \Phi(E)dE}$$

Multigroup Neutron balance Equation

[rate of change of number of neutrons in volume dV with energy within group g] =
[rate of production in volume dV of neutrons with energy within group g] +
[rate of scattering of neutrons in dV into energy group g] -
[rate of absorption in dV of neutrons with energy in group g] -
-[rate of scattering of neutrons in dV outside of energy group g] -
[rate of leakage out of dV of neutrons with energy within group g]

Multigroup Neutron balance Equation

$$\frac{\partial}{\partial t} n_g dV = s_g dV + \sum_{g'=1}^{G} \Sigma_{sg' \to g} \Phi_{g'} dV - \Sigma_{ag} \Phi_g dV - \Sigma_{sg} \Phi_g dV - \nabla \cdot \vec{J}_g dV$$

Dividing by dV:

$$\frac{\partial}{\partial t} n_g = s_g + \sum_{g'=1}^{G} \Sigma_{sg' \to g} \Phi_{g'} - \Sigma_{ag} \Phi_g - \Sigma_{sg} \Phi_g - \nabla \cdot \vec{J}_g$$

Multigroup Neutron balance Equation

Multigroup Fick's Law:

$$\vec{J}_g = -D_g \nabla \Phi_g$$

Multigroup Diffusion Equation

$$\frac{\partial}{\partial t} n_g = s_g + \sum_{g'=1}^{G} \Sigma_{sg' \to g} \Phi_{g'} - \Sigma_{ag} \Phi_g - \Sigma_{sg} \Phi_g + \nabla \cdot \left(D_g \nabla \Phi_g \right)$$

For constant diffusion coefficient:

$$\frac{\partial}{\partial t} n_g = s_g + \sum_{g'=1}^{G} \Sigma_{sg' \to g} \Phi_{g'} - \Sigma_{ag} \Phi_g - \Sigma_{sg} \Phi_g + D_g \nabla^2 \Phi_g$$

Steady state (no time dependence)

$$- D_g \nabla^2 \Phi_g + \Sigma_{ag} \Phi_g + \Sigma_{sg} \Phi_g - \sum_{g'=1}^{G} \Sigma_{sg' \to g} \Phi_{g'} = s_g$$

Particular Cases of the Diffusion Equation

One-Group Diffusion Equation

The entire energy range is included in just one group

Time-dependent:

$$\frac{\partial n}{\partial t} = S + \Sigma_s \Phi - \Sigma_a \Phi - \Sigma_s \Phi + D\nabla^2 \Phi$$

$$\frac{\partial n}{\partial t} = S - \Sigma_a \Phi + D\nabla^2 \Phi \Rightarrow \textit{one group D.E}$$

One-Group Diffusion Equation

Steady State:
The steady-state multigroup diffusion equation

$$-D_g\nabla^2\Phi_g - \sum_{g'=1}^{g-1}\Sigma_{sg'\to g}\Phi_{g'} + \Sigma_{ag}\Phi_g + \sum_{g'=g+1}^{G}\Sigma_{sg\to g'}\Phi_g = s_g$$

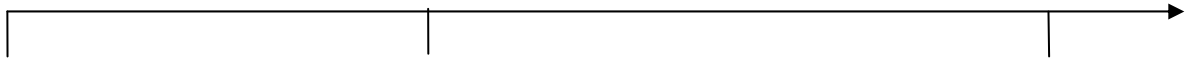
becomes:

$$-D_1\nabla^2\Phi_1 + \Sigma_{a1}\Phi_1 = s_1$$

We can drop the group index to obtain:

$$-D\nabla^2\Phi + \Sigma_a\Phi = s$$

Two-group Diffusion Equation



$E_2=0$ E_1 E_0

Group 1 (fast group): g=1

$$\frac{\partial n_1}{\partial t} = S_1 + \left(\Sigma_{s_{1\to 1}}\Phi_1 + \Sigma_{s_{2\to 1}}\Phi_2\right) - \Sigma_{a_1}\Phi_1 - \Sigma_{s_1}\Phi_1 + D_1\nabla^2\Phi_1$$

Where $\Sigma_{s_1} = \Sigma_{s_{1\to 2}} + \Sigma_{s_{1\to 1}}$

$$\frac{\partial n_1}{\partial t} = S_1 + \Sigma_{s_{1\to 1}}\Phi_1 + \Sigma_{s_{2\to 1}}\Phi_2 - \Sigma_{a_1}\Phi_1 - \Sigma_{s_{1\to 2}}\Phi_1 - \Sigma_{s_{1\to 1}}\Phi_1 + D_1\nabla^2\Phi_1$$

$$\frac{\partial n_1}{\partial t} = S_1 - \Sigma_{a_1}\Phi_1 - \Sigma_{s_{1\to 2}}\Phi_1 + D_1\nabla^2\Phi_1$$

Group 2 (Slow, thermal)

$$\frac{\partial n_2}{\partial t} = S_2 + \Sigma_{s_{1\to 2}}\Phi_1 + \Sigma_{s_{2\to 2}}\Phi_2 - \Sigma_{a_2}\Phi_2 - \Sigma_{s_2}\Phi_2 + D_2\nabla^2\Phi_2$$

Where $\Sigma_{s_2} = \Sigma_{s_{2\to 1}} + \Sigma_{s_{2\to 2}}$

$$\frac{\partial n_2}{\partial t} = S_2 + \Sigma_{s_{1\to 2}}\Phi_1 + \Sigma_{s_{2\to 2}}\Phi_2 - \Sigma_{a_2}\Phi_2 - \Sigma_{s_{2\to 1}}\Phi_2 - \Sigma_{s_{2\to 2}}\Phi_2 + D_2\nabla^2\Phi$$

$$\frac{\partial n_2}{\partial t} = S_2 + \Sigma_{s_{1\to 2}}\Phi_1 - \Sigma_{a_2}\Phi_2 + D_2\nabla^2\Phi_2$$

Two-group diffusion equation

$$\frac{\partial n_1}{\partial t} = S_1 - \Sigma_{a_1}\Phi_1 - \Sigma_{s_{1\to 2}}\Phi_1 + D_1\nabla^2\Phi_1$$

$$\frac{\partial n_2}{\partial t} = S_2 + \Sigma_{s_{1\to 2}}\Phi_1 - \Sigma_{a_2}\Phi_2 + D_2\nabla^2\Phi_2$$

$$\Sigma_{a_1} + \Sigma_{s_{1\to 2}} = \Sigma_r \quad \text{(Removal cross section)}$$

Two-group diffusion equation

Steady state $\dfrac{\partial n}{\partial t} = 0$

$$\begin{cases} -D_1 \nabla^2 \Phi_1 + \Sigma_{r_1} \Phi_1 = S_1 \\ -D_2 \nabla^2 \Phi_2 + \Sigma_{a_2} \Phi_2 = S_2 + \Sigma_{s_{1\to 2}} \Phi_1 \end{cases}$$

Two-Group Diffusion Equation

We could have started directly with the steady-state multigroup diffusion equation

$$-D_g\nabla^2\Phi_g-\sum_{g'=1}^{g-1}\Sigma_{sg'\to g}\Phi_{g'}+\Sigma_{ag}\Phi_g+\sum_{g'=g+1}^{G}\Sigma_{sg\to g'}\Phi_g=s_g$$

Group 1 (fast):

$$-D_1\nabla^2\Phi_1+\Sigma_{a1}\Phi_1+\Sigma_{s1\to 2}\Phi_1=s_1$$

Group 2 (slow, thermal):

$$-D_2\nabla^2\Phi_2-\Sigma_{s1\to 2}\Phi_1+\Sigma_{a2}\Phi_2=s_2$$

Two-group diffusion equations:

$$-D_1\nabla^2\Phi_1+\Sigma_{a1}\Phi_1+\Sigma_{s1\to 2}\Phi_1=s_1$$

$$-D_2\nabla^2\Phi_2-\Sigma_{s1\to 2}\Phi_1+\Sigma_{a2}\Phi_2=s_2$$

Solving the Diffusion Equation for Simple Cases

One Group, Infinite Homogeneous Medium, Uniformly Distributed Source

$$-D\nabla^2\Phi(\vec{r})+\Sigma_a\Phi(\vec{r})=s$$

Infinite, homogeneous medium

$$\Phi(\vec{r})=\Phi=ct$$

$$\nabla^2\Phi=0$$

The equation becomes:

$$\Sigma_a\Phi=s$$

Solving for the flux, we obtain:

$$\Phi=\frac{s}{\Sigma_a}\ \text{(constant)}$$

Two Groups, Infinite Homogeneous Medium, Uniformly Distributed Source

$$-D_1\nabla^2\Phi_1 + \Sigma_{a1}\Phi_1 + \Sigma_{s1\to 2}\Phi_1 = s_1$$
$$-D_2\nabla^2\Phi_2 - \Sigma_{s1\to 2}\Phi_1 + \Sigma_{a2}\Phi_2 = s_2$$

For an infinite and homogeneous medium with uniformly-distributed source:

$$\Phi_1(\vec{r}) = \Phi_1 = ct$$
$$\Phi_2(\vec{r}) = \Phi_2 = ct$$
$$\nabla^2\Phi_1 = 0$$
$$\nabla^2\Phi_2 = 0$$

The two-group equations become:

$$\Sigma_{a1}\Phi_1 + \Sigma_{s1\to 2}\Phi_1 = s_1$$
$$-\Sigma_{s1\to 2}\Phi_1 + \Sigma_{a2}\Phi_2 = s_2$$

The first equation can be easily solved to yield:

$$\Phi_1 = \frac{s_1}{\Sigma_{a1} + \Sigma_{s1\to 2}} = \frac{s_1}{\Sigma_r}$$

$\Sigma_r =$ removal cross section

The second equation can be rewritten as:

$$\Sigma_{a2}\Phi_2 = s_2 + \Sigma_{s1\to 2}\Phi_1$$

$$\Sigma_{s1\to 2}\Phi_1 = q_T = \text{slowing down density}$$

Using the expression found for the fast flux, we have:

$$\Sigma_{a2}\Phi_2 = s_2 + \Sigma_{s1\to 2}\frac{s_1}{\Sigma_r}$$

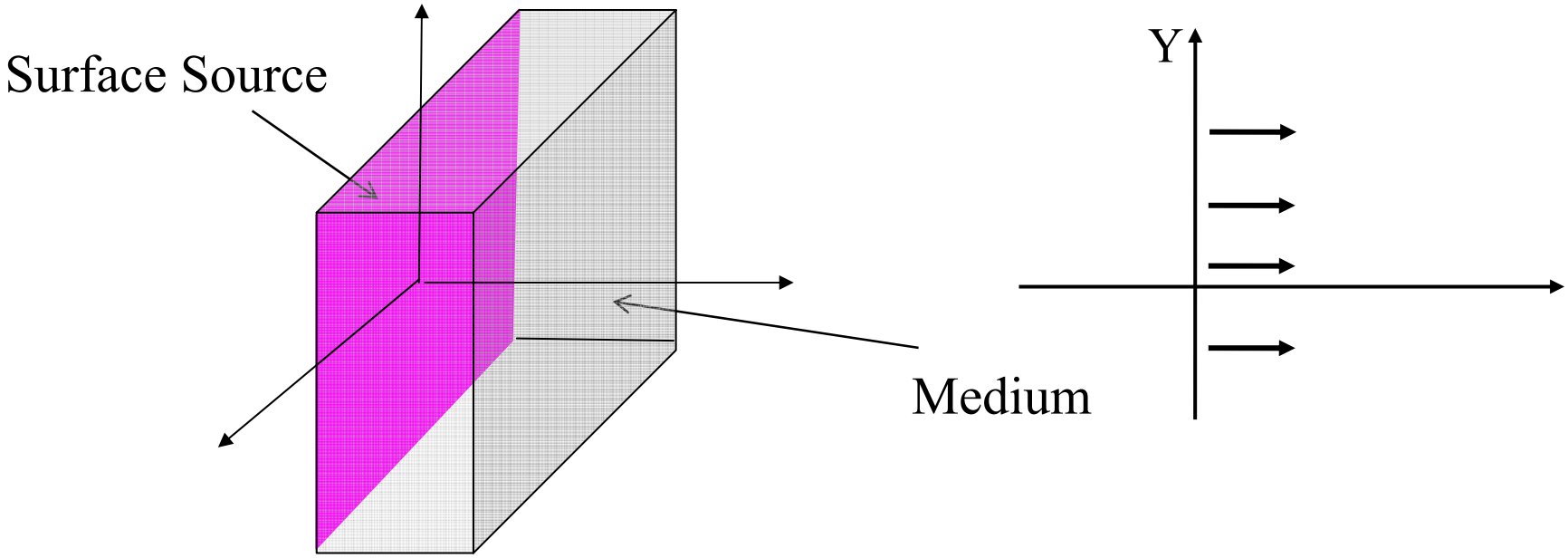
The thermal flux is hence:

$$\Phi_2 = \frac{s_2 + \Sigma_{s1\to 2}\dfrac{s_1}{\Sigma_r}}{\Sigma_{a2}}$$

If there is no external thermal source $(s_2 = 0)$, then the solution simplifies to:

$$\Phi_2 = \Sigma_{s1\to 2}\frac{s_1}{\Sigma_r \Sigma_{a2}}$$

One-group diffusion equation for a semi-infinite medium (half space)



$$\Phi(x, y, z) = \Phi(z)$$

$$-D\frac{d^2\Phi}{dz^2} + \Sigma_a\Phi = 0$$

→ Assume no volume sources

$$-\frac{d^2\Phi}{dz^2}+\frac{\Sigma_a\Phi}{D}=0$$

$$L^2=\frac{D}{\Sigma_a}$$ → Diffusion Length

$$-\frac{d^2\Phi}{dz^2}+\frac{\Sigma_a\Phi}{L^2}=0$$

Characteristic Equation

$$-r^2+\frac{1}{L^2}=0$$

$$r^2=\frac{1}{L^2}$$

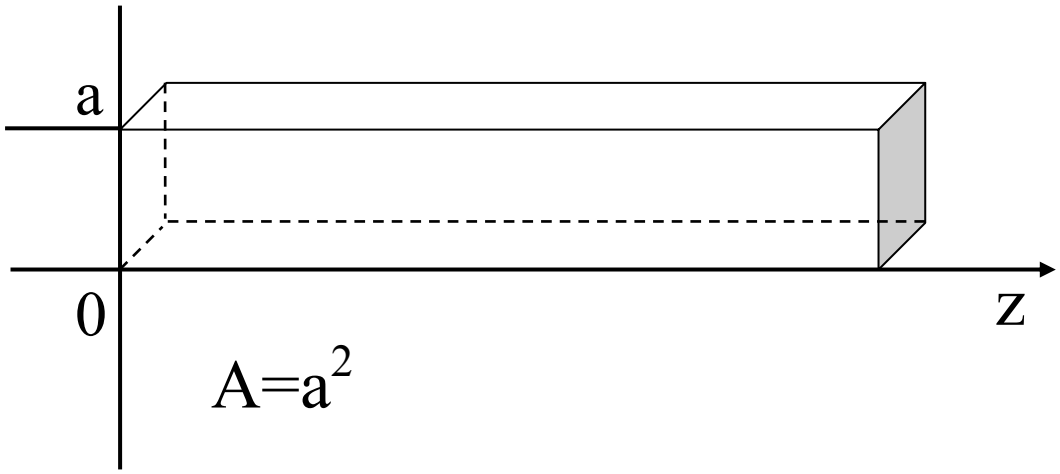
$$r=\pm\frac{1}{L}$$

General solution is

$$ae^{\frac{z}{L}} + be^{-\frac{z}{L}} \Rightarrow be^{-\frac{z}{L}}$$

Finding the constant b:

Consider an infinite parallelipiped of cross section area A and extending from zero to infinity in the z direction :



Express the equality between total absorption in the parallelipiped and the source of neutrons coming in from the boundary source S_b at Z=0.

Absorption rate from 0 to ∞ in a prism of cross-section area A

$$\left[\int_0^{\infty} \Sigma_a \Phi(z) dz \right] A$$

Source rate

$$S_b A$$

Equality between source and absorption:

$$S_b A = \left[\int_0^\infty \Sigma_a \Phi(z) dz \right] A$$

$$S_b = \int_0^\infty \Sigma_a \Phi(z) dz = \int_0^\infty \Sigma_a b e^{-\frac{z}{L}} dz = \Sigma_a b \int_0^\infty e^{-\frac{z}{L}} dz$$

$$\int_0^\infty e^{-\frac{z}{L}} dz = \left[-L e^{-\frac{z}{L}} \right]_0^\infty = -\lim_{z \to 0} \left(L e^{-\frac{z}{L}} \right) + L e^{-\infty} = L$$

$$S_b = \Sigma_a b L \Rightarrow b = \frac{S_b}{\Sigma_a L}$$

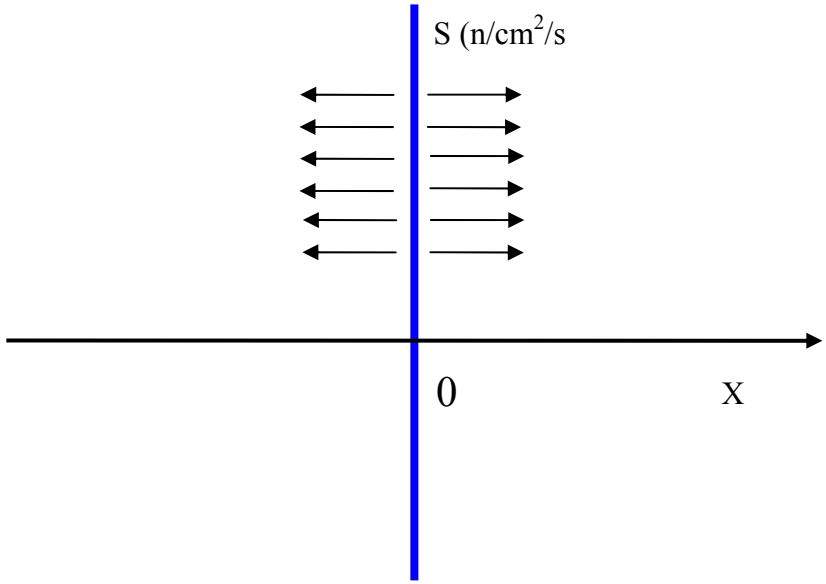
Hence:

$$\Phi(z) = \frac{S_b}{\Sigma_a L} e^{-\frac{z}{L}}$$

Units:

$$[\Phi] = \frac{[S]}{[\Sigma][L]} = \frac{\frac{n}{cm^2 s}}{\frac{1}{cm} cm} = \frac{n}{cm^2 s}$$

One Group Diffusion for an Infinite Planar Source Situated in an Infinite Homogeneous Medium at x=0



Equivalent to two half-spaces (left and right)

$$-D\nabla^2\Phi(x,y,z)+\Sigma_a\Phi(x,y,z)=0 \quad \text{for x} \neq 0$$

Because of the planar (y-z) symmetry, $\Phi = \Phi(x)$

The equation becomes:

$$-D\frac{\partial^2\Phi(x)}{\partial\Phi^2(x)} + \Sigma_a\Phi(x) = 0$$

Using the diffusion length notation:

$$\frac{d^2\Phi}{dx^2} - \frac{1}{L^2}\Phi = 0, \quad x \neq 0$$

This is a homogeneous second order linear differential equation with constant coefficients. The general solution is of the type:

$$\Phi(x) = Ae^{\frac{-x}{L}} + Ce^{\frac{x}{L}}$$

Because the flux needs to be finite, we have C=0. Hence:

$$\Phi(x) = Ae^{-x/L}$$

The current is:

$$J(x) = -D\frac{d}{dx}\left(Ae^{-x/L} \right) = A\frac{D}{L}e^{-x/L}$$

To find A, we use the boundary condition:

$$\lim_{x \to +0} J(x) = \frac{S}{2}$$

The initial condition yields:

$$J(x) = A\frac{D}{L}e^{0/L} = \frac{s}{2} \Rightarrow A\frac{D}{L} = \frac{s}{2} \Rightarrow A = \frac{sL}{2D}$$

The flux for x>0 is hence:

$$\Phi = \frac{SL}{2D}e^{-x/L}$$

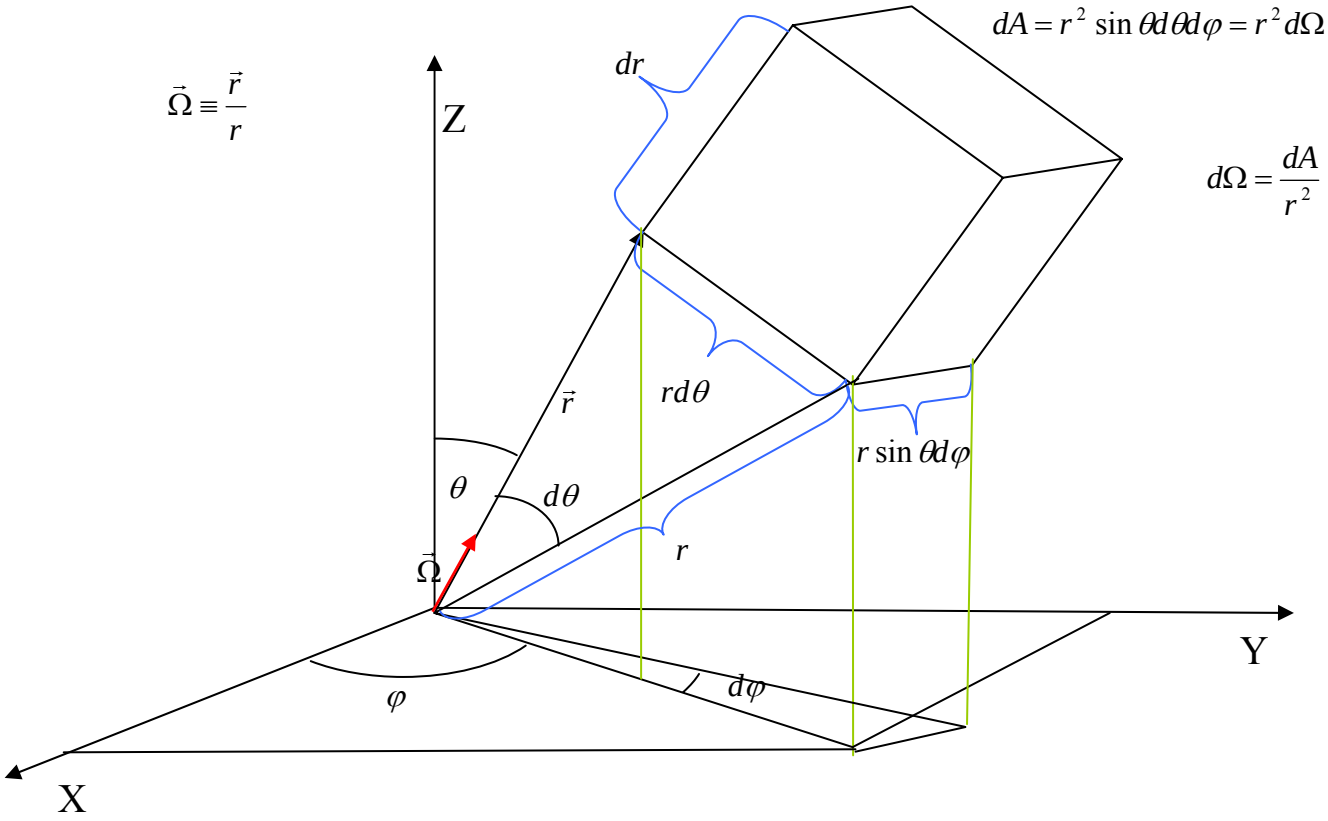
Analogously, the flux for x<0 is:

$$\Phi = \frac{SL}{2D}e^{x/L}$$

One Group Diffusion for a Point Source Situated in an Infinite Homogeneous Medium

Use spherical coordinates with the source placed at the center

$d^3r = dV = r^2 dr \sin\theta d\theta d\varphi$



Because the problem is symmetrical with respect to both θ and φ (spherical symmetry), the flux will only depend on r.

$$\Phi(x, y, z) \Rightarrow \Phi(r, \theta, \varphi)$$

$$\Phi = \Phi(r)$$

Expression of Laplacian in spherical coordinates for a function with spherical symmetry, f(r).

$$\nabla^2 f(r) = \frac{1}{r^2} \frac{d}{dr}\left(r^2 \frac{df}{dr} \right)$$

The diffusion equation becomes:

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\Phi}{dr}\right)-\frac{1}{L^2}\Phi=0$$

This is a homogeneous second order differential equation with constant coefficients.

The boundary condition is

$$J4\pi\varepsilon^2=S\Rightarrow\varepsilon^2J=\frac{S}{4\pi}$$

$$\lim_{r\to0}\left(r^2J(r)\right)=\frac{S}{4\pi}$$

Rate at which neutrons exit a very small sphere of radius ε, surrounding the origin: $S=J4\pi\varepsilon^2$

This is equal to the rate at which neutrons are produced, because ε is so small that absorption in this very small sphere can be ignored.

To solve the equation, we make the substitution:

$$w = r\Phi \Leftrightarrow \Phi = \frac{w}{r}$$

The equation becomes:

$$\frac{1}{r^2}\frac{d}{dr}\left[r^2\frac{d}{dr}\left(\frac{w}{r}\right)\right] - \frac{1}{L^2}\frac{w}{r} = 0$$

which yields:

$$\frac{d^2w}{dr^2} - \frac{1}{L^2}w = 0$$

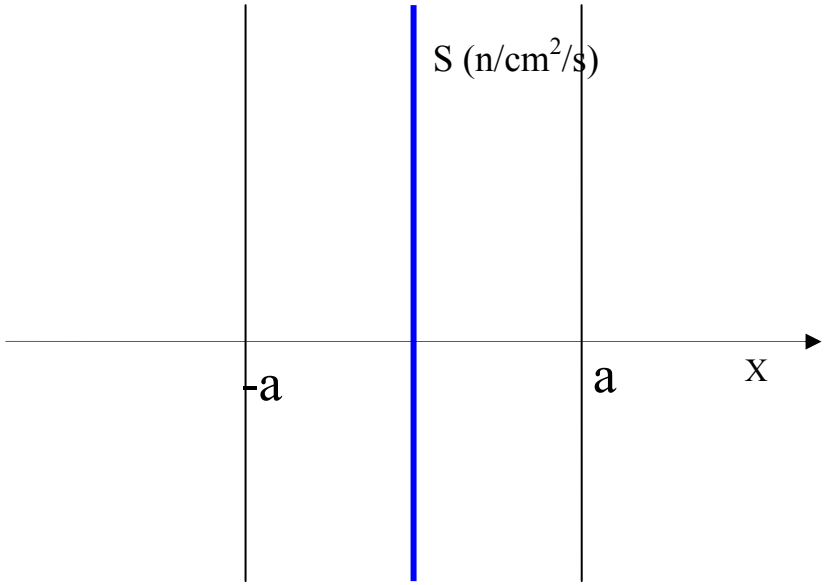
Solution is:

$$w = Ae^{-\frac{r}{L}}; \quad \Phi = \frac{Ae^{-\frac{r}{L}}}{r}$$

Following a similar treatment as for the plane source, we find:

$$\Phi = \frac{S}{4\pi D}\frac{e^{-\frac{r}{L}}}{r}$$

One Group Diffusion for a Bare Slab with an Infinite Planar Source Situated in the Middle



The problem is symmetric with respect to the source and also has planar symmetry

$$\Phi = \Phi(x)$$

Diffusion equation:

$$\frac{d^2\Phi}{dx^2} - \frac{1}{L^2}\Phi = 0, \quad x \neq 0$$

Will treat the right half.

This is a homogeneous second order linear differential equation with constant coefficients. The general solution is of the type:

$$\Phi(x) = Ae^{\frac{-x}{L}} + Ce^{\frac{x}{L}}$$

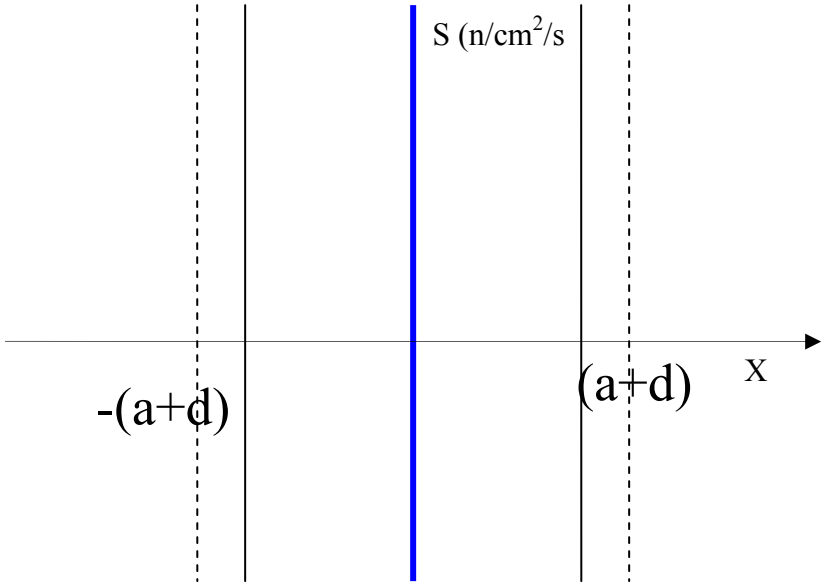
The left boundary condition is, just as before:

$$\lim_{x \to +0} J(x) = \frac{S}{2}$$

yielding:

$$J(x) = -\frac{D}{L} A e^{\frac{-x}{L}} + \frac{D}{L} C e^{\frac{x}{L}} \Bigg|_{x=0} = -\frac{D}{L} A + \frac{D}{L} C = \frac{S}{2}$$

The right boundary condition is now a vacuum boundary condition, that is the flux vanishes at the extrapolated boundary.



$$\phi(\tilde{a}) = 0$$

where

$$\tilde{a} = a + d$$

The above yields:

$$\Phi(\tilde{a}) = Ae^{\frac{-\tilde{a}}{L}} + Ce^{\frac{\tilde{a}}{L}} = 0$$

We obtain A and C by solving the system:

$$-\frac{D}{L}A + \frac{D}{L}C = \frac{S}{2}$$

$$\Phi(\tilde{a}) = Ae^{\frac{-\tilde{a}}{L}} + Ce^{\frac{\tilde{a}}{L}} = 0$$

The final solution is:

$$\Phi(x) = \frac{SL}{2D} \frac{e^{\frac{x}{L}} - e^{\frac{x-2\tilde{a}}{L}}}{1 + e^{-\frac{2\tilde{a}}{L}}}$$

Neutron Moderation (two group treatment)

Two-group diffusion

Assume $\Sigma_a = 0$ (good moderator)

$$-D_1\nabla^2\Phi_1 + \Sigma_{s1\to 2}\Phi_1 = 0$$
$$-D_2\nabla^2\Phi_2 - \Sigma_{s1\to 2}\Phi_1 + \Sigma_{a2}\Phi_2 = 0$$

The two equations can be rearranged to:

$$-D_1\nabla^2\Phi_1 + \Sigma_{s1\to 2}\Phi_1 = 0$$
$$-D_2\nabla^2\Phi_2 + \Sigma_{a2}\Phi_2 = \Sigma_{s1\to 2}\Phi_1$$

We make the following notations:

$$\frac{D_1}{\Sigma_{s1\to 2}} = \tau_T \ = \text{age}$$

$$\frac{D_2}{\Sigma_{a2}} = L_T^2 \ = \text{thermal diffusion area}$$

$$L_T = \text{thermal diffusion lenght}$$

With the new notations, the equations are written:

$$-\nabla^2\Phi_1 + \frac{1}{\tau_T}\Phi_1 = 0$$

$$-\nabla^2\Phi_2 + \frac{1}{L_T^2}\Phi_2 = \frac{D_1}{D_2\tau_T}\Phi_1$$

These can be solved for different configurations.

Nuclear Reactor Theory

Multiplication Constant

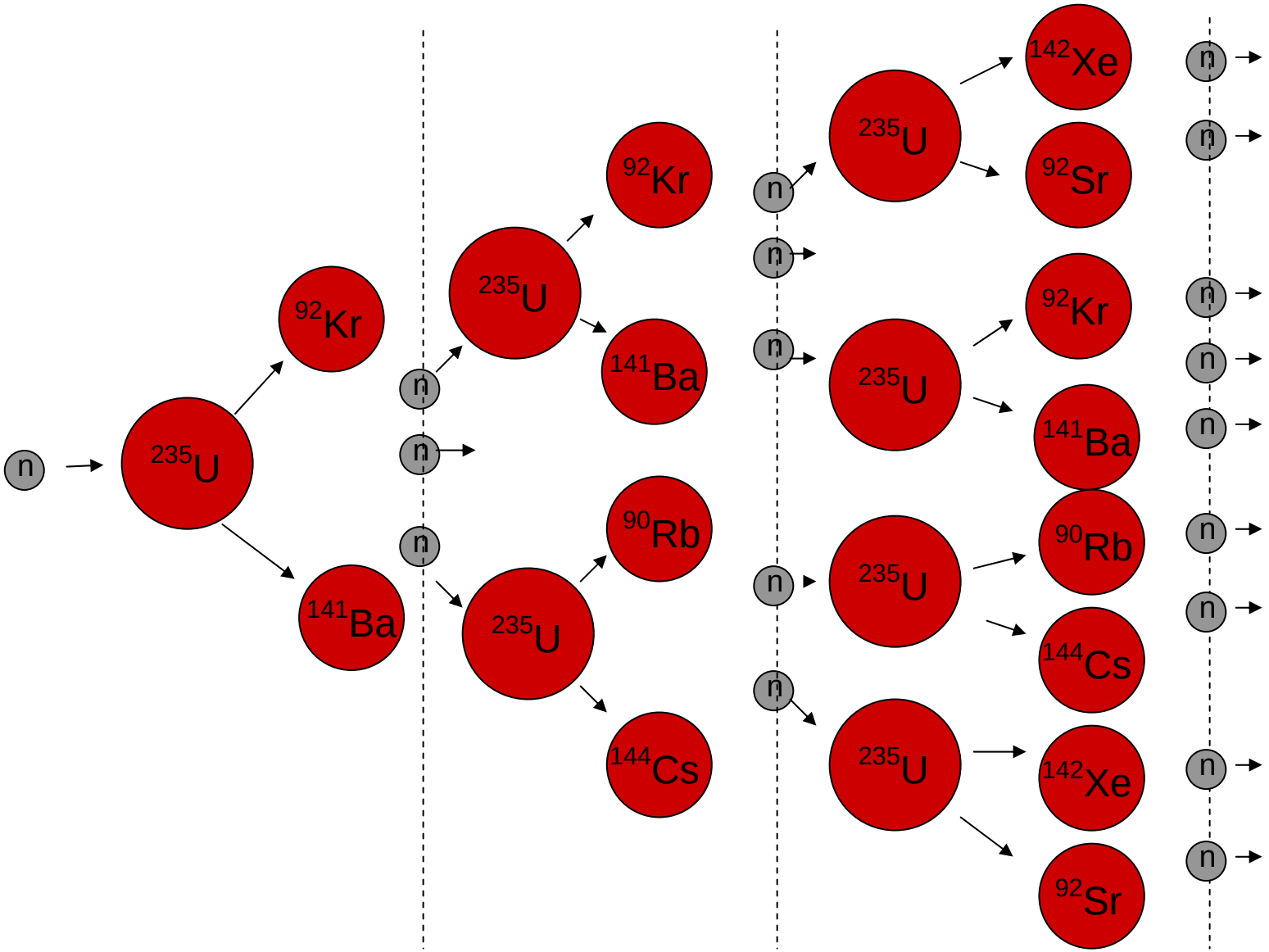
Preliminaries - Neutron Fluence

Neutron *fluence* is defined as the time integral of the flux

$$\psi = \int_{t_1}^{t_2} \Phi(t)dt$$

Where units for ψ are $\frac{n}{cm^2}$ and units for Φ are $\frac{n}{cm^2 s}$

Fission Chain Reaction



Each fission produces 2-3 more neutrons which can, in principle, induce new fissions in avalanche. This is not desirable.

However, not all neutrons resulting from fission induce new fissions. Some undergo gamma capture.

If two few neutrons (less than one per fission) induce new fissions the fission reaction dies down. Not desirable either.

The trick is to only allow one of the secondary neutrons to induce a new fission and thus have a fission rate that is constant in time. A reactor operating at a constant fission rate is said to be *critical*.

Infinite Homogeneous Reactor
(One-Group Diffusion Approximation)

Multiplicative medium ($\Sigma_f > 0$).

Non-Multiplicative medium ($\Sigma_f = 0$).

The steady-state diffusion equation is written:

$$-D\nabla^2\Phi + \Sigma_a\Phi = S$$

The source now consists of fission neutrons:

$$S = \nu\Sigma_f\Phi$$

So the equation becomes:

$$-D\nabla^2\Phi + \Sigma_a\Phi = \nu\Sigma_f\Phi$$

The flux is constant in space because the medium is infinite and homogeneous, so the equation becomes.

$$\Sigma_a \Phi = \nu \Sigma_f \Phi$$

It is obvious that the above cannot be satisfied, unless

$$\Sigma_a = \nu \Sigma_f$$

If that is not the case, then the source is artificially divided by a factor k, just to balance the equation.

$$\Sigma_a \Phi = \frac{1}{k} \nu \Sigma_f \Phi$$

k is called the *multiplication constant (factor)*. For an infinite medium, it is called the *infinite multiplication constant* and denoted by k_∞.

It is obvious that, for the one-group homogeneous reactor case:

$$k_\infty = \frac{\nu\Sigma_f}{\Sigma_a}$$

It is also obvious that the value of the flux cannot be determined because once the appropriate k is used, any value of the flux will satisfy the balance equation.

$$\Sigma_a\Phi = \frac{1}{k_\infty}\nu\Sigma_f\Phi \Rightarrow \Sigma_a\Phi = \frac{1}{\dfrac{\nu\Sigma_f}{\Sigma_a}}\nu\Sigma_f\Phi$$

Interpretation of k

Since the balance equation is written:

$$\Sigma_a \Phi = \frac{1}{k_\infty} \nu \Sigma_f \Phi$$

We have:

$$k_\infty = \frac{\nu \Sigma_f \Phi}{\Sigma_a \Phi} = \frac{production\ rate}{loss\ rate}$$

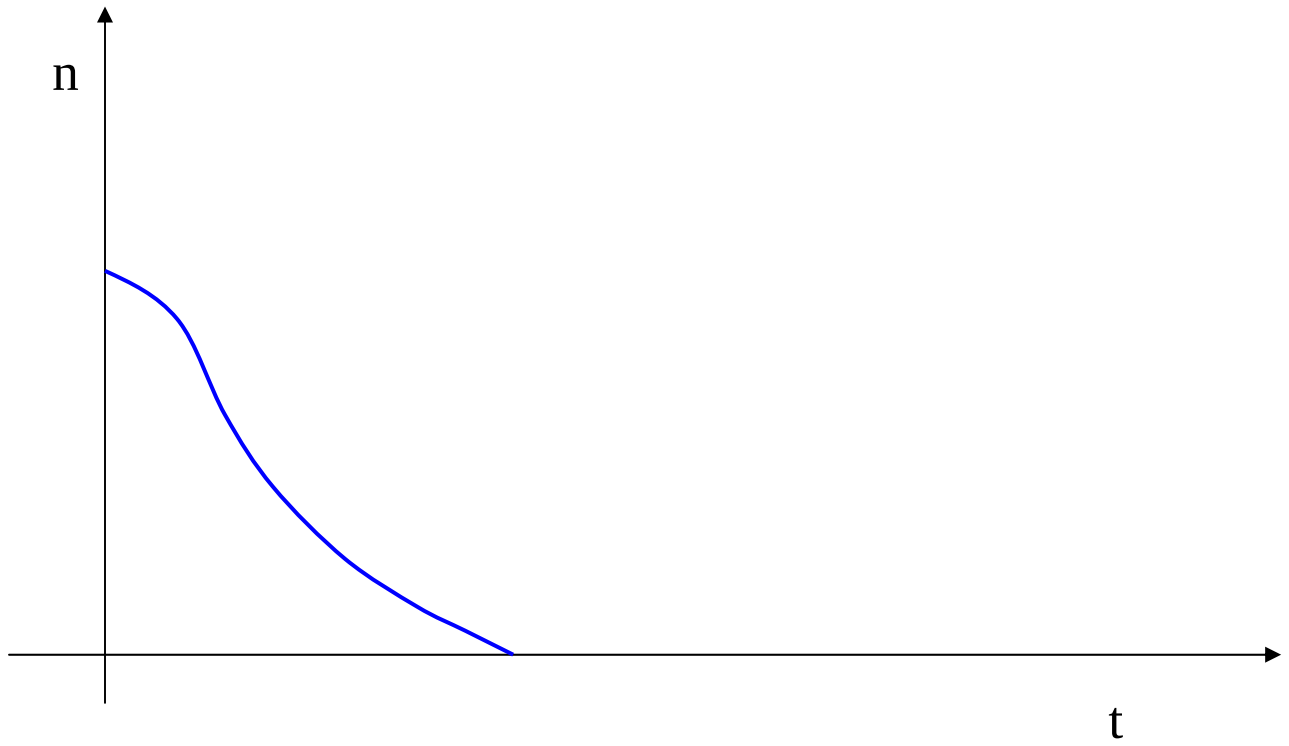
So k can be interpreted as the ratio of the neutron production rate and the neutron loss rate.

The name "multiplication factor" is used because k represents the ratio between the neutron density for one generation of neutrons, divided by the neutron density for the previous generation. This needs some explaining.

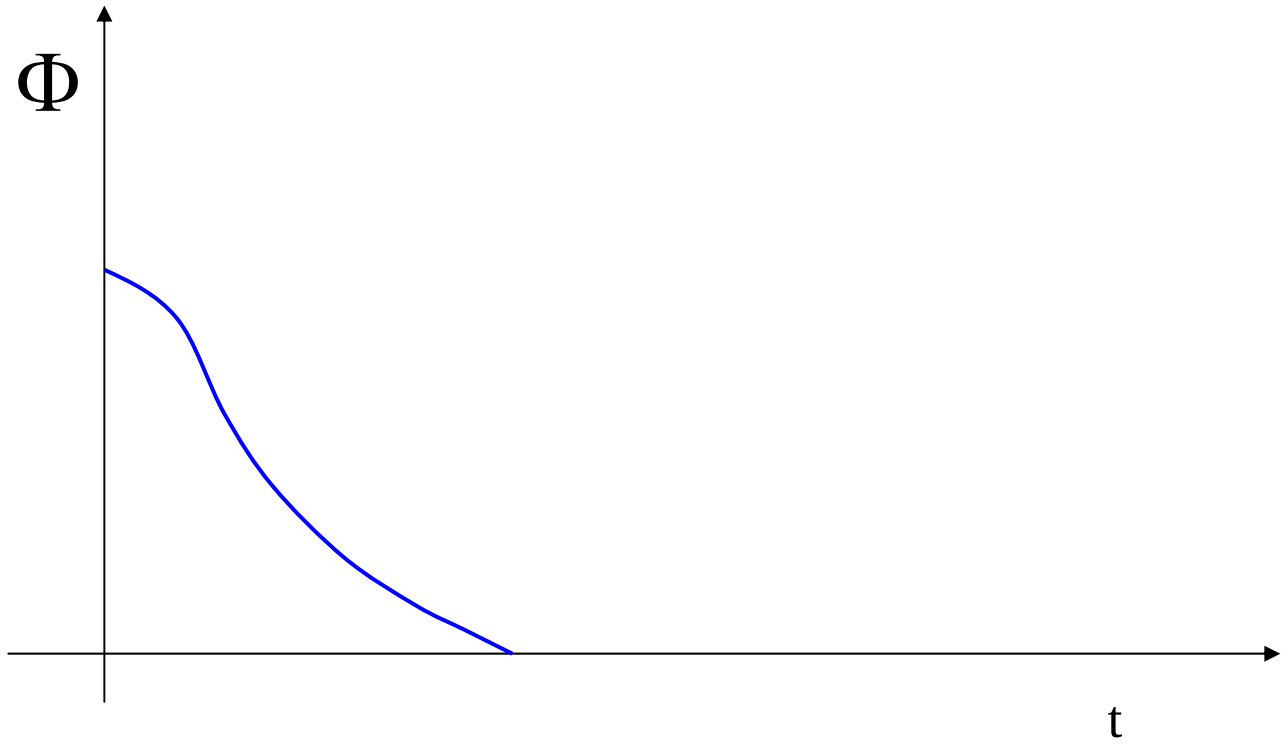
Consider a bare infinite homogeneous reactor. Initially there are no neutrons present.

Now, assume some neutrons, with density n_0 are introduced in the reactor. Let's call these "generation 0" neutrons. These neutrons will fly around, producing a flux $\Phi_0(t) = n_0(t)\mathrm{v}$ which will decrease as the neutrons are absorbed, until all neutrons are eventually absorbed.

The time dependence of the zero-generation neutrons looks something like this:



The flux, has a similar shape



As these zeroth-generation neutrons are absorbed, some of them produce fissions. We consider the neutrons born out of these fissions first generation neutrons. They are produced at a rate:

$$\nu\Sigma_f\Phi_0(t)$$

and are absorbed at a rate

$$\Sigma_a\Phi_1(t)$$

Overall, the number of first-generation neutrons that are produced per unit volume is:

$$n_1 = \int_0^\infty \nu\Sigma_f \Phi_0(t)dt = \nu\Sigma_f \int_0^\infty \Phi_0(t)dt = \nu\Sigma_f \psi_0$$

The total number of absorptions of first-generation neutrons is:

$$\int_0^\infty \Sigma_a \Phi_1(t)dt = \Sigma_a \int_0^\infty \Phi_1(t)dt = \Sigma_a \psi_1$$

Since, in the end, all first-generation neutrons get absorbed, we have:

$$\Sigma_a \psi_1 = \nu \Sigma_f \psi_0$$

which yields:

$$\psi_1 = \frac{\nu \Sigma_f}{\Sigma_a} \psi_0 = k_\infty \psi_0$$

The first-generation neutrons, in turn, produce second generation neutrons. Their number is:

$$n_2 = \nu \Sigma_f \psi_1 = \nu \Sigma_f k_\infty \psi_0 = k_\infty n_1$$

The process continues:

$$n_3 = k_\infty n_2$$

and so on.

The number of neutrons in each generation is equal to the number in the previous generation multiplied by k_∞. Hence the name *multiplication factor*.

Infinite Homogeneous Reactor
(Two-Group Diffusion Approximation)

Diffusion equations:

$$-D_1\nabla^2\Phi_1+\Sigma_{a1}\Phi_1+\Sigma_{s1\to2}\Phi_1=\nu\Sigma_{f1}\Phi_1+\nu\Sigma_{f2}\Phi_2$$
$$-D_2\nabla^2\Phi_2-\Sigma_{s1\to2}\Phi_1+\Sigma_{a2}\Phi_2=0$$

Because the reactor is infinite and the flux (both fast and thermal) is constant in space, we have:

$$\Sigma_{a1}\Phi_1+\Sigma_{s1\to2}\Phi_1=\nu\Sigma_{f1}\Phi_1+\nu\Sigma_{f2}\Phi_2$$
$$-\Sigma_{s1\to2}\Phi_1+\Sigma_{a2}\Phi_2=0$$

Attempt to solve the system:

Group 2 equation yields:

$$\Phi_2 = \frac{\Sigma_{s1\to 2}}{\Sigma_{a2}}\Phi_1$$

Substituting into the group 1 equation, we obtain:

$$\Sigma_{a1}\Phi_1 + \Sigma_{s1\to 2}\Phi_1 = \nu\Sigma_{f1}\Phi_1 + \nu\Sigma_{f2}\frac{\Sigma_{s1\to 2}}{\Sigma_{a2}}\Phi_1$$

Obviously, the above is only satisfied if:

$$\Sigma_{a1} + \Sigma_{s1\to 2} = \left(\nu\Sigma_{f1} + \nu\Sigma_{f2} \frac{\Sigma_{s1\to 2}}{\Sigma_{a2}} \right) \frac{1}{k}$$

which may not always be the case. This means that unless the above is satisfied, we cannot have a steady-state solution to our diffusion equations.

To force the system of equations to have a (steady-state) solution, we resort to the same trick as before: use a "fudge factor" 1/k that multiplies fission productions.

Thus, our equations become:

$$\Sigma_{a1}\Phi_1 + \Sigma_{s1\to 2}\Phi_1 = \frac{1}{k_\infty}\left(\nu\Sigma_{f1}\Phi_1 + \nu\Sigma_{f2}\Phi_2\right)$$

$$-\Sigma_{s1\to 2}\Phi_1 + \Sigma_{a2}\Phi_2 = 0$$

And, by substituting $\Phi_2 = \frac{\Sigma_{s1\to 2}}{\Sigma_{a2}}\Phi_1$ into the fast-group equation, we obtain:

$$\Sigma_{a1}\Phi_1 + \Sigma_{s1\to 2}\Phi_1 = \frac{1}{k}\left(\nu\Sigma_{f1}\Phi_1 + \nu\Sigma_{f2}\frac{\Sigma_{s1\to 2}}{\Sigma_{a2}}\Phi_1\right)$$

Dividing by the flux, we obtain:

$$\Sigma_{a1} + \Sigma_{s1\to 2} = \frac{1}{k_\infty}\left(\nu\Sigma_{f1} + \nu\Sigma_{f2}\frac{\Sigma_{s1\to 2}}{\Sigma_{a2}}\right)$$

We can now solve for k_∞ .

$$k_\infty = \frac{\nu\Sigma_{f1} + \nu\Sigma_{f2}\dfrac{\Sigma_{s1\to 2}}{\Sigma_{a2}}}{\Sigma_{a1} + \Sigma_{s1\to 2}}$$

Choosing k_∞ to have the above value ensures the system admits a solution.

That solution is

$$\Phi_2 = \frac{\Sigma_{s1\to 2}}{\Sigma_{a2}} \Phi_1$$

We cannot find the fast flux explicitly.

A close look at the system of equations

$$\Sigma_{a1}\Phi_1 + \Sigma_{s1\to 2}\Phi_1 = \frac{1}{k_\infty}\left(\nu\Sigma_{f1}\Phi_1 + \nu\Sigma_{f2}\Phi_2\right)$$

$$-\Sigma_{s1\to 2}\Phi_1 + \Sigma_{a2}\Phi_2 = 0$$

reveals that it is a homogeneous system of linear equations which defines an eigenvalue/eigenvector problem. The eigenvalue is $1/k_\infty$ and, as expected, the eigenvector can only be determined up to a multiplicative constant which, in our solution, is Φ_1.

k_∞ can, in the two-group case be interpreted in three different ways:

1. the eigenvalue that allows the system of equations to have a solution
2. the ratio of productions over losses
3. the factor by which the number of neutrons gets multiplied from one generation to the next

Criticality

K<1 - Subcritical

- Number of neutrons decreases form one generation to the next
- Rate of neutron production smaller than rate of neutron loss

K=1 - Critical

- Number of neutrons stays constant form one generation to the next
- Rate of neutron production equals rate of neutron loss

K>1 - Supercritical

- Number of neutrons increases form one generation to the next
- Rate of neutron production larger than rate of neutron loss

Neutron Life Cycle, Four Factor Formula, Six Factor Formula

The Four-Factor Formula

Let us look at the group 1 equation in the two-group approximation.

$$\Sigma_{a1}\Phi_1 + \Sigma_{s1\to 2}\Phi_1 = \frac{1}{k_\infty}\left(\nu\Sigma_{f1}\Phi_1 + \nu\Sigma_{f2}\Phi_2\right)$$

Solving for the multiplication factor, we obtain:

$$k_\infty = \frac{\nu\Sigma_{f1}\Phi_1 + \nu\Sigma_{f2}\Phi_2}{\Sigma_{a1}\Phi_1 + \Sigma_{s1\to 2}\Phi_1}$$

The above can be processed as follows:

$$k_\infty = \frac{\nu\Sigma_{f1}\Phi_1 + \nu\Sigma_{f2}\Phi_2}{\Sigma_{a1}\Phi_1 + \Sigma_{s1\to 2}\Phi_1} = \frac{\nu\Sigma_{f1}\Phi_1 + \nu\Sigma_{f2}\Phi_2}{\Sigma_{a1}\Phi_1 + \Sigma_{s1\to 2}\Phi_1}\frac{\nu\Sigma_{f2}\Phi_2}{\nu\Sigma_{f2}\Phi_2} =$$

$$= \frac{\nu\Sigma_{f1}\Phi_1 + \nu\Sigma_{f2}\Phi_2}{\nu\Sigma_{f2}\Phi_2}\frac{\nu\Sigma_{f2}\Phi_2}{\Sigma_{a1}\Phi_1 + \Sigma_{s1\to 2}\Phi_1}$$

By making the notation:

$$\varepsilon = \frac{\nu\Sigma_{f1}\Phi_1 + \nu\Sigma_{f2}\Phi_2}{\nu\Sigma_{f2}\Phi_2}$$

We obtain:

$$k_\infty = \varepsilon \frac{\nu\Sigma_{f2}\Phi_2}{\Sigma_{a1}\Phi_1 + \Sigma_{s1\to 2}\Phi_1}$$

We can continue the processing:

$$k_\infty = \varepsilon \frac{\nu\Sigma_{f2}\Phi_2}{\Sigma_{a1}\Phi_1 + \Sigma_{s1\to 2}\Phi_1} \frac{\Sigma_{a2}\Phi_2}{\Sigma_{a2}\Phi_2} = \varepsilon \frac{\Sigma_{a2}\Phi_2}{\Sigma_{a1}\Phi_1 + \Sigma_{s1\to 2}\Phi_1} \frac{\nu\Sigma_{f2}\Phi_2}{\Sigma_{a2}\Phi_2}$$

Denoting:

$$p = \frac{\Sigma_{a2}\Phi_2}{\Sigma_{a1}\Phi_1 + \Sigma_{s1\to 2}\Phi_1}$$

We have:

$$k_{\infty} = \varepsilon p \frac{\nu \Sigma_{f2} \Phi_2}{\Sigma_{a2} \Phi_2}$$

We can, moreover divide the thermal absorption cross section into the absorption cross section for fuel, and the one for moderator.

$$\Sigma_{a2} = \Sigma_{a2}^{\text{fuel}} + \Sigma_{a2}^{\text{moderator}}$$

With this, we can rewrite the formula for the multiplication factor as follows:

$$k_\infty = \varepsilon p \frac{\nu\Sigma_{f2}\Phi_2}{\Sigma_{a2}\Phi_2}\frac{\Sigma_{a2}^{\text{fuel}}\Phi_2}{\Sigma_{a2}^{\text{fuel}}\Phi_2} = \varepsilon p \frac{\Sigma_{a2}^{\text{fuel}}\Phi_2}{\Sigma_{a2}\Phi_2}\frac{\nu\Sigma_{f2}\Phi_2}{\Sigma_{a2}^{\text{fuel}}\Phi_2}$$

Denoting:

$$f = \frac{\Sigma_{a2}^{\text{fuel}}\Phi_2}{\Sigma_{a2}\Phi_2}$$

and

$$\eta = \frac{\nu\Sigma_{f2}\Phi_2}{\Sigma_{a2}^{\text{fuel}}\Phi_2}$$

We obtain:

$$k_\infty = \varepsilon p f \eta$$

This is known as the *four factor formula*.

The names and interpretation of the factors are as follows:

Fast fission factor

$$\varepsilon = \frac{\nu\Sigma_{f1}\Phi_1 + \nu\Sigma_{f2}\Phi_2}{\nu\Sigma_{f2}\Phi_2} = \frac{total\ \ fission\ rate}{thermal\ \ fission\ rate}$$

Resonance escape probability

$$p = \frac{\Sigma_{a2}\Phi_2}{\Sigma_{a1}\Phi_1 + \Sigma_{s1\to 2}\Phi_1} = \frac{\Sigma_{s1\to 2}\Phi_2}{\Sigma_{a1}\Phi_1 + \Sigma_{s1\to 2}\Phi_1}$$

$$= \frac{rate\ of\ \ slowing\ \ down}{rate\ of\ \ slowing\ \ down + absorptions}$$

Thermal utilization factor

$$f = \frac{\Sigma_{a2}^{\text{fuel}} \Phi_2}{\Sigma_{a2} \Phi_2} = \frac{rate\ of\ thermal\ absorption\ in\ fuel}{total\ rate\ of\ thermal\ absorption\ s}$$

η (number of neutrons produced per neutron absorbed in fuel)

$$\eta = \frac{\nu \Sigma_{f2} \Phi_2}{\Sigma_{a2}^{\text{fuel}} \Phi_2}$$

$$= \frac{rate\ of\ neutron\ production\ through\ thermal\ fission}{rate\ of\ thermal\ absorption}$$

Six Factor Formula

For a finite reactor, in addition to the processes we studied above, fast neutrons, as well as thermal neutrons can leak out of the reactor.

We define the following two factors to account for the leakage:

α_f = fast non-leakage probability

α_t = thermal non-leakage probability

Our expression for k then becomes the **six**-factor formula:

$$k_{eff} = \varepsilon p f \eta \alpha_f \alpha_t$$

One-Group Treatment of Finite Reactors

Diffusion Equation

$$D\nabla^2\Phi - \Sigma_a\Phi + \frac{1}{k}\nu\Sigma_f\Phi = 0$$

$$\nabla^2\Phi + \frac{1}{D}\left(-\Sigma_a + \frac{1}{k}\nu\Sigma_f\right)\Phi = 0$$

Notation:
(B^2 is called **Buckling**)

$$B^2 = \frac{1}{D}\left(-\Sigma_a + \frac{1}{k}\nu\Sigma_f\right)$$

The equation can be rewritten:

$$\nabla^2 \Phi + B^2 \Phi = 0$$

B depends on k. It turns out that B cannot take just any value. It has to be equal to the value imposed by the geometry, called the geometrical buckling.

$$B^2 = B_g^2$$

Then:

$$\frac{1}{D}\left(-\Sigma_a + \frac{1}{k}\nu\Sigma_{\mathrm{f}}\right) = B_g^2$$

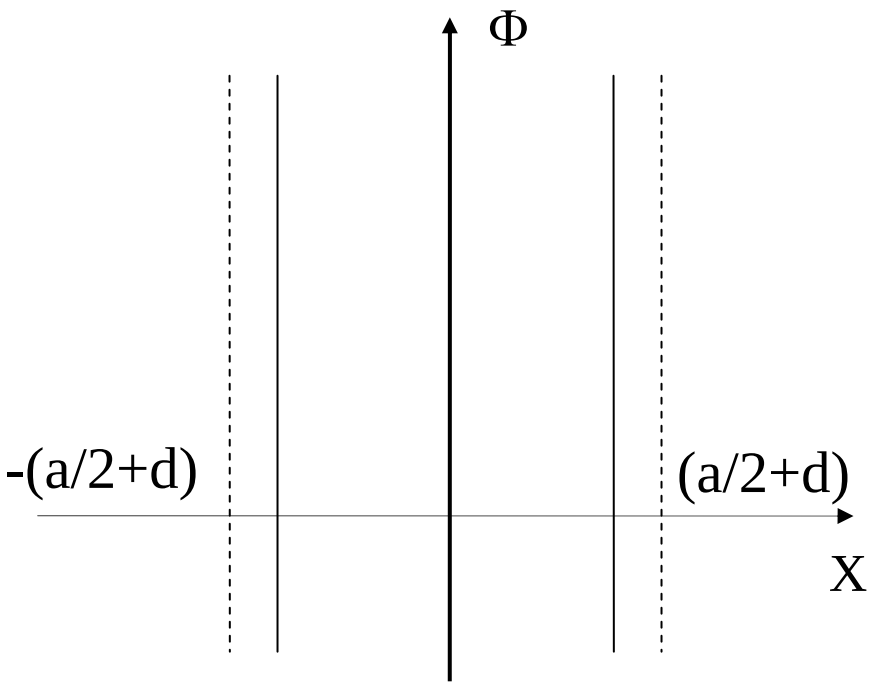
offers an equation for k.

$$k = \frac{\nu\Sigma_{\mathrm{f}}}{B_g^2 D + \Sigma_a}$$

Where $B_g^2 D$ is the leakage.

Things will become clearer by showing an example.

Infinite Slab Reactor



$$\Phi(\frac{\tilde{a}}{2}) = 0$$

where

$$\tilde{a} = a + 2d$$

We then have:

$$\frac{d^2\Phi}{dx^2} + B^2\Phi = 0$$

Boundary conditions:

$$\Phi\left(\frac{\tilde{a}}{2}\right) = \Phi\left(\frac{-\tilde{a}}{2}\right) = 0$$

The symmetry of problem implies:

$$\left.\frac{d\Phi}{dx}\right|_{x=0} = 0$$

General Solution:

$$\Phi(x) = A\cos Bx + C\sin Bx$$

$$\left.\frac{d\Phi(x)}{dx}\right|_{x=0} = \left[-AB\sin Bx + CB\cos Bx\right] =$$

$$= CB = 0 \Rightarrow C = 0$$

Hence:

$$\Phi(x) = A\cos Bx$$

Vacuum B.C.

$$\Phi\left(\frac{\tilde{a}}{2}\right) = A\cos\left(\frac{B\tilde{a}}{2}\right) = 0$$

Implies:

$$\cos\left(\frac{B\tilde{a}}{2}\right) = 0$$

$$\frac{B\tilde{a}}{2} = \frac{\pi}{2} + k\pi$$

$$B\tilde{a} = \pi + 2k\pi = (2k+1)\pi = n\pi$$

Yields:

$$B_n = \frac{n\pi}{\tilde{a}}\ ; \qquad B_1 = \frac{\pi}{\tilde{a}}$$

Fundamental solution

$$\Phi(x) = A\cos B_1 x = A\cos\left(\frac{\pi x}{\tilde{a}}\right)$$

B_1 is the geometrical buckling

A cannot be determined from the diffusion equation. It can be determined from the condition on the reactor power.

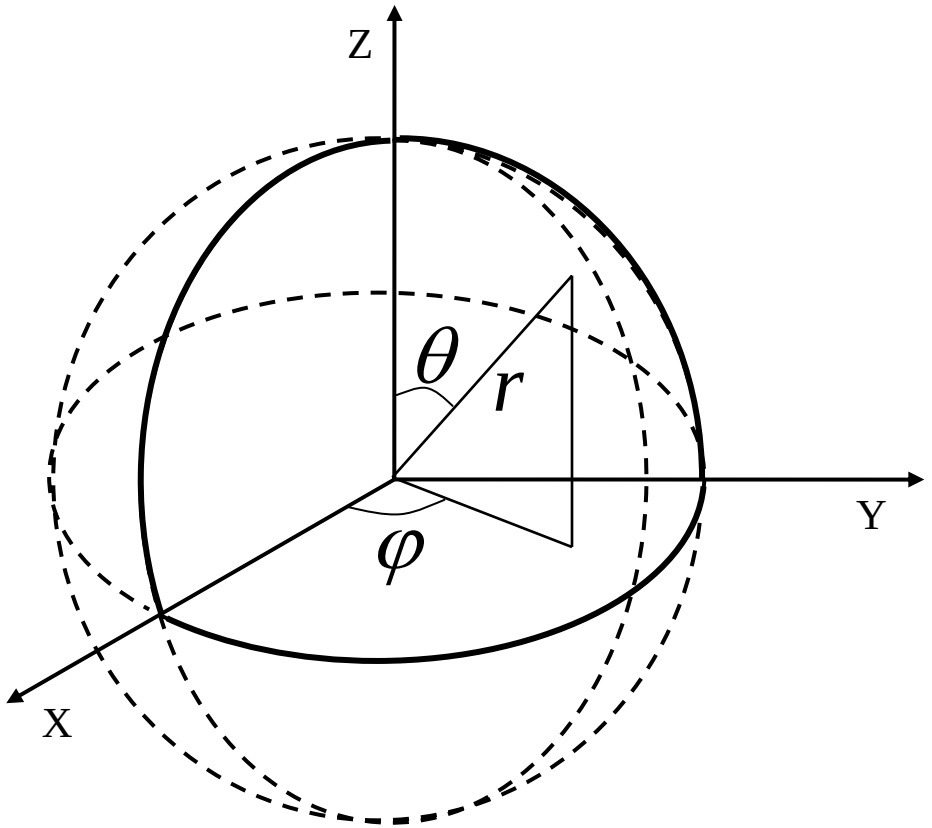
$$P = E_R \Sigma_f \int_{-a/2}^{a/2} \Phi(x)dx$$

$$P = \frac{2\tilde{a}E_R\Sigma_f A\sin\left(\frac{\pi a}{2\tilde{a}}\right)}{\pi}$$

$$\Phi(x) = \frac{\pi P}{2aE_R\Sigma_f}\cos\left(\frac{\pi x}{a}\right)$$

$$k = \frac{\nu\Sigma_{\mathrm{f}}}{B_g^2 D + \Sigma_a} = \frac{\nu\Sigma_{\mathrm{f}}}{\left(\frac{\pi}{\tilde{a}}\right)^2 D + \Sigma_a}$$

Spherical Reactor



$$w = \Phi r \text{ (change of variable)}$$

$$\frac{d^2 w}{dr^2} + B^2 w = 0$$

$$w(r) = A\sin Br + C\cos Br$$

$$\Phi(r) = \frac{w}{r}$$

We have, in sequence:

$$\frac{1}{r^2}\frac{d}{dr}r^2\frac{d\Phi}{dr}+B^2\Phi=0$$

$$\Phi=A\frac{\sin Br}{r}+C\frac{\cos Br}{r}$$

Because the flux has to be finite at r=0, we have:

$$C=0$$

$$\Phi=A\frac{\sin Br}{r}$$

$$\Phi\left(\tilde{R}\right)=0$$

B.C. $\tilde{R}=R+d$

$$B_1^2=\left(\frac{\pi}{\tilde{R}}\right)^2$$

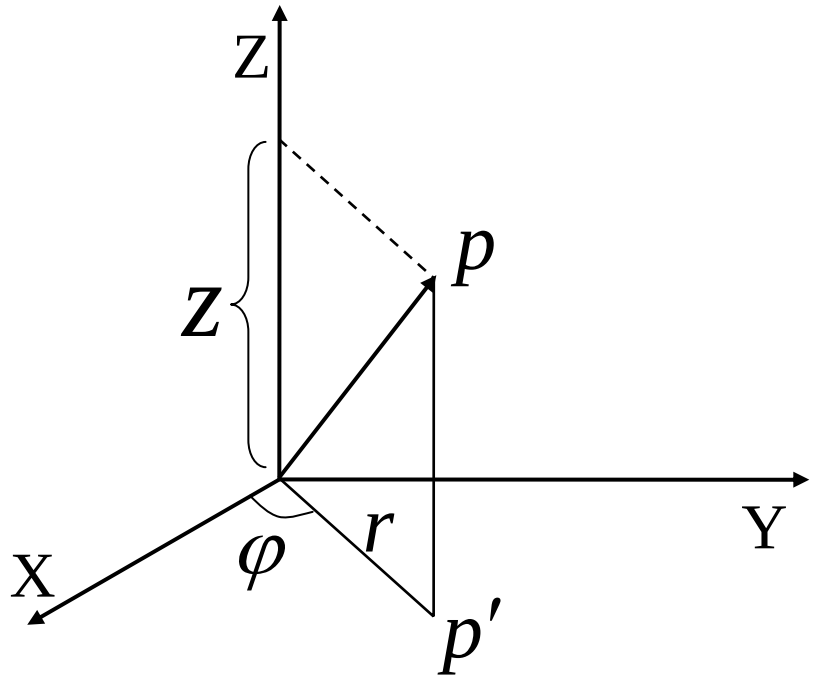
$$\Phi=A\frac{\sin \pi r/\tilde{R}}{r}$$

The total power can be used to find A.

$$P=E_R\Sigma_f\int\Phi(r)dV \qquad dV=4\pi r^2dr$$

$$\Phi=\frac{P}{4E_R\Sigma_f R^2}\frac{\sin\left(\pi r/\tilde{R}\right)}{r}$$

Infinite Cylinder



Cylindrical coordinates

We have in sequence:

$$\frac{1}{r}\frac{d}{dr}r\frac{d\Phi}{dr}+B^2\Phi=0$$

$$\frac{d^2\Phi}{dr^2}+\frac{1}{r}\frac{d\Phi}{dr}+B^2\Phi=0$$

Bessel's Equation:

$$\frac{d^2\Phi}{dr^2}+\frac{1}{r}\frac{d\Phi}{dr}+\left(B^2-\frac{m^2}{r^2}\right)\Phi=0$$

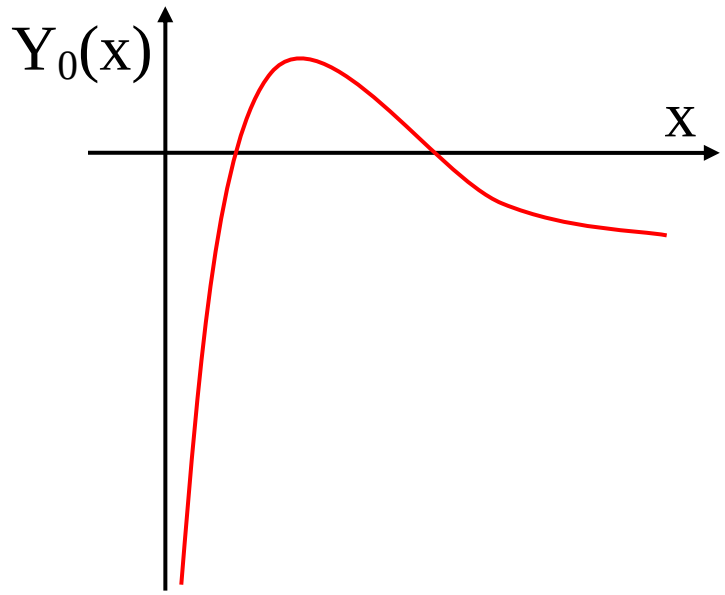
Our equation is Bessel's equation for m=0.

Solution: Bessel functions of first and second kind:

$$\Phi = AJ_0(Br) + CY_0(Br)$$

J – Bessel Function

Y- Modified Bessel Function

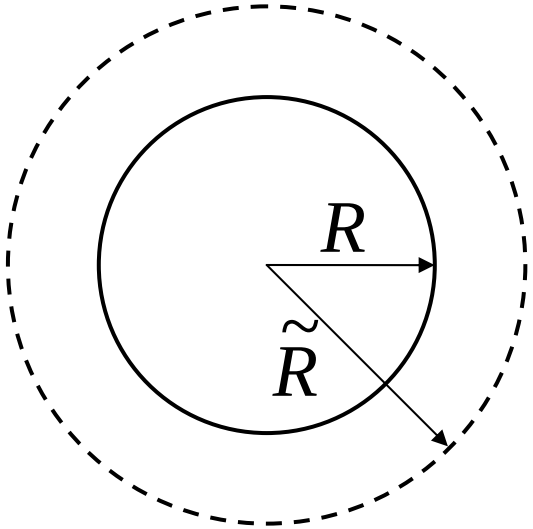


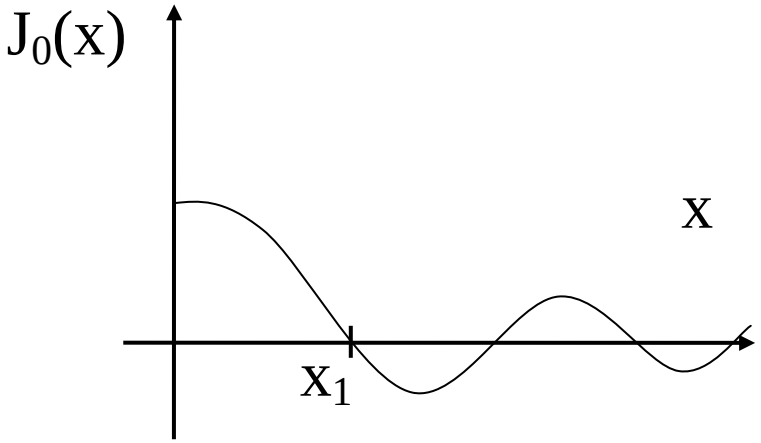
Y_0 infinite at origin hence C=0

$$\Phi = AJ_0(Br)$$

B.C.

$$\Phi(\tilde{R}) = AJ_0(B\tilde{R}) = 0$$





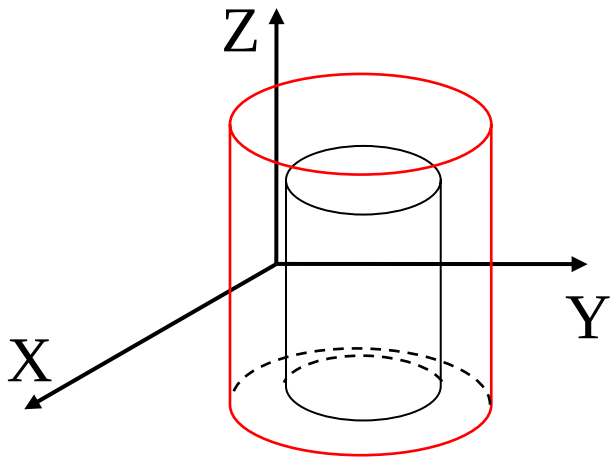
Final Solution

$$B\tilde{R} = x_1 \Rightarrow B = \frac{x_1}{\tilde{R}}$$

$$B_1^2 = \left(\frac{x_1}{\tilde{R}}\right)^2 = \left(\frac{2.405}{\tilde{R}}\right)^2$$

$$\Phi = AJ_0\left(\frac{2.405r}{\tilde{R}}\right)$$

Finite Cylinder



$$\frac{d^2\phi}{\partial r^2}+\frac{1}{r}\frac{\partial\phi}{\partial r}+\frac{\partial^2\phi}{\partial z^2}+B^2\phi=0$$

B.C.

$$\Phi(\tilde{R}, z) = 0$$

$$\Phi(r, \frac{\tilde{H}}{2}) = 0$$

Separation of Variables

$$\Phi(r, z) = R(r)Z(z)$$

$$\frac{\partial Z}{\partial r} = 0, \qquad \frac{\partial R}{\partial r} = 0$$

$$Z\frac{1}{r}\frac{d}{\partial r}\left(r\frac{\partial R}{\partial r}\right)+R\frac{\partial^2 Z}{\partial z^2}+B^2RZ=0$$

$$\frac{1}{R}\frac{1}{r}\frac{d}{\partial r}r\frac{\partial R}{\partial r}+\frac{1}{Z}\frac{\partial^2 Z}{\partial z^2}=-B^2$$

$$\frac{1}{R}\frac{1}{r}\frac{d}{\partial r}r\frac{\partial R}{\partial r}=-B_r^2;\qquad R=AJ_0\left(B_r r\right)$$

$$\frac{1}{Z}\frac{\partial^2 Z}{\partial z^2}=-B_z^2;\qquad Z=A\cos\left(B_z z\right)$$

Solution:

$$\Phi(r,z) = AJ_0\left(\frac{2.405r}{\tilde{R}}\right)\cos\frac{\pi z}{\tilde{H}}$$

$$\Phi(r,z) = AJ_0\left(B_r r\right)\cos\left(B_z z\right)$$

Point Kinetics

Point Kinetics Equations

part 1: all neutrons emitted in a fission are assumed prompt

Static One-energy-group diffusion equation

- time-dependent diffusion (results from neutron balance)

$$\frac{\partial n(\vec{r},t)}{\partial t} = \nu\Sigma_f \Phi(\vec{r},t) + D\nabla^2\Phi(\vec{r},t) - \Sigma_a \Phi(\vec{r},t)$$

- If sources are exactly equal to sinks, then the static equation results (no time dependence)

$$0 = \nu\Sigma_f \Phi(\vec{r}) + D\nabla^2\Phi(\vec{r}) - \Sigma_a \Phi(\vec{r}) \Leftrightarrow -D\nabla^2\Phi(\vec{r}) + \Sigma_a \Phi(\vec{r}) = \nu\Sigma_f \Phi(\vec{r})$$

- To keep the static form of the diffusion equation even when the sources do not exactly equal the sinks, we introduced K, (multiplication factor) to artificially adjust the sources.

$$0 = \frac{1}{k}\nu\Sigma_f \Phi(\vec{r}) + D\nabla^2\Phi(\vec{r}) - \Sigma_a \Phi(\vec{r}) \Leftrightarrow -D\nabla^2\Phi(\vec{r}) + \Sigma_a \Phi(\vec{r}) = \frac{1}{k}\nu\Sigma_f \Phi(\vec{r})$$

Time – dependent one-energy-group diffusion equation

- Now, we will not use k any more, but rather concentrate on the time-dependent equation
- Time-dependent one-energy-group diffusion equation

$$\frac{\partial n(\vec{r},t)}{\partial t} = \nu\Sigma_f \Phi(\vec{r},t) + D\nabla^2\Phi(\vec{r},t) - \Sigma_a \Phi(\vec{r},t)$$

- But let's remember:

$$\Phi = n\bar{\mathrm{v}} \Leftrightarrow n = \frac{\Phi}{\bar{\mathrm{v}}}$$

- so:

$$\frac{\partial n}{\partial t} = \frac{1}{\bar{\mathrm{v}}}\frac{\partial \Phi}{\partial t}$$

Time-dependence of the neutron flux and neutron density

- We can now write the time-dependent diffusion: in two separate ways
  - concentrate on the flux

$$\frac{1}{\bar{v}}\frac{\partial\Phi(\vec{r},t)}{\partial t} = \nu\Sigma_f\Phi(\vec{r},t) + D\nabla^2\Phi(\vec{r},t) - \Sigma_a\Phi(\vec{r},t)$$

  - concentrate on the neutron density

$$\frac{\partial n(\vec{r},t)}{\partial t} = \nu\Sigma_f\bar{v}n(\vec{r},t) + D\nabla^2\bar{v}n(\vec{r},t) - \Sigma_a\bar{v}n(\vec{r},t)$$

Some assumptions

- Static one-energy-group diffusion equation for a critical reactor

$$-D\nabla^2\Phi(\vec{r})+\Sigma_a\Phi(\vec{r})=\nu\Sigma_f\Phi(\vec{r})$$

- It can be rewritten as:

$$\nabla^2\Phi(\vec{r})+B^2\Phi(\vec{r})=0$$

- Where

$$\frac{\nu\Sigma_f-\Sigma_a}{D}=B^2$$

- Assume that the equation satisfied by the time-independent flux in a critical reactor is also satisfied, at any time t, by the time-dependent flux in a non-critical reactor.

$$\nabla^2\Phi(\vec{r})+B^2\Phi(\vec{r})=0$$

$$\nabla^2\Phi(\vec{r},t)+B^2\Phi(\vec{r},t)=0\Leftrightarrow\nabla^2\Phi(\vec{r},t)=-B^2\Phi(\vec{r},t)$$

- This is equivalent to assuming that the spatial shape of the flux does not change with time

Back to the time-dependence of the neutron flux and neutron density

- We can now write the time-dependent diffusion
  - for the flux

$$\frac{1}{\bar{v}}\frac{\partial\Phi(\vec{r},t)}{\partial t} = \nu\Sigma_f\Phi(\vec{r},t) - DB^2\Phi(\vec{r},t) - \Sigma_a\Phi(\vec{r},t)$$

  - for the neutron density

$$\frac{\partial n(\vec{r},t)}{\partial t} = \nu\Sigma_f\bar{v}n(\vec{r},t) - DB^2\bar{v}n(\vec{r},t) - \Sigma_a\bar{v}n(\vec{r},t)$$

Time-dependence of the neutron density

$$\frac{\partial n(\vec{r},t)}{\partial t} = \left(\nu\Sigma_f \bar{v} - DB^2\bar{v} - \Sigma_a \bar{v}\right) n(\vec{r},t) \Leftrightarrow \frac{\partial n(\vec{r},t)}{\partial t} = \alpha n(\vec{r},t)$$

- where

$$\alpha = \left(\nu\Sigma_f - DB^2 - \Sigma_a\right)\bar{v}$$

$$\frac{\partial n(\vec{r},t)}{\partial t} = \alpha n(\vec{r},t)$$

- Integrating over the entire reactor we obtain:

$$\int_V \frac{\partial n(\vec{r},t)}{\partial t} d^3\vec{r} = \int_V \alpha n(\vec{r},t) d^3\vec{r}$$

$$\frac{d}{dt}\int_V n(\vec{r},t) d^3\vec{r} = \alpha \int_V n(\vec{r},t) d^3\vec{r}$$

$$d^3\vec{r} = dV$$

Time-dependence of the total neutron population

- Total neutron population

$$n(t) = \int_V n(\vec{r}, t) d^3\vec{r}$$

- Equation governing the time behavior of the total neutron population

$$\frac{d}{dt} n(t) = \alpha n(t) \Leftrightarrow \dot{n}(t) = \alpha n(t)$$

- solution

$$n(t) = n_0 e^{\alpha t}$$

Time dependence of the neutron flux

$$\frac{1}{\bar{v}}\frac{\partial\Phi(\vec{r},t)}{\partial t} = \nu\Sigma_f\Phi(\vec{r},t) - DB^2\Phi(\vec{r},t) - \Sigma_a\Phi(\vec{r},t)$$

$$\frac{1}{\bar{v}}\frac{\partial\Phi(\vec{r},t)}{\partial t} = \left(\nu\Sigma_f - DB^2 - \Sigma_a\right)\Phi(\vec{r},t)$$

$$\frac{\partial\Phi(\vec{r},t)}{\partial t} = \left(\nu\Sigma_f - DB^2 - \Sigma_a\right)\bar{v}\Phi(\vec{r},t) \Leftrightarrow \frac{\partial\Phi(\vec{r},t)}{\partial t} = \alpha\Phi(\vec{r},t)$$

- The results are analogous to those for the neutron density.
- Integrating over the volume of the reactor:

$$\int_V \frac{\partial \Phi(\vec{r},t)}{\partial t} d^3\vec{r} = \int_V \alpha \Phi(\vec{r},t) d^3\vec{r}$$

$$\frac{d}{dt}\int_V \Phi(\vec{r},t) d^3\vec{r} = \alpha \int_V \Phi(\vec{r},t) d^3\vec{r}$$

$$\frac{d}{dt}\hat{\phi}(t) = \alpha\hat{\phi}(t) \Leftrightarrow \dot{\hat{\phi}}(t) = \alpha\hat{\phi}(t) \Rightarrow \hat{\phi}(t) = \hat{\phi}_0 e^{\alpha t}$$

- where:

$\hat{\phi}(t) = \int_V \Phi(\vec{r},t) d^3\vec{r}$ is the volume-integrated flux

Observations

- The total neutron population and the volume integrated flux obey the same equation.
- The relation between the volume integrated flux and the total neutron population is the same as that between the flux and neutron density.

$$\Phi(\vec{r},t) = n(\vec{r},t)\bar{\mathrm{v}}$$

$$\int_V \Phi(\vec{r},t)d^3\vec{r} = \bar{\mathrm{v}}\int_V n(\vec{r},t)d^3\vec{r} \Leftrightarrow \hat{\phi}(t) = n(t)\bar{\mathrm{v}}$$

Point Kinetics Equation without Delayed Neutrons

- Just a special way of arranging the coefficients.
- Usually written for the neutron population, but similar equation can be written for the volume-integrated flux.
- Multiplication constant

$$k_{eff} = \frac{\nu\Sigma_f}{\Sigma_a + DB^2}$$

- Reactivity

$$\rho = \frac{k-1}{k} = 1 - \frac{1}{k}$$

- we can write:

$$\alpha = \left(\nu\Sigma_f - DB^2 - \Sigma_a\right)\bar{v} = \nu\Sigma_f \bar{v}\frac{\nu\Sigma_f - DB^2 - \Sigma_a}{\nu\Sigma_f} =$$

$$\nu\Sigma_f \bar{v}\left(\frac{\nu\Sigma_f}{\nu\Sigma_f} - \frac{DB^2 + \Sigma_a}{\nu\Sigma_f}\right) = \nu\Sigma_f \bar{v}\left(1 - \frac{1}{k_{eff}}\right) = \nu\Sigma_f \bar{v}\rho$$

- Notation:

$$\Lambda = \frac{1}{\nu \Sigma_f \bar{v}}$$

- It follows that:

$$\alpha = \frac{\rho}{\Lambda}$$

- The equation for the neutron population can then be written

$$\frac{dn(t)}{dt} = \frac{\rho}{\Lambda} n(t)$$ = Point kinetics eq. w/o delayed neutrons

- A similar equation can be written for the volume-integrated flux.

$$\frac{d\hat{\phi}(t)}{dt} = \frac{\rho}{\Lambda} \hat{\phi}(t)$$

Alternative Processing Leading to the Point Kinetics Equation

$$\alpha = \left(\nu\Sigma_f - DB^2 - \Sigma_a\right)\bar{v} = \left(DB^2 + \Sigma_a\right)\bar{v}\frac{\nu\Sigma_f - DB^2 - \Sigma_a}{DB^2 + \Sigma_a} =$$

$$\left(DB^2 + \Sigma_a\right)\bar{v}\left(\frac{\nu\Sigma_f}{DB^2 + \Sigma_a} - \frac{DB^2 + \Sigma_a}{DB^2 + \Sigma_a}\right) = \left(DB^2 + \Sigma_a\right)\bar{v}(k-1)$$

- New notation

$$\ell = \frac{1}{\left(DB^2 + \Sigma_a\right)\bar{v}}$$

- With the new notation the point kinetics eq. can be written (a less common form):

$$\frac{dn(t)}{dt} = \frac{k-1}{\ell}n(t)$$

- and, for the volume-integrated flux:

$$\frac{d\hat{\phi}(t)}{dt} = \frac{k-1}{\ell}\hat{\phi}(t)$$

Point kinetics equation(s)

- Nomenclature – called point-kinetics because the reactor is reduced to a point – no accounting for spatial or energy dependence.
- Can be derived starting from a more general, space and energy dependent, flux.

Names and interpretations of symbols

- Neutron generation time

$$\Lambda = \frac{1}{\bar{\nu} \nu \Sigma_f}$$

- Interpretations
  - Average time between two neutron births in successive generations
  - Time it would take to generate the current number of neutrons at the current generation rate.
  - Average “age” of neutrons in the reactor. (Note that this is a time, and not the Fermi age).

- Neutron life time

$$\ell = \frac{1}{\overline{\nu}} \frac{1}{\Sigma_a + DB^2}$$

- For an infinite reactor:

$$\ell_\infty = \frac{1}{\overline{\nu}} \frac{1}{\Sigma_a}$$

- Interpretations
  - average time between the birth and death of a neutron
  - Time necessary to lose all the neutrons in the reactor at the current loss rate.
  - Average life expectancy for neutrons in the reactor.

Important Notes

- For a critical reactor the generation time and neutron lifetime are equal.

- For a supercritical reactor, the generation time is shorter that the neutron lifetime. Neutrons live longer than the time it takes a new generation to appear. The neutron population increases.

- For a subcritical reactor, the generation time is longer that the neutron lifetime. Neutrons live less than the time it takes a new generation to appear. The neutron population decreases.

Point Kinetics Equations

part 2: Accounting for Delayed Neutrons

Point Kinetics with Only One Delayed Neutron Group

(Equivalent to assuming that all precursors have the same half life)

- We make the same assumptions about the buckling staying constant as in the case with no delayed neutrons.
- We write directly the equation for the entire reactor (volume-integrated quantities)
- Some neutrons are emitted directly from fission
- Some neutrons come from the decay of precursors.

Neutron Balance Equation for the Entire Reactor

- Sources
  - Prompt neutrons from fission

$$\int_V \nu_p \Sigma_f \Phi(\vec{r}) d^3\vec{r} = \nu_p \Sigma_f \int_V \Phi(\vec{r}) d^3\vec{r} = \nu_p \Sigma_f \hat{\phi} = (\nu - \nu_d)\Sigma_f \hat{\phi} = \nu(1-\beta)\Sigma_f \hat{\phi}$$

  - Delayed neutrons from the decay of precursors

$\lambda \hat{C}$ ($\hat{C}$ = total number of precursors in the reactor)

$$\hat{C}(t) = \int_V C(\vec{r}, t) d^3\vec{r}$$

- Sinks
  - Absorption

$$\int_V \Sigma_a \Phi(\vec{r}) d^3\vec{r} = \Sigma_a \int_V \Phi(\vec{r}) d^3\vec{r} = \Sigma_a \hat{\phi}$$

  - Leakage

$$\int_V DB^2 \Phi(\vec{r}) d^3\vec{r} = DB^2 \int_V \Phi(\vec{r}) d^3\vec{r} = DB^2 \hat{\phi}$$

Precursor Balance Equation for the Entire Reactor

- Source

$$\int_V \nu_d \Sigma_f \Phi(\vec{r}) d^3\vec{r} = \nu_d \Sigma_f \int_V \Phi(\vec{r}) d^3\vec{r} = \nu_d \Sigma_f \hat{\phi} = \nu\beta\Sigma_f \hat{\phi}$$

- Sink

$$\lambda \hat{C}$$

Neutron and Precursor balance Equations

- Neutron Balance

$$\frac{dn(t)}{dt} = \nu_p \Sigma_f \hat{\phi} - \Sigma_a \hat{\phi} - DB^2 \hat{\phi} + \lambda\, \hat{C}$$

- Precursor Balance

$$\frac{d\hat{C}(t)}{dt} = \nu_d \Sigma_f \hat{\phi} - \lambda \hat{C}(t)$$

- We now have a system of two (coupled) differential equations.

Point Kinetics Equations with One Group of Delayed Neutrons

- Rearrange the first equation in a few steps

$$\frac{dn(t)}{dt} = \left[\nu(1-\beta)\Sigma_f - \Sigma_a - DB^2\right]\hat{\phi} + \lambda\,\hat{C}$$

$$\frac{dn(t)}{dt} = \nu\Sigma_f \frac{\left[\nu(1-\beta)\Sigma_f - \Sigma_a - DB^2\right]}{\nu\Sigma_f}\hat{\phi} + \lambda\,\hat{C}$$

$$\frac{dn(t)}{dt} = \bar{\nu}\nu\Sigma_f \left[\frac{\nu\Sigma_f - \Sigma_a - DB^2}{\nu\Sigma_f} - \frac{\beta\nu\Sigma_f}{\nu\Sigma_f}\right]\frac{\hat{\phi}}{\bar{\nu}} + \lambda\,\hat{C}$$

$$\frac{dn(t)}{dt} = \bar{\nu}\nu\Sigma_f \left[1 - \frac{\Sigma_a + DB^2}{\nu\Sigma_f} - \beta\right]\frac{\hat{\phi}}{\bar{\nu}} + \lambda\,\hat{C}$$

- Rearrange the second equation

$$\frac{d\hat{C}(t)}{dt} = \nu_d \Sigma_f \hat{\phi} - \lambda \hat{C}(t)$$

$$\frac{d\hat{C}(t)}{dt} = \nu \beta \Sigma_f \hat{\phi} - \lambda \hat{C}(t)$$

$$\frac{d\hat{C}(t)}{dt} = \beta \bar{\mathrm{v}} \nu \Sigma_f \frac{\hat{\phi}}{\bar{\mathrm{v}}} - \lambda \hat{C}(t)$$

Make the same notations and observations as for the case with no delayed neutrons

$$\Lambda = \frac{1}{\nu\Sigma_f \bar{\mathrm{v}}}$$

$$\hat{\phi}(t) = n(t)\bar{\mathrm{v}} \Leftrightarrow n(t) = \frac{\hat{\phi}(t)}{\bar{\mathrm{v}}}$$

$$k = \frac{\nu\Sigma_f}{\Sigma_a + DB^2} \Rightarrow \frac{1}{k} = \frac{\Sigma_a + DB^2}{\nu\Sigma_f}$$

$$\rho = 1 - \frac{1}{k} = 1 - \frac{\Sigma_a + DB^2}{\nu\Sigma_f}$$

- Neutron Balance Equation

$$\frac{dn(t)}{dt} = \bar{\nu}\nu\Sigma_f \left[1 - \frac{\Sigma_a + DB^2}{\nu\Sigma_f} - \beta \right] \frac{\hat{\phi}}{\bar{\nu}} + \lambda \hat{C}$$

$$\frac{dn(t)}{dt} = \frac{\rho - \beta}{\Lambda} n(t) + \lambda \hat{C}$$

- Precursor Balance Equation

$$\frac{d\hat{C}(t)}{dt} = \beta\bar{\nu}\nu\Sigma_f \frac{\hat{\phi}}{\bar{\nu}} - \lambda\hat{C}(t)$$

$$\frac{d\hat{C}(t)}{dt} = \frac{\beta}{\Lambda} n(t) - \lambda\hat{C}(t)$$

- Final form of kinetics equations using the neutron population

$$\frac{dn(t)}{dt} = \frac{\rho - \beta}{\Lambda} n(t) + \lambda \hat{C}$$

$$\frac{d\hat{C}(t)}{dt} = \frac{\beta}{\Lambda} n(t) - \lambda\hat{C}(t)$$

- Final form of the point kinetics equations using the volume-integrated flux

$$\hat{\phi}(t) = n(t)\bar{v} \Leftrightarrow n(t) = \frac{\hat{\phi}(t)}{\bar{v}}$$

$$\frac{d\hat{\phi}(t)}{dt} = \frac{\rho - \beta}{\Lambda}\hat{\phi}(t) + \bar{v}\lambda\,\hat{C} \Leftrightarrow \frac{d\hat{\phi}(t)}{dt} = \frac{\rho - \beta}{\Lambda}\hat{\phi}(t) + \frac{1}{\Lambda \nu\Sigma_f}\lambda\,\hat{C}$$

$$\frac{d\hat{C}(t)}{dt} = \frac{\beta}{\Lambda}\frac{1}{\bar{v}}\hat{\phi}(t) - \lambda\hat{C}(t) \Leftrightarrow \frac{d\hat{C}(t)}{dt} = \beta\nu\Sigma_f\hat{\phi}(t) - \lambda\hat{C}(t)$$

Point Kinetics Equations with Six Groups of Delayed Neutrons

- Equations using the neutron population
  (7 coupled differential equations)

$$\frac{dn(t)}{dt} = \frac{\rho - \beta}{\Lambda} n(t) + \sum_{k=1}^{6} \lambda_k \hat{C}_k$$

$$\frac{d\hat{C}_k(t)}{dt} = \frac{\beta_k}{\Lambda} n(t) - \lambda_k \hat{C}_k(t), \quad k = 1...6$$

- Equations using the volume-integrated flux
  (7 coupled differential equations)

$$\frac{d\hat{\phi}(t)}{dt} = \frac{\rho - \beta}{\Lambda} \hat{\phi}(t) + \bar{\nu} \sum_{k=1}^{6} \lambda_k \hat{C}_k \Leftrightarrow \frac{d\hat{\phi}(t)}{dt} = \frac{\rho - \beta}{\Lambda} \hat{\phi}(t) + \frac{1}{\Lambda \nu \Sigma_f} \sum_{k=1}^{6} \lambda_k \hat{C}_k$$

$$\frac{d\hat{C}_k(t)}{dt} = \frac{\beta_k}{\Lambda} \frac{1}{\bar{\nu}} \hat{\phi}(t) - \lambda_k \hat{C}_k(t) \Leftrightarrow \frac{d\hat{C}_k(t)}{dt} = \beta_k \nu \Sigma_f \hat{\phi}(t) - \lambda_k \hat{C}_k(t)$$

Inhour Equation

Inhour Equation

Start with the point kinetics equation

$$\frac{dn(t)}{dt} = \frac{\rho - \beta}{\Lambda} n(t) + \sum_{k=1}^{6} \lambda_k \hat{C}_k$$

$$\frac{d\hat{C}_k(t)}{dt} = \frac{\beta_k}{\Lambda} n(t) - \lambda_k \hat{C}_k(t), \quad k = 1...6$$

This is a system of seven coupled differential equations with constant coefficients.

Solutions are of the form $\begin{bmatrix} ne^{\omega t} \\ c_1 e^{\omega t} \\ \vdots \\ c_6 e^{\omega t} \end{bmatrix}$

Substituting the above form we obtain:

$$\omega n = \frac{\rho - \beta}{\Lambda} n + \sum_{k=1}^{6} \lambda_k c_k$$

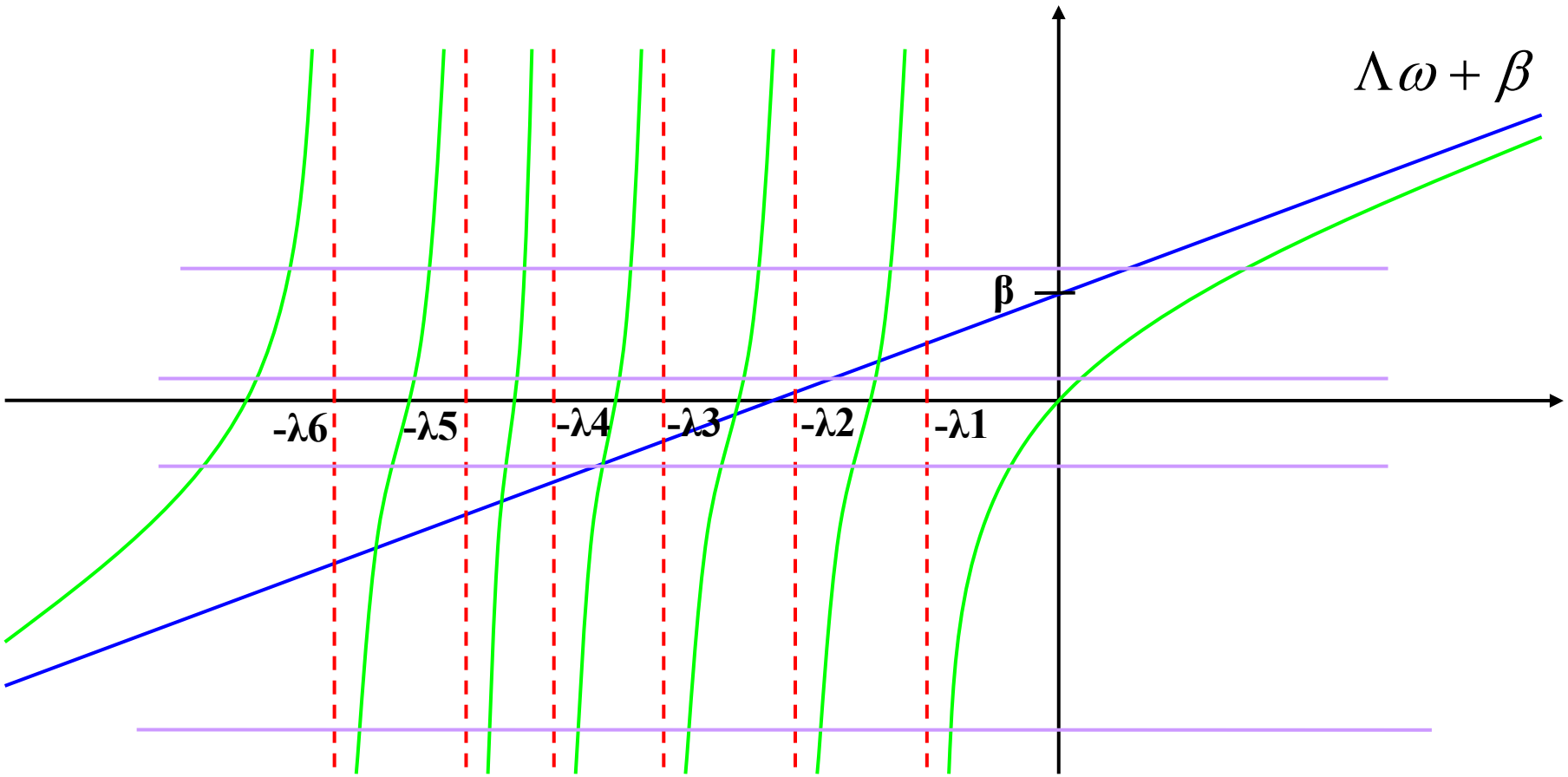
$$\omega c_k = \frac{\beta}{\Lambda} n - \lambda_k c_k, \quad k = 1...6$$

Solving for c_k in the precursor equations, we obtain:

$$c_k = \frac{\beta_k}{\Lambda(\omega + \lambda_k)} n, \quad k = 1...6$$

$$\rho = \Lambda\omega + \beta - \sum_{k=1}^{6} \lambda_k \frac{\beta_k}{(\omega + \lambda_k)}$$

We can solve graphically for ω by plotting the RHS and intersecting it with a horizontal line at y=ρ.



Reactor Period:

$$T = \frac{1}{\omega_{\max}}$$

Discussion

If $\rho = 0$

$$\omega_1 = 0 \rightarrow e^0 = 1$$

$$\omega_2 \ldots \omega_7 < 0 \rightarrow e^{-\sim t}$$

$$n(t) = a_1 + a_2 e^{\omega_2 t} + \ldots = a_1 + a_2 e^{-\sim t} + \ldots$$

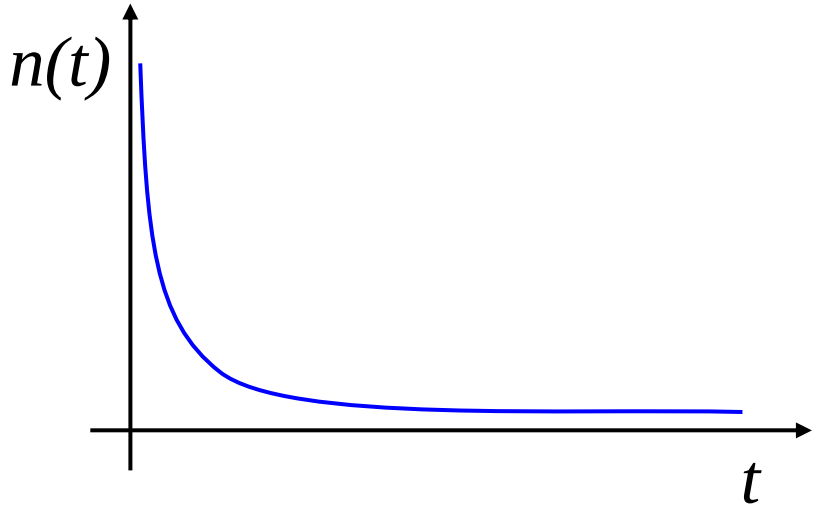
$n(t) \rightarrow$ after a long time n becomes constant

If $\rho < 0$

$$\omega_k < 0; \quad k = 1,2,\ldots 7$$

$$n(t) = a_1 e^{-\sim t} + a_2 e^{-\sim t} + a_3 e^{-\sim t} \ldots a_7 e^{-\sim t}$$

$as \quad t \rightarrow \infty; \quad n(t) \rightarrow 0$



$$n(t) = a_1 e^{\omega_1 t}\left(1 + \sum_{k=2}^{7} \frac{a_k}{a_1} e^{(\omega_k - \omega_1)t}\right)$$

For large t

$$n(t) \approx a_1 e^{\omega_1 t} = a_1 e^{\frac{t}{T}}$$

Asymptotic behavior of the neutron population. Here is where the reactor period comes in.

If $\rho > 0$

$\omega_1 > 0$

$\omega_k > 0 \quad k = 1,2,\ldots 7$

$$n(t) = a_1 e^{+\sim t} + a_2 e^{-\sim t} + a_3 e^{-\sim t} \ldots a_7 e^{-\sim t}$$

$$n(t) = a_1 e^{\omega_1 t}\left(1 + \sum_{k=2}^{7} \frac{a_2}{a_1} e^{(\omega_k - \omega_1)t}\right)$$

For large t

$$n(t) \approx a_1 e^{\omega_1 t} = a_1 e^{\frac{t}{T}}$$

$$T = \frac{1}{\omega_1}$$

The asymptotic behavior of the neutron population is governed by the reactor period.

Fission Product Poisoning

Fission Product Poisons

- Poisons are isotopes with large absorption cross sections for thermal neutrons
- Some poisons are introduced intentionally to control the reactor, such as B or Gd.
- Some poisons are produced as fission products during normal operation of the reactor.
- Xe and Sm are the most important of these
- We will only study Xe in detail

Effects of Poisons on Reactivity

For a homogeneous reactor, in a one-energy-group formalism:

$$k_{eff}^{0} = \frac{\nu\Sigma_f}{\Sigma_{a0} + DB^2}$$

If we ***add*** a poison (say Xe) with a uniform concentration (number density) X, we have:

$$\Sigma_{aX} = X\sigma_{aX}$$

$$\Sigma_a = \Sigma_{a0} + \Sigma_{aX}$$

It follows that:

$$k_{eff} = \frac{\nu\Sigma_f}{\Sigma_a + DB^2} = \frac{\nu\Sigma_f}{\Sigma_{a0} + \Sigma_{aX} + DB^2}$$

The insertion of the poison induces a change in reactivity:

$$\Delta\rho = \rho - \rho_0 = \left(1 - \frac{1}{k_{eff}}\right) - \left(1 - \frac{1}{k_{eff}^0}\right) = \frac{1}{k_{eff}^0} - \frac{1}{k_{eff}}$$

If we assume the initial reactor to be critical $(\rho_0 = 0)$, then the reactivity of the reactor with poison is:

$$\rho = \Delta\rho = \frac{1}{k_{eff}^{0}} - \frac{1}{k_{eff}} = \frac{\Sigma_{a0} + DB^2}{\nu\Sigma_f} - \frac{\Sigma_{a0} + \Sigma_{aX} + DB^2}{\nu\Sigma_f} = -\frac{\Sigma_{aX}}{\nu\Sigma_f} = -\frac{X\sigma_{aX}}{\nu\Sigma_f}$$

In order to calculate the reactivity inserted by the poison, we need to be able to calculate the concentration of poison nuclei, X.

Effects of Non-Uniform Poison Concentration

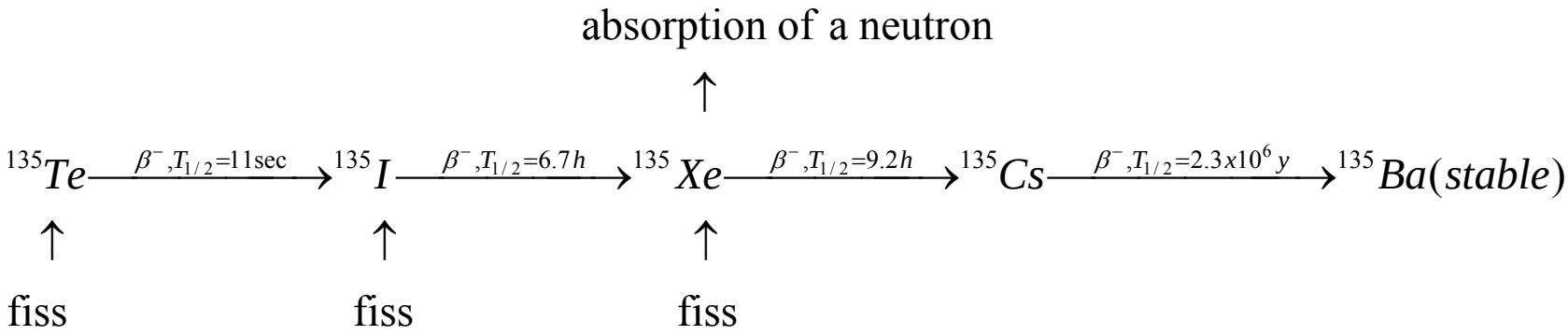
In the case of non-uniform poison concentration we need to apply the perturbation formula for reactivity:

$$\Delta\rho = \frac{1}{k_0} - \frac{1}{k} = -\frac{\int_V \Phi^2(\vec{r})\delta\Sigma_a(\vec{r})dV}{\int_V \Phi^2(\vec{r})\nu\Sigma_f(\vec{r})dV} - \frac{\int_V \Phi^2(\vec{r})\Sigma_{aX}(\vec{r})dV}{\int_V \Phi^2(\vec{r})\nu\Sigma_f(\vec{r})dV} = -\frac{\int_V \Phi^2(\vec{r})X(\vec{r})\sigma_{aX}dV}{\int_V \Phi^2(\vec{r})\nu\Sigma_f(\vec{r})dV}$$

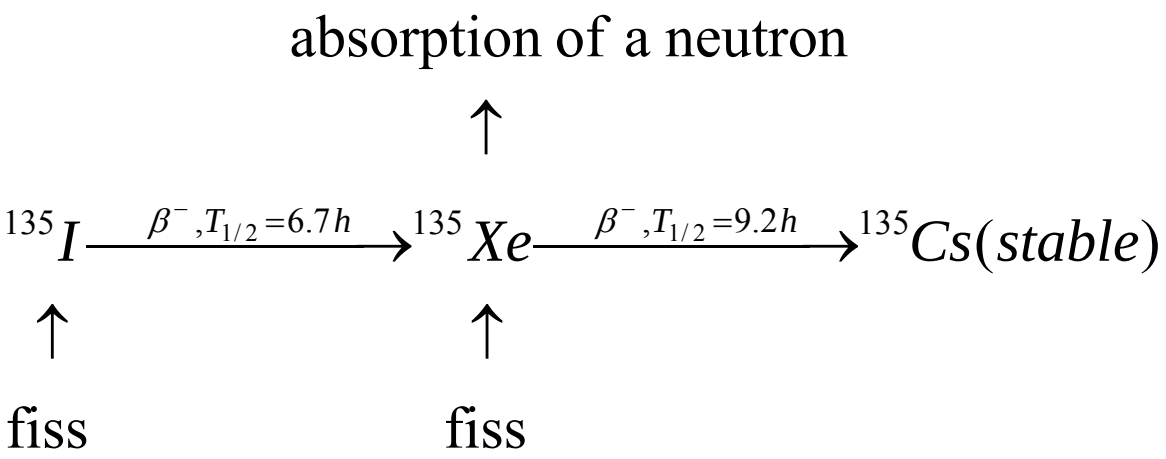
It is easily seen that for a uniform distribution of poison, we recover:

$$\Delta\rho = \frac{1}{k_0} - \frac{1}{k} = -\frac{\Sigma_{aX}\int_V \Phi^2(\vec{r})dV}{\nu\Sigma_f\int_V \Phi^2(\vec{r})dV} = \frac{\Sigma_{aX}}{\nu\Sigma_f}$$

Xe Production and Destruction



Simplified Xe production/destruction



Finding the Xe Concentration

To find the number density of Xe nuclei, we first write the balance equation for Iodine nuclei:

$$\frac{dI}{dt} = \gamma_I \Sigma_f \phi - \lambda_I I$$

Where γ is called the fission product yield and equals the average number of I nuclides created per fission.

Next, we write the balance equation for Xe nuclei

$$\frac{dX}{dt} = \lambda_I I + \gamma_X \Sigma_f \phi - \lambda_X X - \sigma_{aX} \phi X$$

I Concentration for Equilibrium (Steady-State) Conditions

$$I_{\infty} = \frac{\gamma_I \Sigma_f \phi}{\lambda_I}$$

Xe Concentration for Equilibrium (Steady-State) Conditions

$$X_\infty = \frac{\lambda_I I_\infty + \gamma_{\infty f}\phi}{\lambda_X + \sigma_{aX}\phi}$$

$$X_\infty = \frac{(\gamma_1 + \gamma_X)\Sigma_f \phi}{\lambda_X + \sigma_{aX}\phi}$$

Note that both I and Xe concentrations depend on the flux level.

Xe Absorption Macroscopic Cross Section

$$\Sigma_{aX} = X_{\infty}\sigma_{aX} = \frac{(\gamma_1 + \gamma_X)\Sigma_f \phi\, \sigma_{aX}}{\lambda_X + \sigma_{aX}\phi}$$

By making the notation:

$$\phi_X = \frac{\lambda_X}{\sigma_{aX}} = 0.770 \times 10^{13}\, cm^{-2}\,\mathrm{sec}^{-1}$$

We can rewrite the Xe macroscopic cross section as:

$$\Sigma_{aX} = \frac{(\gamma_1 + \gamma_X)\Sigma_f \phi}{\phi_X + \phi}$$

If Xe is Assumed Uniformly Distributed:

$$\rho = -\frac{\Sigma_{aX}}{\nu\Sigma_f} = \frac{1}{\nu\Sigma_f}\frac{(\gamma_1+\gamma_X)\Sigma_f\phi}{\phi_X+\phi} = -\frac{(\gamma_1+\gamma_X)}{\nu}\frac{\phi}{\phi_X+\phi}$$

$$\rho = -\frac{(\gamma_1+\gamma_X)}{\nu}\frac{\phi}{\phi_X+\phi}$$

For $\Phi >> \Phi_X$ we have:

$$\rho \cong -\frac{\gamma_I+\gamma_X}{\nu}$$

For $\Phi << \Phi_X$ we have:

$$\rho = -\frac{(\gamma_1+\gamma_X)}{\nu}\frac{\phi}{\phi_X}$$

Xe After Shutdown-Reactor Dead Time

Shutdown means $\Phi = 0$

Iodine

$$I(t) = I_0 e^{-\lambda_I t}$$

Xe

$$X(t) = X_0 e^{-\lambda_X t} + \frac{\lambda_I I_0}{\lambda_1 - \lambda_X}(e^{-\lambda_X t} - e^{-\lambda_I t})$$

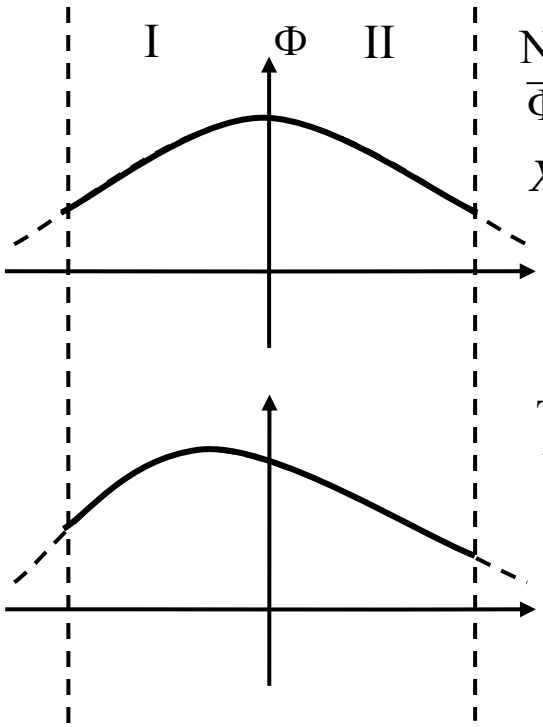
If Xe is assumed to be uniformly distributed:

$$\rho = -\frac{1}{\nu}\left[\frac{(\gamma_I + \gamma_X)\phi}{\phi_X + \phi} e^{-\lambda_X t} + \frac{\gamma_I \phi}{\phi_I - \phi_X}(e^{-\lambda_x t} - e^{-\lambda_I t})\right]$$

Where:

$$\phi_I = \frac{\lambda_{1I}}{\sigma_{aX}} = 1.055 \times 10^{13}\, cm^{-2}\,\mathrm{sec}^{-1}$$

Xe Oscillations



Normal flux shape

$\overline{\Phi}_I = \overline{\Phi}_{II}$

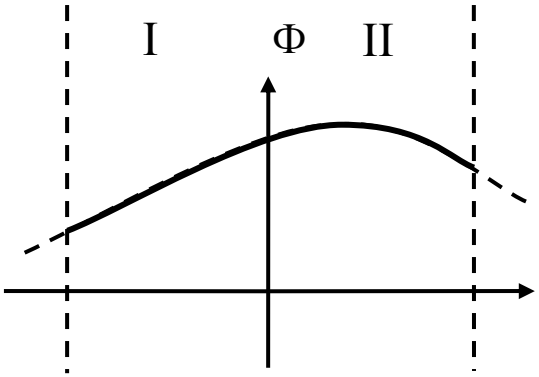
$X_I = X_{II}$

Tilted flux shape

$\overline{\Phi}_I > \overline{\Phi}_{II}$

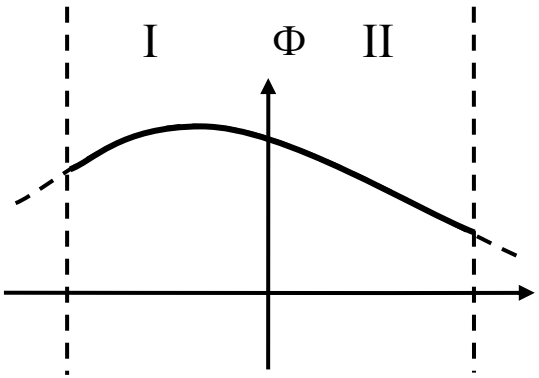
$X_I > X_{II}$

Consequence of $X_I > X_{II}$



$\overline{\Phi}_I < \overline{\Phi}_{II}$ => Over time : $X_I < X_{II}$

Consequence:



In CANDU Reactors the liquid zone controllers are used to dampen Xe oscillations.