UN 0802
Reactor Physics

Tentative Dates: November 12 - December 17, 2005

Instructors:

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Study materials:

Course notes

Textbook: Introduction to Nuclear Engineering (third edition)
J.R. Lamarsh & A.J.Baratta
Prentice-Hall, 2001
ISBN: 0-201-82498-1

Additional reading: Nuclear Reactor Analysis
James J. Duderstadt & Louis J. Hamilton
ISBN: 0471223638

Prerequisite Topics in Mathematics

- Vector algebra
- Differential operators (gradient, divergence, curl, Laplacian)
- First and second order differential equations with constant coefficients
- Partial differential equations with constant coefficients
- Systems of linear algebraic equations

Assignments: Assignments will consist on homework problems and projects.
Assignments make up 30% of the final grade.

Grading:

Assignments: 30%
Midterm exam: 35%
Final exam: 35%

Grading scale: 0-100%.

Schedule:

Nov 12 - lecture
Nov 13 - lecture
Nov.19 - lecture
Nov 20 - lecture
Nov 26 & 27 - weekend off
Dec 3 - Midterm + lecture
Dec 4 - lecture
Dec 10 - lecture
Dec 11 - lecture
Dec 17 - Final

Learning objective

To provide students with primary knowledge on the following topics:

- nuclear structure
- radioactivity
- interaction of radiation with matter
- nuclear reactions
- nuclear fission as used for power production
- basic quantities and methods used to describe the behaviour of neutrons in a nuclear reactor.
- static and time-dependent diffusion equation
- elements of CANDU-reactor design
- basic codes used for nuclear reactor neutronic design

Learning outcomes

After taking the course, students should be able to:

- Understand the structure of the atom and the main components of the nucleus.
- Find isotopes on the Table of Nuclei, and identify the nature of radioactive decay (if any) of a given isotope.
- Describe the main differences between alpha, beta and gamma decay
- Understand the concept of binding energy and how nuclei of different binding energies may undergo fission or fusion.
- Compute the energy released in fission or fusion reactions.
- Describe the way different types of radiation interact with matter.
- Understand the concept of chain reaction and each component of the four(six)-factor formula.
- Formulate the neutron diffusion equation.
- Describe and use methods of solution for the static diffusion equation.
- Describe and use methods of solution for the time-dependent diffusion equation
- Describe fission-product poisoning (Xe, Sm)
- Describe reactivity effects of temperature and void
- Describe the components of a CANDU reactor
- Utilize simple codes employed in reactor neutronic design.

Academic misconduct

Academic misconduct includes, but is not limited to:

> Cheating on examinations, assignments, reports, or other work used to evaluate student performance. Cheating includes copying from another student's work or allowing one's own work to be copied, submitting another person's work as one's own, fabrication of data, consultation with an unauthorized person during an examination, or use of unauthorized aids.

Tentative Course Outline

1. Introduction

2. Atomic And Nuclear Physics
   - 2.1. Photoelectric Effect
   - 2.2. Compton Effect
   - 2.3. Atomic Spectra
   - 2.4. Bohr’s Atomic Model
   - 2.5. De Broglie Waves
   - 2.6. Elements of Relativity
   - 2.7. Relativistic Mass Formula
   - 2.8. Relativistic Energy
   - 2.9. Relativistic Momentum
   - 2.10. Nuclear Constituents
   - 2.11. Notations of Isotopes
   - 2.12. Descriptions of Nuclear Particles (Mass, Charge, Spin)
   - 2.13. Binding Energy
   - 2.14. The Liquid Drop Nuclear Model
   - 2.15. The Decay Process
   - 2.16. Natural Radioactivity
   - 2.17. Induced Radioactivity
   - 2.18. Radioactive Families

3. Interaction of Radiation With Matter
   - 3.1. Interactions of Heavy Charged Particles
   - 3.2. Interactions of Light Charged Particles
   - 3.3. Interactions of Gamma Radiation
   - 3.4. Interactions of Neutrons
   - 3.5. Types of Nuclear Reactions
   - 3.6. Kinematics of Nuclear Reactions
   - 3.7. Reaction Cross Sections
   - 3.8. Attenuation and Shielding

4. Nuclear Reactors and Nuclear Power
   - 4.1. The Fission Chain Reaction
   - 4.2. Reactor Fuel, Moderator and Coolant
   - 4.3. Main Nuclear Plant Components

5. Basic Concepts of Neutron Physics
   - 5.1. Fission
   - 5.2. Flux, Current, Source
   - 5.3. Reaction Rate Densities
   - 5.4. Fick's Law and the Diffusion Equation
   - 5.5. Solutions to the Diffusion Equation
   - 5.6. The Group Diffusion Model
   - 5.7. Two-Energy-Group Neutron Moderation

6. Nuclear Reactor Theory
    6.1. Fundamental Neutronic Problems (Fixed-Source and Eigenvalue)
    6.2. Criticality
    6.3. Homogeneous Reactors - Flux Separability In Energy And Space
    6.4. One-Group Reactor Equation
    6.5. One-Group Flux Solution for Different-Shape Homogeneous Reactors (Slab, Parallelepiped, Cylinder, Sphere)
    6.6. Multiregion Problems - Reflector

7. Nuclear Reactor Kinetics/Dynamics
    7.1. Classification of Time-Dependent Problems.
    7.2. Reactor Kinetics
    7.3. Reactivity Devices
    7.4. Temperature Effects On Reactivity

8. Discussion of Basic CANDU Design

9. Discussion of CANDU Computational Schemes

10. Discussion of CANDU LOCA Calculations

11. Hands-on Calculations with POWDERPUFS-V Lattice Computer Code, In-Class And Home Exercises

12. Xe-I Kinetics, Calculations and Exercises

Quantum Properties of Matter (and Light)

All figures reproduced from: R. Serway “Physics for Scientists and Engineers with Modern Physics”, third edition, volume II.

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Is light a wave or is it made up of particles?

- Newton
  - particles (cites reflection and propagation in straight line)

- Huyghens
  - wave (cites interference, refraction, diffraction)

Beginning of 20th Century

- The wave theory of light was prevalent as it seemed to explain all phenomena involving light.
  - reflection
  - interference (diffraction as well)
- Moreover, Maxwell had shown light to be an electromagnetic wave (as were X rays).
- Huyghens seemed to have won the dispute, but....

A few phenomena could not be explained by the wave theory

- Black body radiation
- Photoelectric effect
- Compton effect
- Atomic spectra

Black Body Radiation

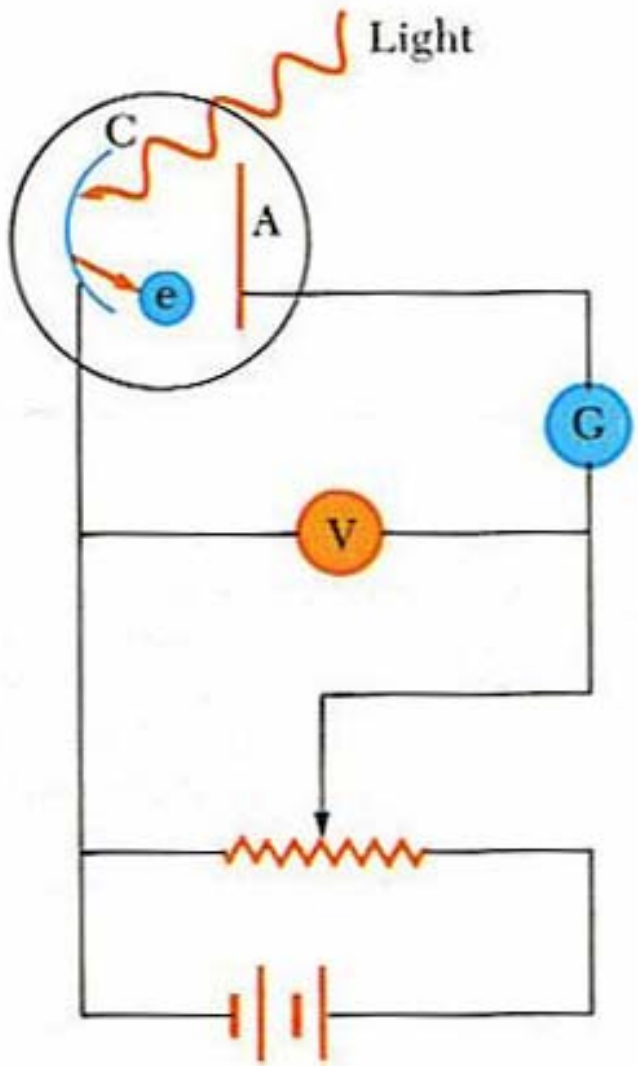
- A “black body” is a body that only absorbs and emits light, but does not reflect it.
- A black body emits light with a continuous spectrum.
- Attempts to explain theoretically the shape of the spectrum of the black body radiation based on classical theory had failed, especially for small wavelengths.
- Max Planck was able to explain the entire spectrum, by assuming that energy could only be absorbed or emitted in discrete units called **quanta**.

Max Planck – Light Quanta

- Energy of one quantum $E = hf$
- $h = 6.626 \times 10^{-34}\, J \cdot s$ Planck's Constant
- Same dimensions as angular momentum
- Very small number

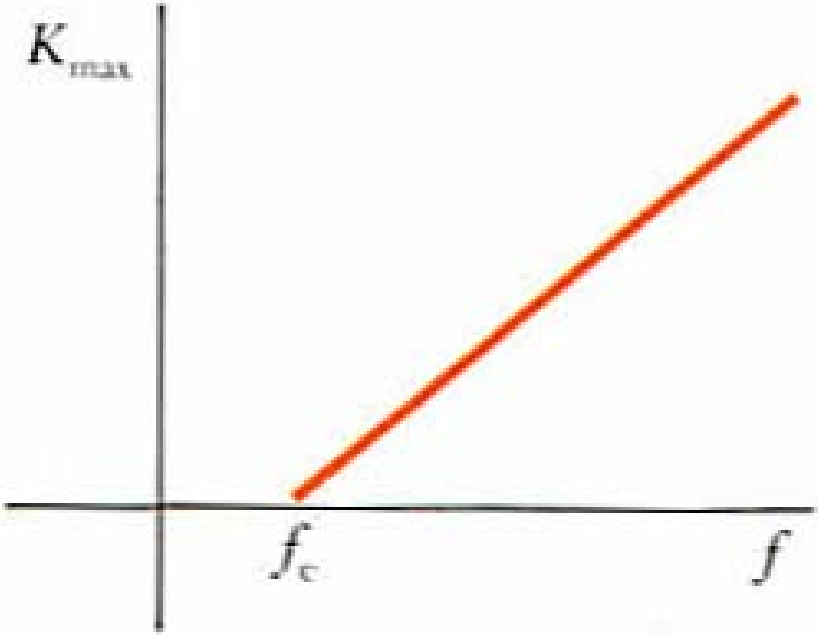
Photoelectric Effect

- When light is incident on the (metallic) cathode, electrons are emitted. (called photoelectrons)



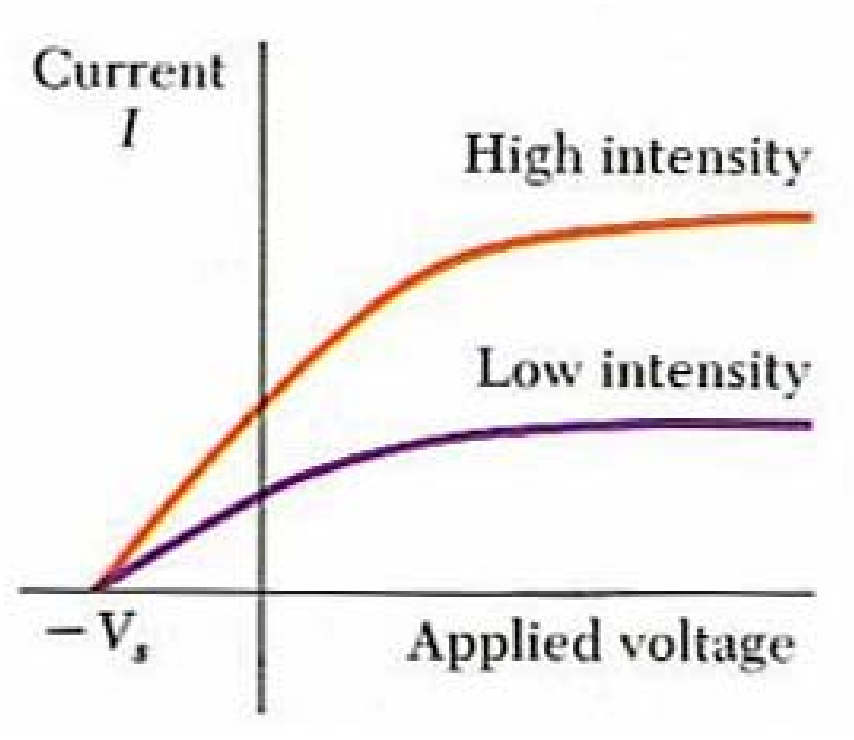
Laws of the Photoelectric Effect

1. No electrons emitted if the frequency of the incident light is lower than a certain value, called the “cutoff frequency”.
2. The maximum KE of electrons increases linearly with light frequency.



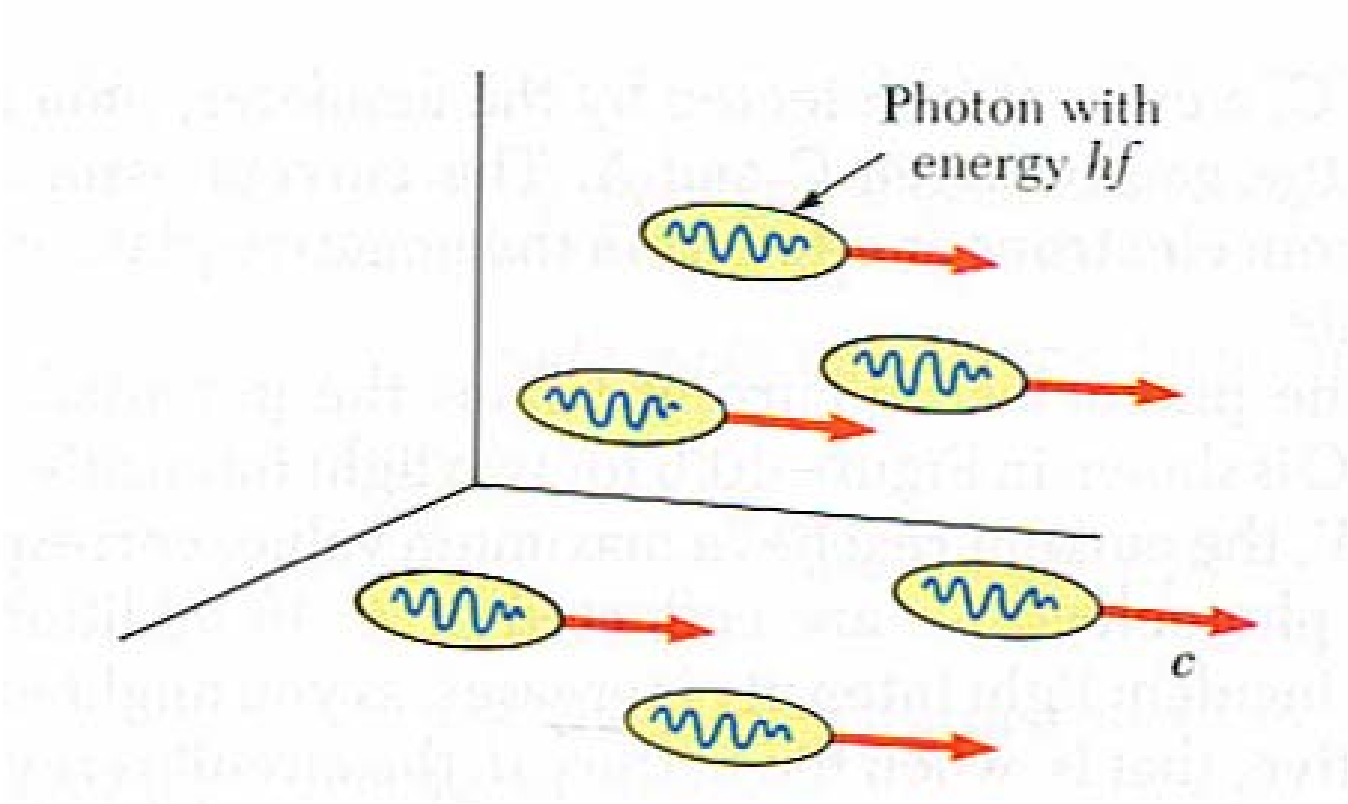
Laws of the Photoelectric Effect cont.

3. Above the cutoff frequency, the maximum number of photoelectrons is proportional to the light intensity.
4. Electrons are emitted almost instantaneously (10^{-9} s after beginning of illumination) although the classical electromagnetic theory would predict some delay.



Einstein's Theory of the PE Effect

1. A light beam consists of quanta (photons), each of energy $E = hf$ (Planck's hypothesis), travelling at the speed of light, c



Einstein's Theory of the PE

2. Each photon gives **all** its energy instantaneously to an electron in the metallic cathode.
   - If, and only if, the photon's energy is higher than the minimum binding energy in the metal (called the **work function,** Φ), an electron is emitted.
   - Consequently, the maximum kinetic energy of an electron is:

$$KE_{\max} = hf - \Phi$$

Values of Work Functions

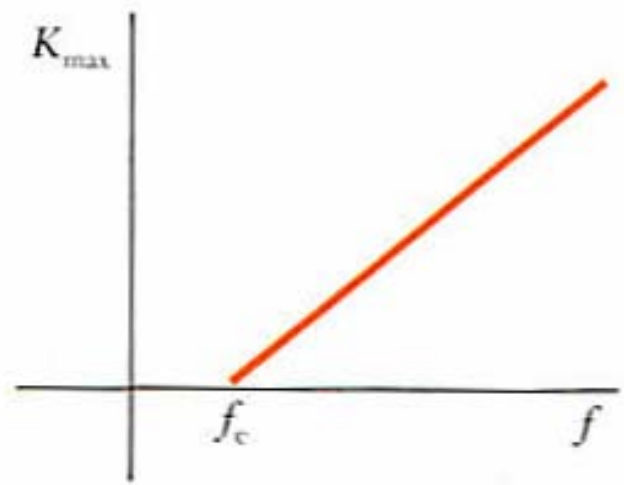
Metal	ϕ (eV)
Na	2.28
Al	4.08
Cu	4.70
Zn	4.31
Ag	4.73
Pt	6.35
Pb	4.14
Fe	4.50

How Einstein's theory explains the four laws of the PE effect

1. Since $KE_{\max} = hf - \Phi$ has to be positive for the electron to be emitted, it follows that nothing happens below a cutoff frequency.

$$KE_{\max} > 0 \Leftrightarrow f > \frac{\Phi}{h} = f_{cutoff}$$

2. $KE_{\max} = hf - \Phi$ describes exactly the linear relationship between the maximum kinetic energy and light frequency that was found experimentally.

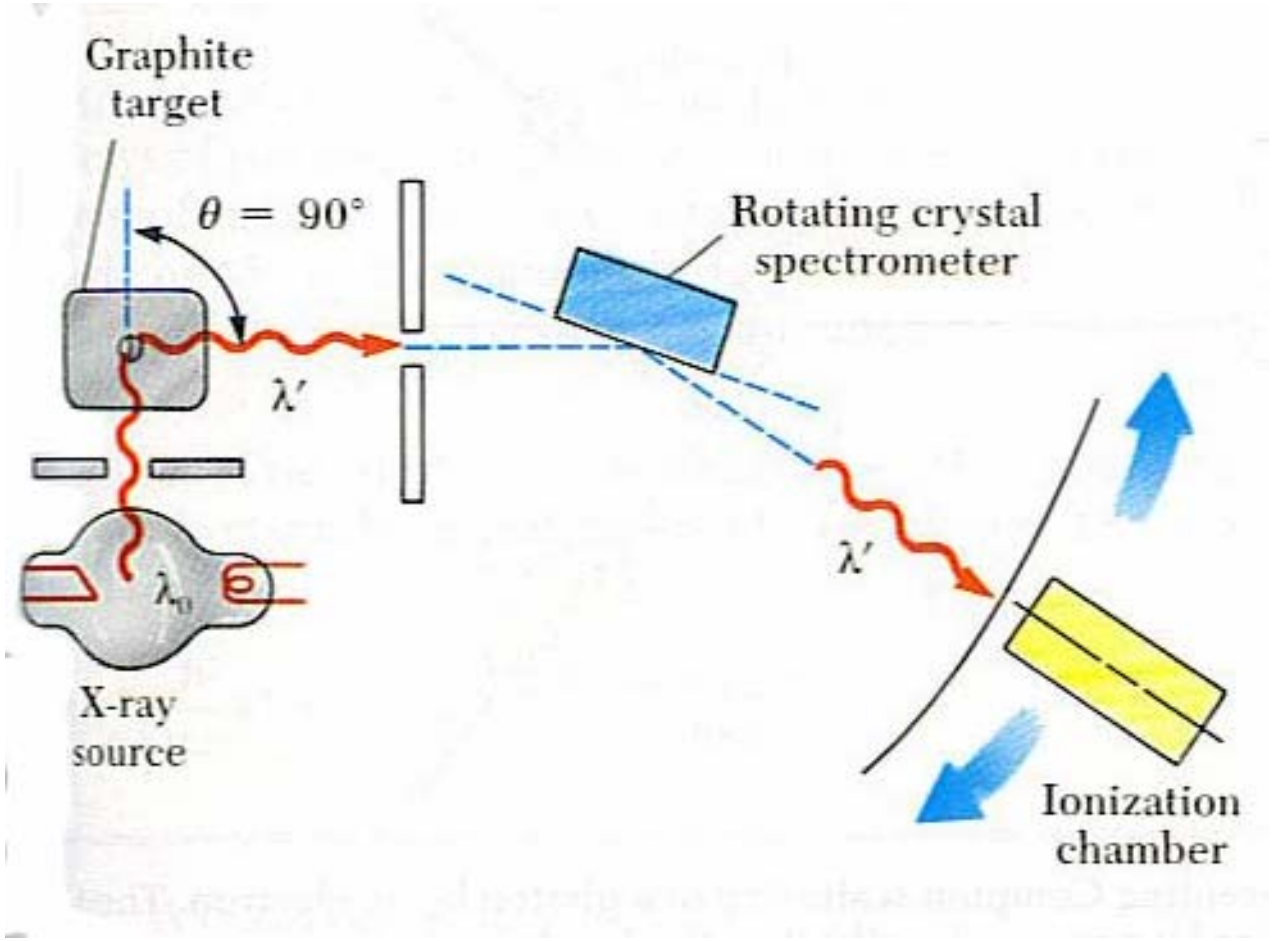


How Einstein's theory explains the four laws of the PE effect

3. Since each photon has the same energy, equal to hf, the intensity of the light is proportional to the number of incident photons per unit time. Since each photon transfers its energy to one electron, it follows that the number of emitted photoelectrons is proportional to the intensity of the incident light.
4. Since each photons interacts with a single electron, the energy transfer happens instantaneously, rather that over a period of time, as would be the case if energy was distributed uniformly in the wave.

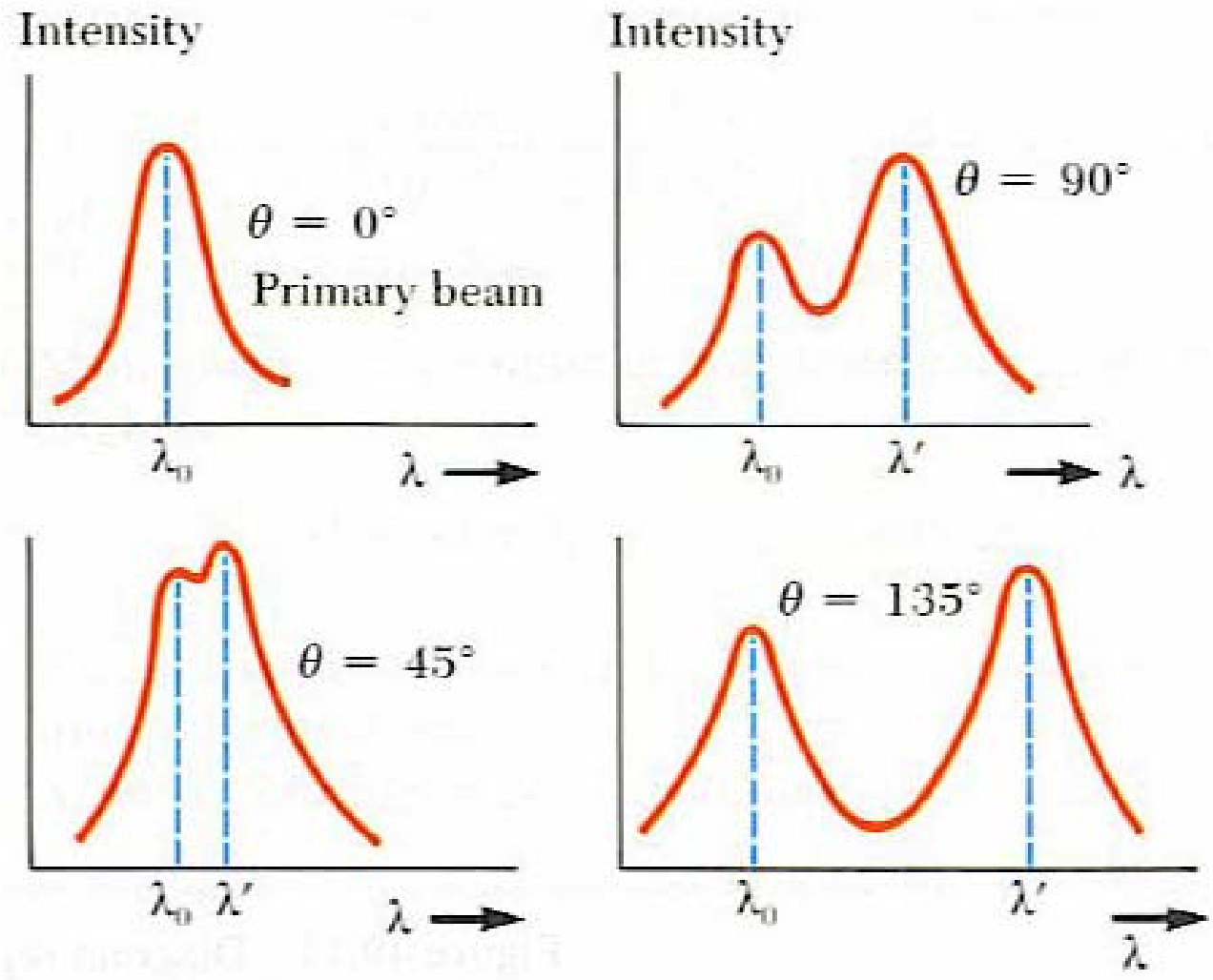
The Compton Effect

- Named after Arthur H. Compton (1892-1962)
- Interaction of em radiation with “free” electrons.



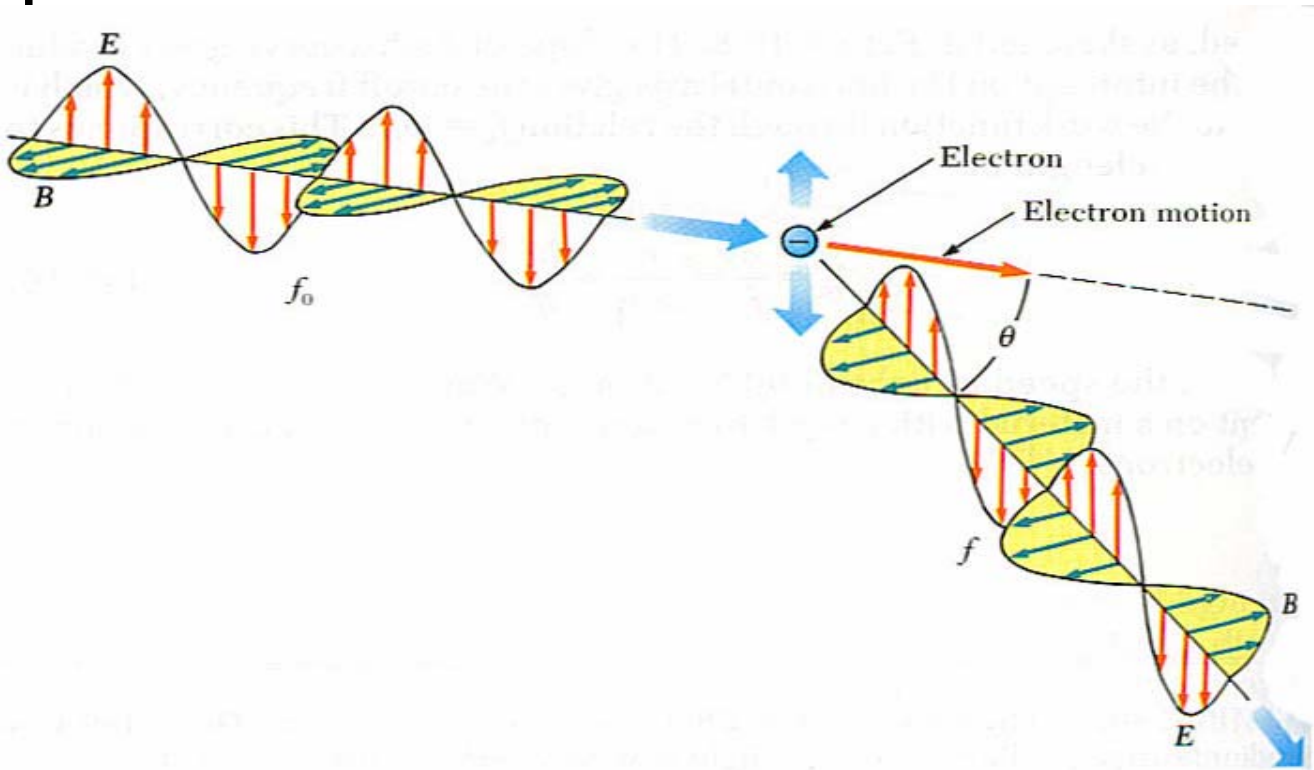
Compton's Measurements

- Frequency of scattered radiation depends only on scattering angle.



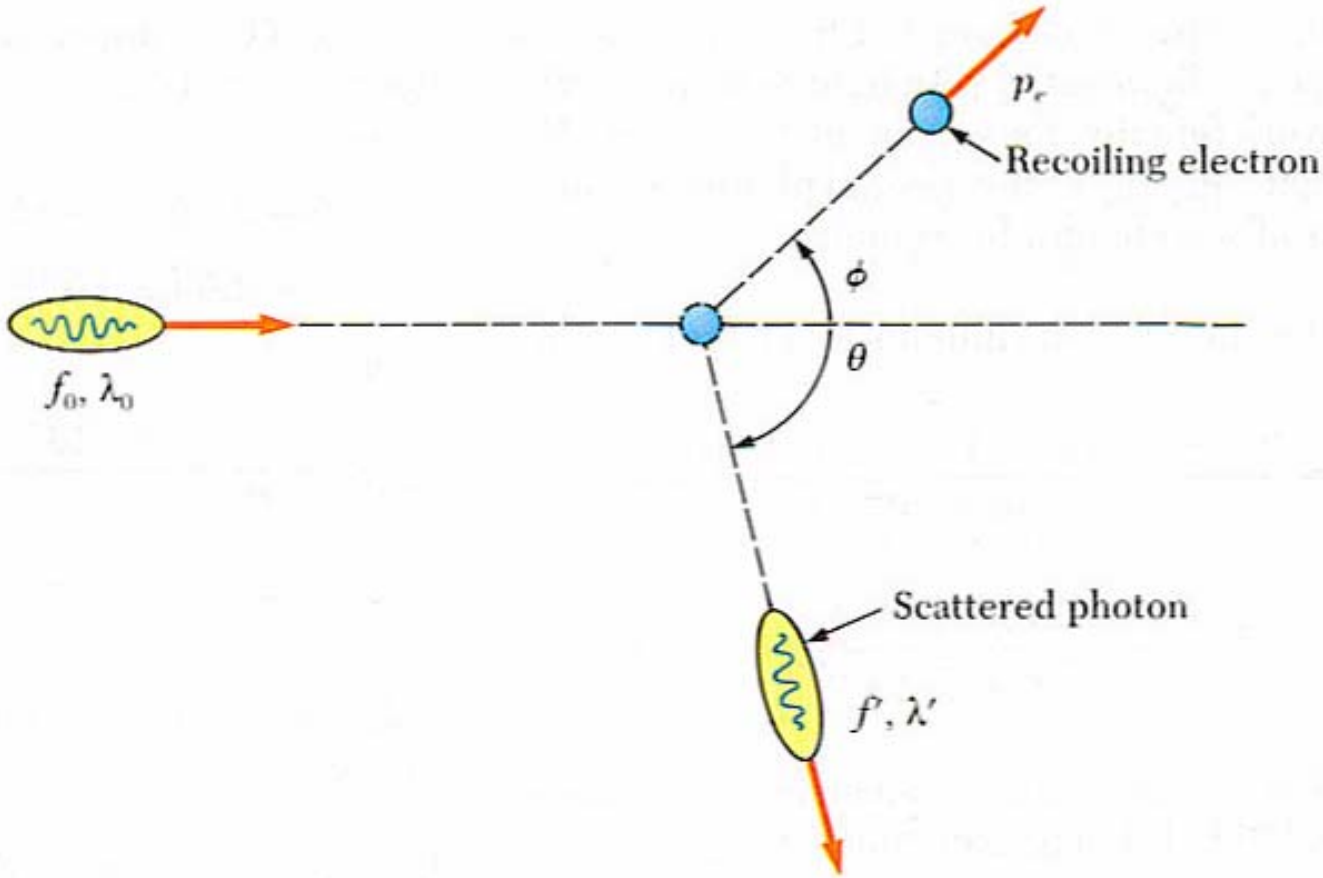
Classical Theory - Inadequate

- Frequency of scattered radiation depends on beam intensity and time of exposure.



Compton's Theory

- Photons undergo elastic collisions with "free" electrons.



Compton's Theory cont.

- To explain the shift in wavelength, the laws of conservation of **relativistic** energy and momentum need to be applied.

$$E_{0\,ph} + E_{0e} = E'_{ph} + E'_e$$

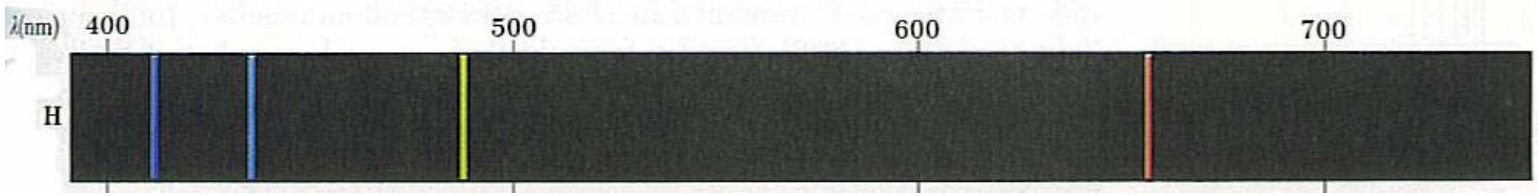
$$\vec{p}_{0\,ph} + \vec{p}_{0e} = \vec{p}'_{ph} + \vec{p}'_e$$

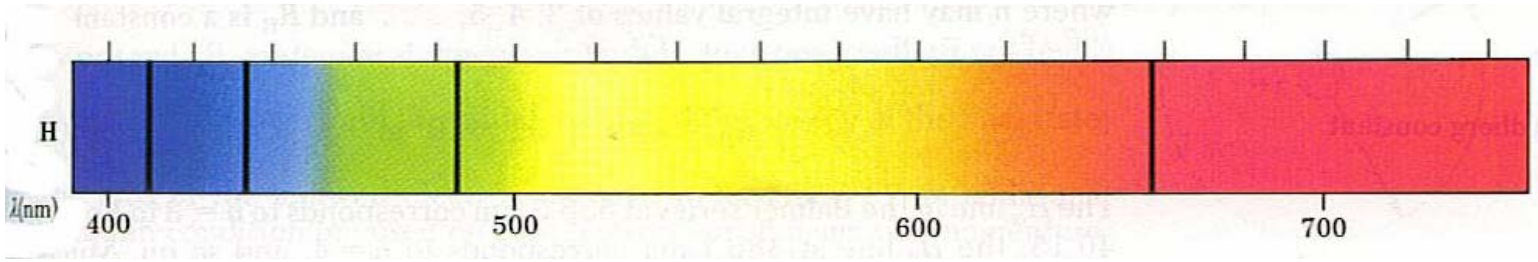
- According to Compton's theory:

$$\lambda' - \lambda_0 = \frac{h}{m_{0e}c}\left(1 - \cos\theta\right)$$

Atomic Spectra

- Emission





- Absorption
- Atomic spectra are discrete (appear as lines)

Hydrogen Spectrum

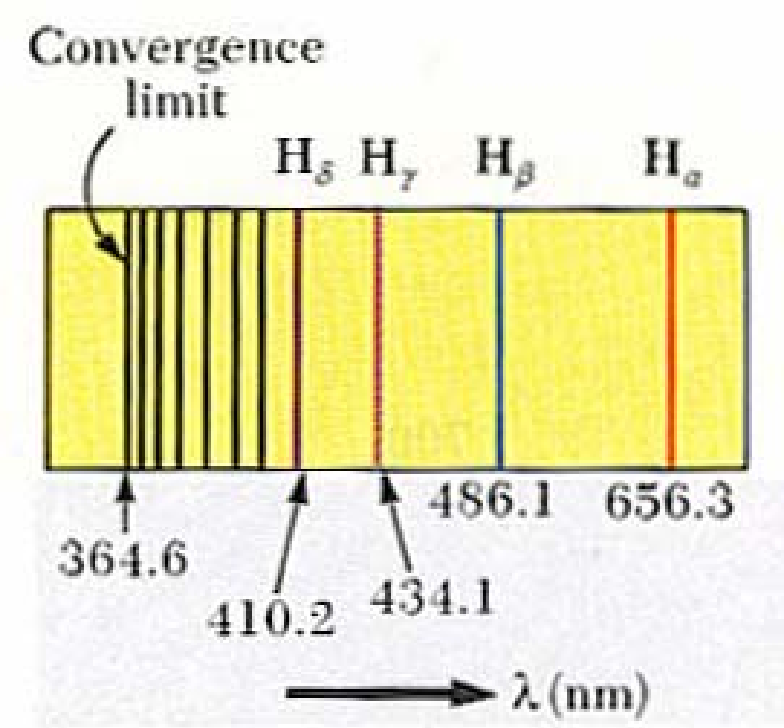
- The wavelengths of emitted/absorbed electromagnetic radiation were found (empirically) to satisfy an interesting relationship:

$$\frac{1}{\lambda} = R_H \left(\frac{1}{m^2} - \frac{1}{n^2} \right) \quad n > m$$

Hydrogen Spectrum

- All lines obtained for a given m, are said to form a **series**.
- Balmer series, n=2 – first one discovered (Johann Balmer)

$$\frac{1}{\lambda_n} = R_H\left(\frac{1}{2^2} - \frac{1}{n^2}\right) \quad n > 2$$



Rutherford's Model of the Hydrogen Atom

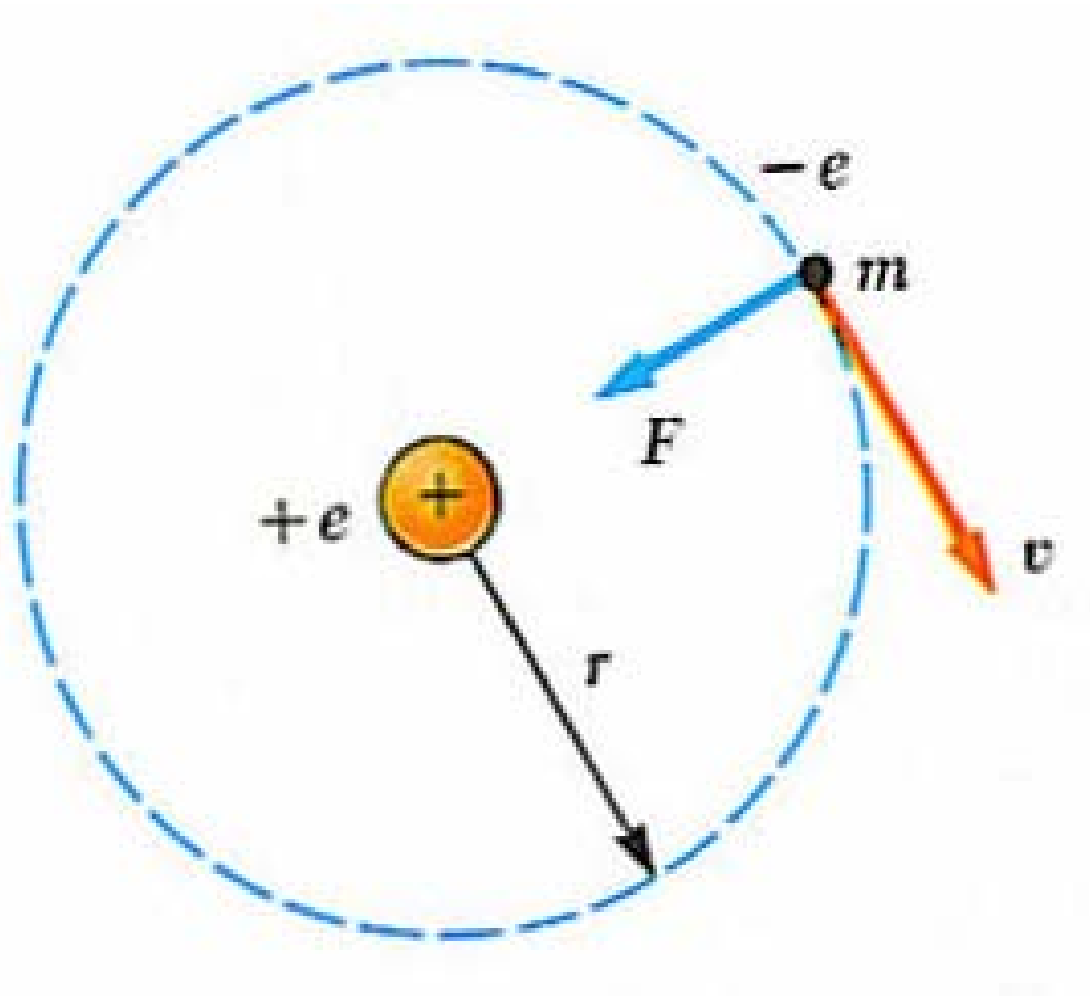
- Electrostatic force

$$F_e = \frac{e^2}{4\pi\varepsilon_0 r^2} = k\frac{e^2}{r^2}$$

- Centripetal force

$$F_c = m_e \frac{v^2}{r}$$

- The two are one and the same



Bohr's Model of the Hydrogen Atom

- The hydrogen atom has only one electron.
- The nucleus consists of only one proton
- Bohr started from Rutherford's model, which assumed the negatively-charged electron to gravitate around the positively-charged proton on a circular orbit.
  - The electrostatic attraction force acts as the centripetal force.
- Rutherford's model had limitations
  - According to the electromagnetic theory, orbiting electrons would radiate light continuously at the frequency they rotated, and in doing so they would lose energy, and eventually fall onto the nucleus.
  - This phenomenon was never observed.

Bohr's Model of the Hydrogen Atom (cont.)

- **Bohr's additional hypotheses (nonrelativistic)**
  - Certain orbits (radii) are stable. No radiative loss of energy occurs for these orbits.
  - The allowed (stable) orbits are those for which the orbital angular momentum has values given by:

$$L = m_e v r = n\hbar; \quad \hbar \equiv \frac{h}{2\pi}$$

  - Electrons can jump from one orbit to another. Only when such a jump occurs energy is either emitted or absorbed, in the form of a photon.

Bohr's Model of the Hydrogen Atom (cont.)

- Need to find the radius and energy of stable orbits (also called Bohr orbits).
- Start from equating the centripetal force with the electrostatic force

$$m_e \frac{v^2}{r} = k \frac{e^2}{r^2}$$

Bohr's Model of the Hydrogen Atom (cont.)

- Relationship between radius and speed

$$m_e \frac{v^2}{r} = k \frac{e^2}{r^2} \Leftrightarrow m_e v^2 = k \frac{e^2}{r}$$

$$v^2 = k \frac{e^2}{m_e r}$$

$$v = \sqrt{\frac{k}{m_e}} \frac{e}{\sqrt{r}}$$

Bohr's Model of the Hydrogen Atom (cont.)

- Expression of angular momentum

$$L = m_e v r = m_e \left(\sqrt{\frac{k}{m_e}} \frac{e}{\sqrt{r}} \right) r = \sqrt{m_e k} e \sqrt{r}$$

- Use the postulated values of the angular momentum to find the radii of the stable orbits.

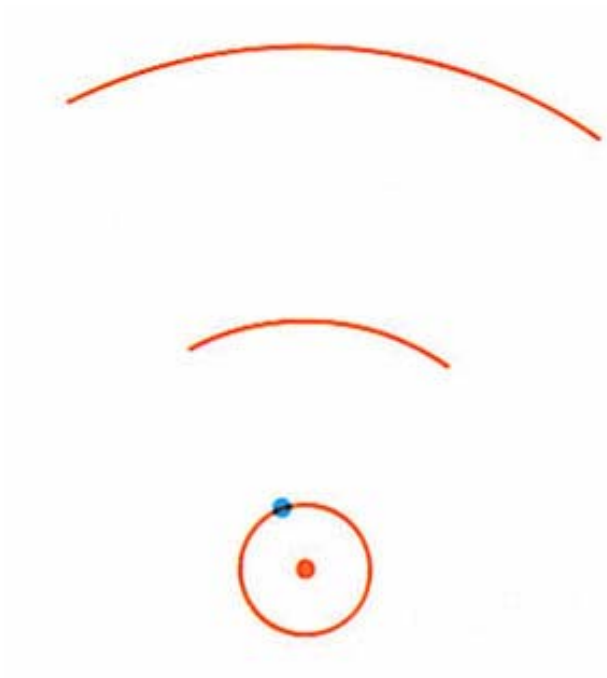
$$L_n = n\hbar \Leftrightarrow \sqrt{m_e k} e \sqrt{r_n} = n\hbar$$ (equation for r)

Bohr's Model of the Hydrogen Atom (cont.)

- Radii of stable orbits

$$\sqrt{m_e k}\, e \sqrt{r_n} = n\hbar \Leftrightarrow \sqrt{r_n} = \frac{n\hbar}{\sqrt{m_e k}\, e}$$

$$r_n = n^2 \frac{\hbar^2}{m_e k e^2}$$



Bohr's Model of the Hydrogen Atom (cont.)

- Energy of electron on a stable orbit

$$E_n = PE_n + KE_n = -\frac{ke^2}{r} + \frac{m_e v^2}{2}$$

- Substitute previously found expression for v^2

$$v^2 = k\frac{e^2}{m_e r}$$

- Find simpler expression for energy (not yet final)

$$E_n = -\frac{ke^2}{r_n} + \frac{m_e}{2}k\frac{e^2}{m_e r_n} = -\frac{ke^2}{2r_n}$$

Bohr's Model of the Hydrogen Atom (cont.)

- Final formula for electron's energy on stable orbit

$$E_n = -\frac{ke^2}{2r_n}$$

$$r_n = \frac{n^2\hbar^2}{m_e ke^2}$$

$$E_n = -\frac{ke^2}{2\dfrac{n^2\hbar^2}{m_e ke^2}} = -\frac{m_e k^2 e^4}{2n^2\hbar^2}$$

$$E_n = -\frac{1}{n^2}\frac{m_e k^2 e^4}{2\hbar^2}$$

Bohr's Model of the Hydrogen Atom (cont.)

- Transitions (jumps)
  - When an electron jumps from one orbit to another, it has to either absorb or emit energy, in the form of a photon.
  - The energy of the photon equals the difference between the energies of the two orbits. For an electron jumping from orbit n to orbit m, we have:

$$hf = E_n - E_m = \frac{1}{m^2}\frac{m_e k^2 e^4}{2\hbar^2} - \frac{1}{n^2}\frac{m_e k^2 e^4}{2\hbar^2} = \frac{m_e k^2 e^4}{2\hbar^2}\left(\frac{1}{m^2} - \frac{1}{n^2}\right)$$

  - If n>m, the potential energy of the initial state (n) is larger than that of the final state (m) and energy is emitted in the form of a photon. If m>n, the situation is reversed and a photon needs to be absorbed.

Bohr's Model of the Hydrogen Atom (cont.)

- Transitions
  - expressing the reciprocal of the wavelength of the photon: $\frac{1}{\lambda} = \frac{f}{c}$

$$hf = \frac{m_e k^2 e^4}{2\hbar^2}\left(\frac{1}{m^2} - \frac{1}{n^2}\right) \Leftrightarrow \frac{f}{c} = \frac{1}{\lambda} = \frac{m_e k^2 e^4}{2\hbar^2 hc}\left(\frac{1}{m^2} - \frac{1}{n^2}\right)$$

$$\times \frac{1}{hc} \qquad \times \frac{1}{hc}$$

$$\frac{1}{\lambda} = \frac{m_e k^2 e^4}{2\hbar^2 hc}\left(\frac{1}{m^2} - \frac{1}{n^2}\right)$$

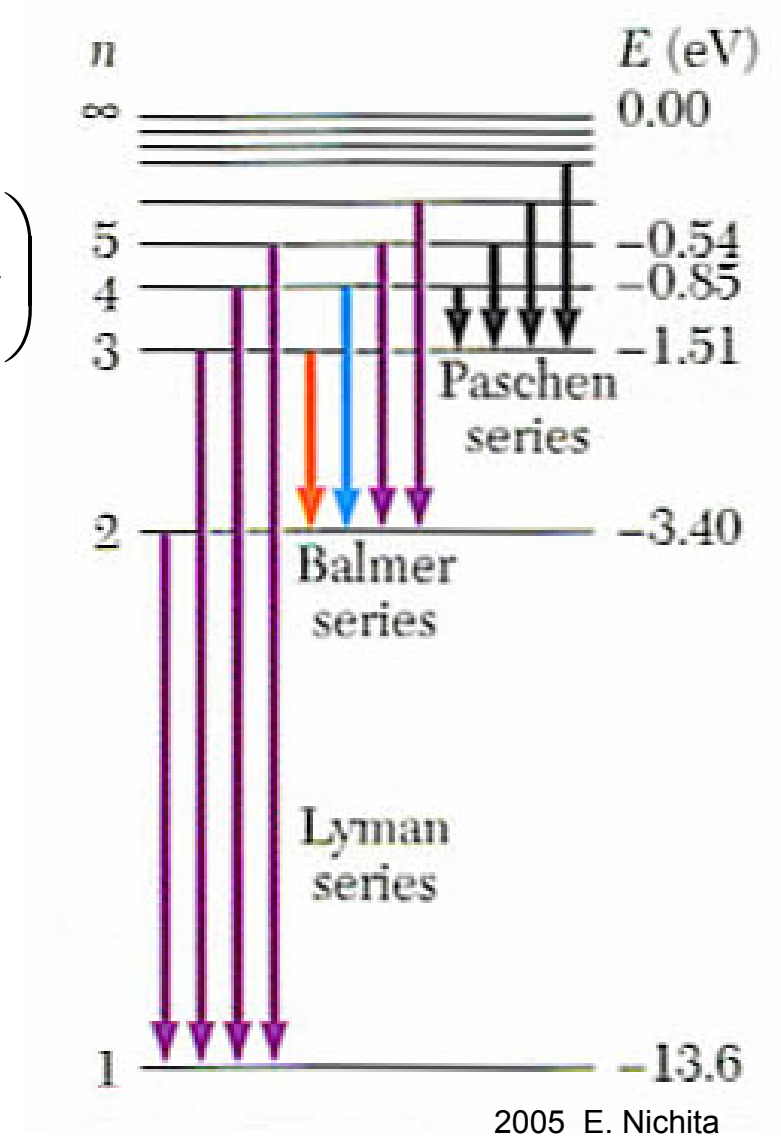
Previously discovered by Balmer for m=2

Bohr's Model of the Hydrogen Atom (cont.)

- Transitions

$$\frac{1}{\lambda} = \frac{m_e k^2 e^4}{2\hbar^2 hc}\left(\frac{1}{m^2} - \frac{1}{n^2}\right) = R_H\left(\frac{1}{m^2} - \frac{1}{n^2}\right)$$

Rydberg's Constant



Light: Wave or Particle?

- Wave
  - Reflection
  - Refraction
  - Interference
  - Diffraction
- Particle
  - Black body radiation
  - Photoelectric effect
  - Compton effect
  - Discrete (line) atomic spectra
- Conclusion
  - Light cannot be described entirely as either wave or particle.
  - Wave behavior is displayed in some situations, while particle behavior is displayed in others.
  - Light displays wave-particle “duality”.

Do other particles display the wave-particle duality?

- Louis de Broglie (1892-1987)
  - Stated that **all** particles display the wave-particle duality.
  - Each particle (not just photons) has a wave associated with it. The associated wave (also called De Broglie wave) satisfies one of the relations found to be true for photons:

$$p = \frac{h}{\lambda} \Leftrightarrow \lambda = \frac{h}{p}$$

De Broglie waves and Bohr's model

- The Bohr orbits are an integer number of electron wavelengths.
- We'll prove this in the following slides

$$v = \sqrt{\frac{k}{m_e}} \frac{e}{\sqrt{r}}$$

$$r_n = \frac{n^2 \hbar^2}{m_e k e^2}$$

$$v = \sqrt{\frac{k}{m_e}} \frac{e}{\sqrt{\frac{n^2 \hbar^2}{m_e k e^2}}} = k \frac{1}{n} \frac{e^2}{\hbar}$$

De Broglie waves and Bohr's model (cont.)

$$v = k\frac{1}{n}\frac{e^2}{\hbar}$$

$$p = m_e v = m_e k\frac{1}{n}\frac{e^2}{\hbar}$$

$$\lambda = \frac{h}{p} = \frac{h}{m_e k\frac{1}{n}\frac{e^2}{\hbar}} = \frac{nh\hbar}{m_e k e^2}$$

$$\lambda = \frac{nh\hbar}{m_e k e^2}$$

De Broglie waves and Bohr's model (cont.)

$$r_n = \frac{n^2\hbar^2}{m_e k e^2}$$

$$C_n = 2\pi r_n = 2\pi \frac{n^2\hbar^2}{m_e k e^2} = \frac{n^2 h\hbar}{m_e k e^2}$$

Circumference of orbit n

$$\lambda = \frac{nh\hbar}{m_e k e^2}$$

$$\frac{C_n}{\lambda} = \frac{\dfrac{n^2 h\hbar}{m_e k e^2}}{\dfrac{nh\hbar}{m_e k e^2}} = n$$

$$C_n = n\lambda$$

De Broglie waves and Bohr's model (cont.)

- The Bohr orbits are an integer number of de Broglie wavelengths
- Immediate consequence:

$$C_n = n\lambda$$

$$\lambda = \frac{h}{p} = \frac{h}{mv}$$

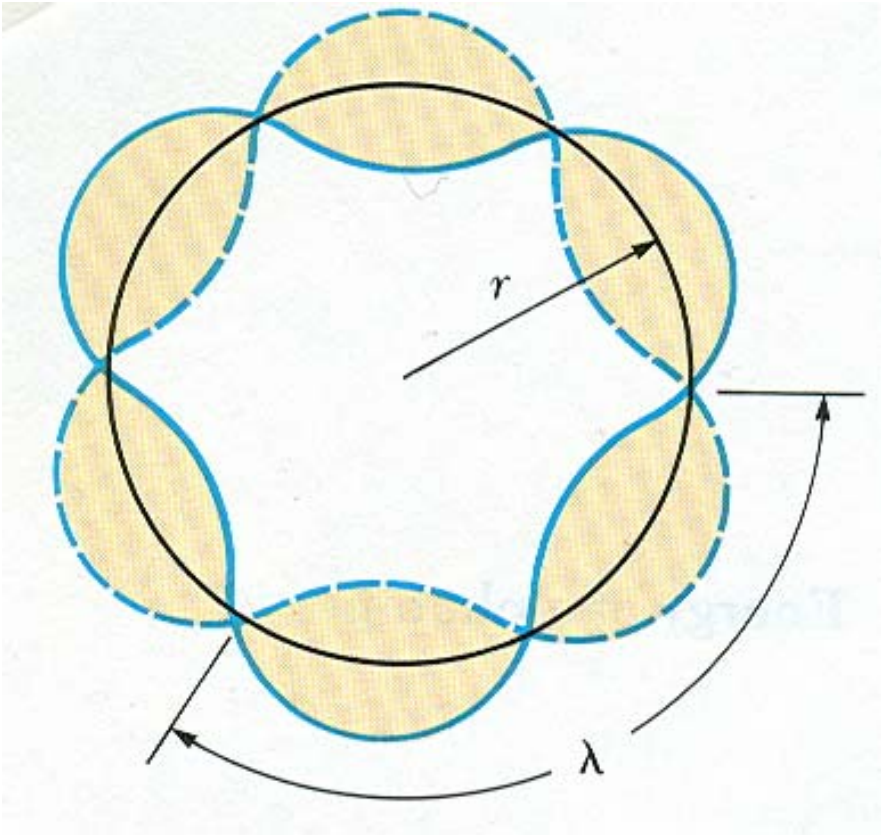
$$2\pi r_n = n\frac{h}{mv}$$

$$m_e v r_n = n\frac{h}{2\pi} = n\hbar$$

We have recovered Bohr's quantization of angular momentum

De Broglie waves and Bohr's model (cont.)

- The Bohr orbits are an integer number of de Broglie wavelengths
- We can picture a stationary wave that goes along the Bohr orbit.



Summary

- Each particle (not just photons) has an associated wave, called De Broglie wave.
- The existence of De Broglie waves for electrons is consistent with Bohr's atomic model.
- De Broglie waves were demonstrated for other particles as well, by diffraction experiments.
- Quantum mechanics expands on these ideas (but we won't go into quantum mechanics) and shows that each system of particles has an associated **wave function** ψ which describes the system's properties.
- The wave function (for any system) is found by solving the **Schrödinger Equation**.

Special Relativity

$\frac{v}{c} = \beta$

Special Relativity - Formulas

- Relativistic mass

$$m = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} m_0$$

rest mass

γ

- Relativistic momentum

$$\vec{p} = m\vec{v} = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} m_0 \vec{v}$$

- Relativistic (total) energy

$$E = mc^2$$

- Relativistic kinetic energy

$$KE = mc^2 - m_0 c^2$$

Structure and States of the Atom

Hydrogenoid Atoms

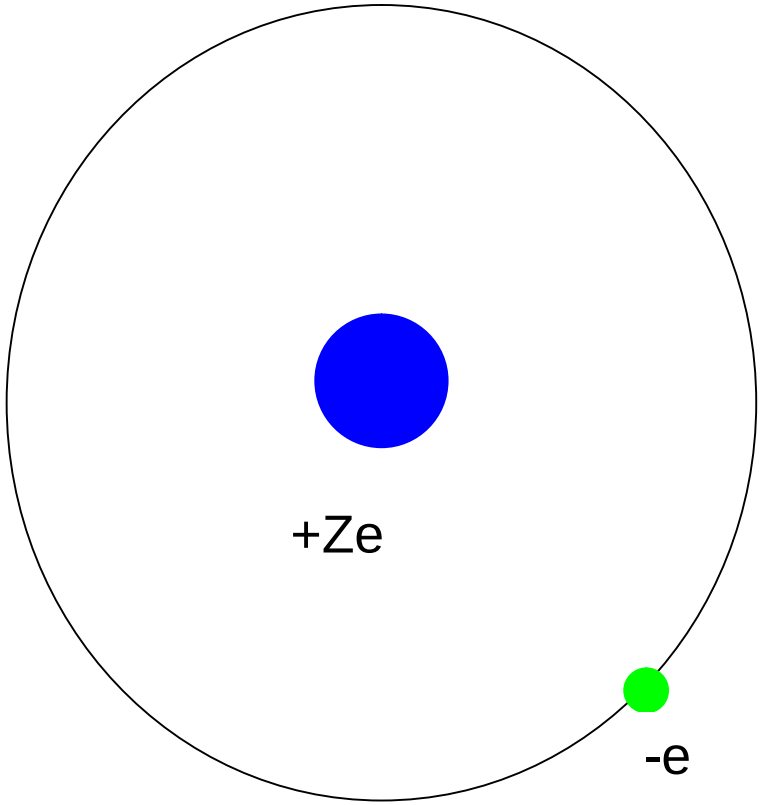
- The Hydrogen atom is the simplest possible.
- Next level of complexity: Hydrogenoid atoms (ions)
  - Atoms that have lost all but one of their electrons.
  - Nucleus made of Z protons and N neutrons
  - Bohr's model applies very well to hydrogenoid atoms.

Hydrogenoid Atoms

$$r_n = \frac{1}{Z}\frac{n^2\hbar^2}{m_e k e^2}$$

$$E_n = -\frac{1}{n^2} Z^2 \frac{m_e k^2 e^4}{2\hbar^2}$$

$$\frac{1}{\lambda} = Z^2 \frac{m_e k^2 e^4}{2\hbar^2 hc}\left(\frac{1}{m^2} - \frac{1}{n^2}\right)$$



Bohr's model for Hydrogen and Hydrogenoid Atoms

- The orbit of the electron, and its energy is characterized by one integer, called **principal quantum number: n**

$$E_n = -\frac{1}{n^2} Z^2 \frac{m_e k^2 e^4}{2\hbar^2}$$

- The state of the atom is characterized completely by the principal quantum number, n.

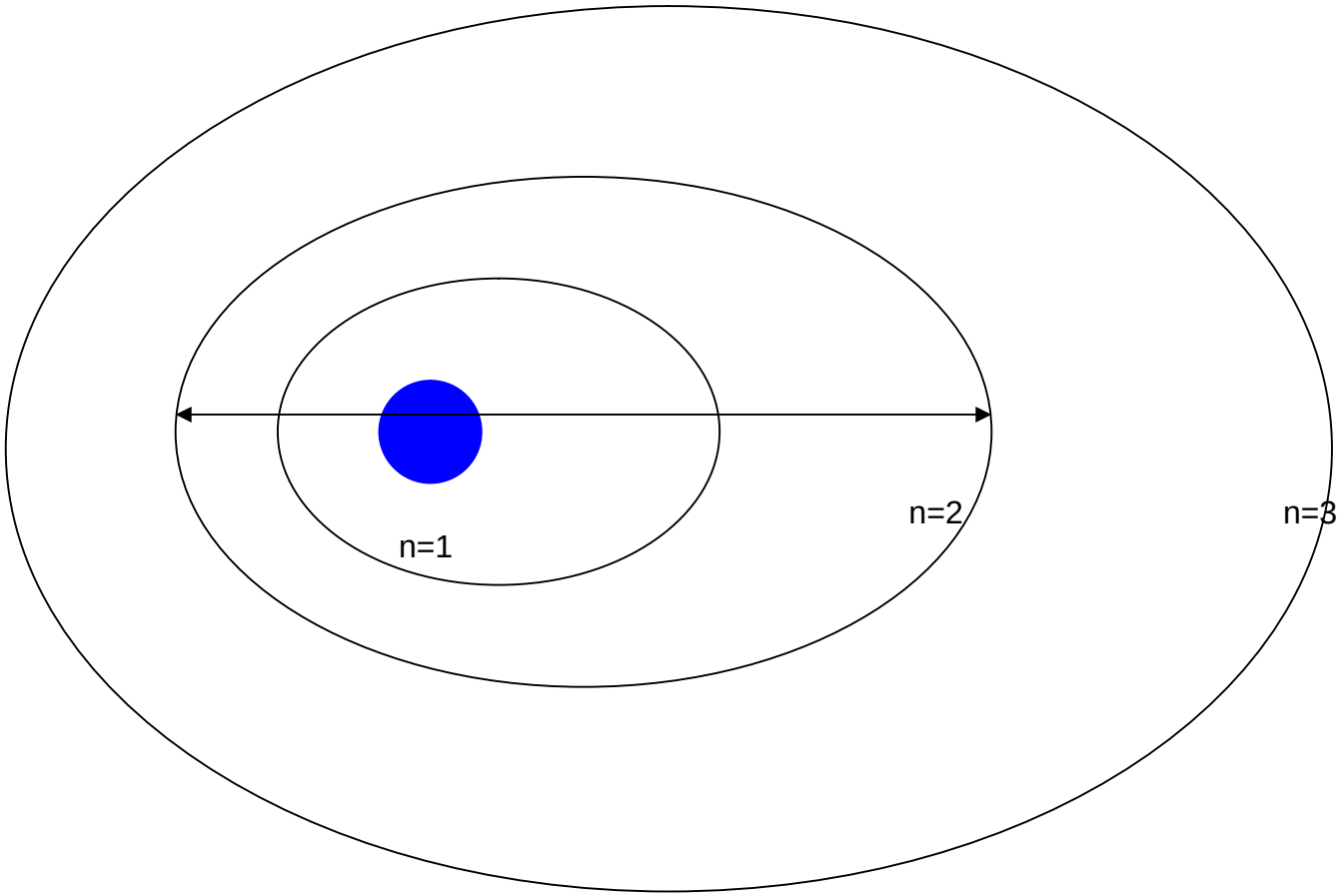
More accurate model (quantum mechanics)

- Four quantum numbers for each electron
  - n = 1,2,3.... principal quantum number
  - l = 0,1, 2...n orbital quantum number
  - m_l = -l, -(l-1), ...0 ... (l-1), l orbital magnetic quantum number
  - m_s = -1/2 or +1/2 spin magnetic quantum number
  - Remember that electrons have an intrinsic "spin" angular momentum, equal to 1/2.

Significance and simplistic interpretation of quantum numbers

- We can have a slightly more complicated model of the atom, whereby orbits are elliptical.
  - n indexes the size of the major axis or, equivalently, the energy of the orbit
  - l indexes how “oval” the orbit is (l=0 corresponds to a circle) or, equivalently, the angular momentum of the electron
  - m_l indexes the spatial orientation of the orbit plane or, equivalently, the z component of the angular momentum.
  - m_s indexes the orientation (or z component) of the spin
    - So there is an additional quantum number, s, the spin angular momentum quantum number, which we don’t usually specify, because it is always fixed at ½ for electrons.

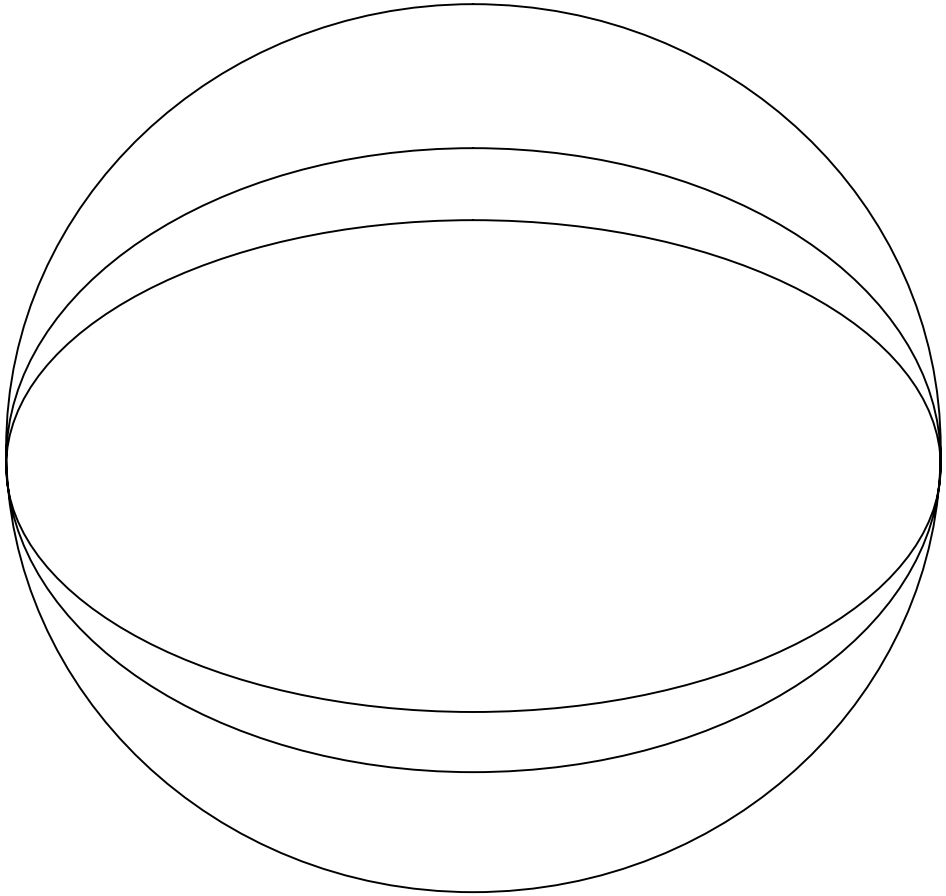
n indexes the major axis of the ellipse (the energy)



l indexes the minor axis of the ellipse (the orbital angular momentum)

- Quantum mechanical expression for angular momentum:

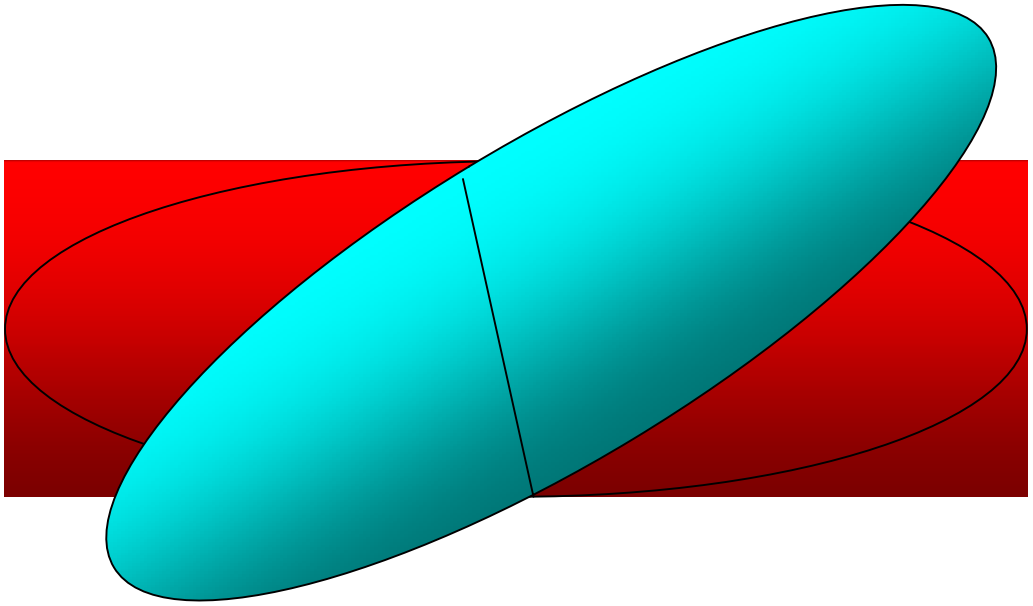
$$L = \sqrt{l(l+1)}\hbar$$



m_l indexes the orientation of the plane of the orbit (z component of the angular momentum)

- QM expression for Lz

$$L_z = m_l \hbar$$

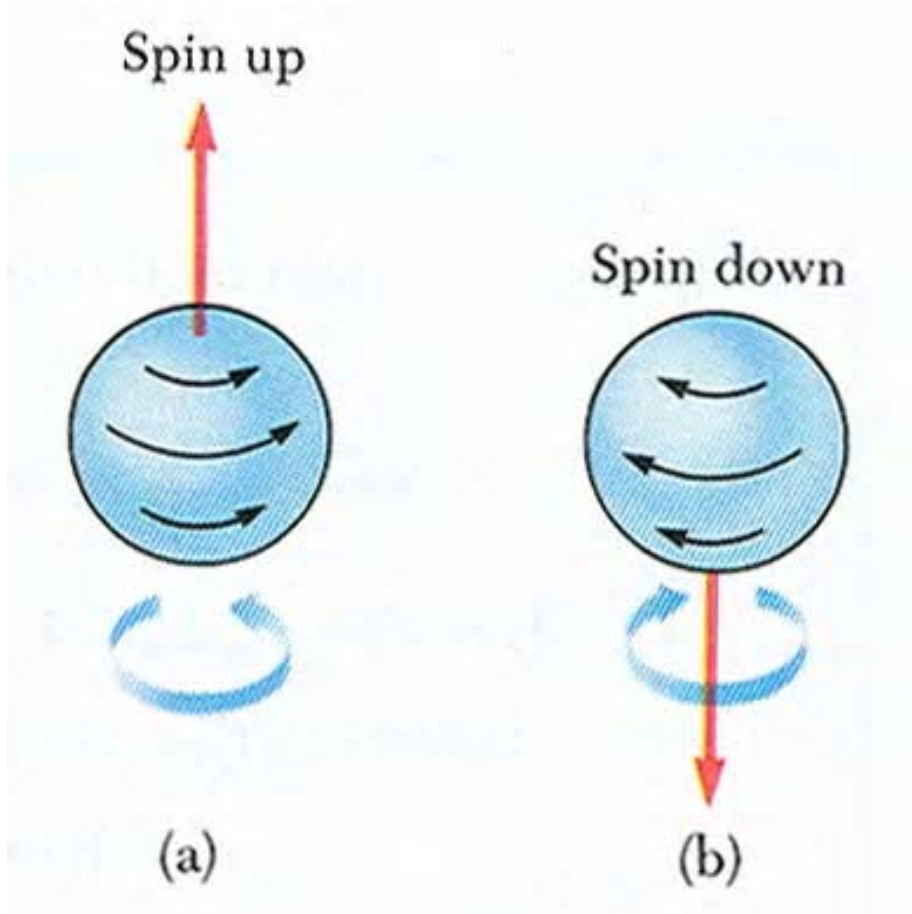


s_l indexes the orientation of the spin

- The spin angular momentum number is the same in both cases, and equal to 1/2.
- QM gives:

$$S = \sqrt{s(s+1)}\hbar$$

$$S_z = m_s \hbar$$



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Atoms with more than one electron (Shell Model)

- Pauli's exclusion principle:
  - There cannot be more than one electron in a state characterized by the same combination of quantum numbers.
  - Shell
    - All states with the same n
  - Subshell
    - All states with same n and l
  - Orbital
    - All states with same n, l, and m_l
    - Can "hold" two electrons (corresponding to $m_s = \pm 1/2$)

Atomic and Nuclear Constituents

- Atom
  - Electrons
  - Nucleus (made up of nucleons)
    - protons
    - neutrons
- The nucleus is “held together” by nuclear attraction forces. These have to be stronger than the repulsive electrostatic forces.
- For neutral atoms, the number of protons in the nucleus equals the number of electrons in orbit.

Some subatomic particles

- proton
- neutron
- electron (beta particle)
- positron
- photon (gamma particle)
- neutrino
- antineutrino
- alpha particle (2 protons + 2 neutrons)

Properties of fundamental particles

- Mass (rest mass)
- charge
- spin (denoted by s)
- parity
  - property resulting from Quantum Mechanics.
  - describes the parity of the wave function

$$+ \Leftrightarrow \psi(\vec{r}) = \psi(-\vec{r})$$

$$- \Leftrightarrow \psi(\vec{r}) = -\psi(-\vec{r})$$

- All these quantities are important because they are conserved in nuclear reactions.

Properties of Nuclei

- Atomic number – Z = number of protons
- Mass number – A = total number of nucleons (protons and neutrons)
- Number of neutrons – N
- The atomic number Z identifies the nuclear species.
- Two nuclei with the same Z but different N are called **isotopes**.
- Notation: $^{A}_{Z}X$ X is the chemical symbol
- Alternative notation: (Z,A)

Other properties of nuclei (parallel those of particles)

- Mass
- charge (+Ze)
- spin (s)
- parity

Atomic Mass Unit (amu)

$^{12}_{6}C$

- Defined as 1/12 of the mass of a C12 **atom**
  - That means that it is 1/12 of the C12 nucleus, plus the mass of 1/2 electron.
- Atomic weight = Ratio between the mass of the atom and 1 amu (dimensionless number)
- Molecular weight = Ratio between the mass of a molecule and 1 amu (dimensionless number)
- 1 Mole – Quantity of a pure substance that has the same mass expressed in grams as the atom's (or molecule's) mass expressed in amu.
- 1 Mole Has $N_A = 6.023 \times 10^{23}$ atoms (molecules)
- Na is the ratio between 1g and 1 amu. (There are N_A amus in a gram)

Atomic Mass vs. Atomic Weight

- Atomic mass has dimensions of mass (e.g. Kg, g, amu, etc.)
- Atomic weight has no dimensions.
- Atomic weight is numerically equal to the atomic mass expressed in amus.
- ^{12}C has an atomic mass of 12 amus and an atomic weight of 12.

amu expressed in Kg

- N_A atoms of ^{12}C weigh 12 g. It follows that 1 amu weighs $1/N_A$ grams.

$$1\,\text{amu} = \frac{12(g)}{12 \times N_A} = \frac{1}{N_A}(g) = \frac{1}{6.023 \times 10^{23}}(g) = 1.66 \times 10^{-24}(g) = 1.66 \times 10^{-27}(kg)$$

Other means of expressing mass

- Because of the mass-energy equivalence expressed by Einstein's formula , $E = mc^2$ mass can also be expressed in units of energy over c^2.
- For example: $1kg = \frac{1kg \times c^2}{c^2} = \frac{1 \times \left(3 \times 10^8\right)^2 (J)}{c^2} = 9 \times 10^{16} \left(\frac{J}{c^2} \right)$
- Often in nuclear physics the energy is measured in MeV, and the mass in MeV/c^2. To find the relation between 1kg and one MeV/c^2 we write:

$$1\frac{MeV}{c^2} = \frac{10^6 eV}{\left(3 \times 10^8 (m/s)\right)^2} = \frac{10^6 \times 1.602 \times 10^{-19} C \times V}{\left(3 \times 10^8 (m/s)\right)^2}$$

$$= \frac{10^6 \times 1.602 \times 10^{-13} J}{\left(3 \times 10^8 (m/s)\right)^2} = 1.78 \times 10^{-24} Kg$$

Examples of elementary particle mass

particle	mass		
	kg	amu	MeV/c^2
proton	1.6726E-27	1.007276	938.28
neutron	1.6750E-27	1.008665	939.57
electron	9.1090E-31	5.486E-4	0.511

Atomic Weight for a Mixture of Atoms

- Consider a mixture of 30% (by atom) C and 70% (by atom) Al. What is the average atomic weight of the mixture?
- <u>Answer</u>
  - Assume there are N atoms in total
  - of these
    - N_C=0.3N are C
    - N_{Al}=0.7N are Al

Atomic Weight for a Mixture of Atoms (cont)

- The total mass of the mixture (in amu) is:

$$m = N_C M_C + N_{Al} M_{Al} = 0.3NM_C + 0.7NM_{Al}\ (amu)$$

- The average mass of one atom (in amu) is:

$$\overline{M} = \frac{m}{N} = \frac{0.3NM_C + 0.7NM_{Al}}{N} =$$
$$= 0.3M_C + 0.7M_{Al} = 0.3 \times 12 + 0.7 \times 13 = 12.7\ (amu)$$

Atomic Weight for a Mixture of Atoms (cont)

- In general
  - For a mixture of n types of atoms, each with atomic fraction $X_i = N_i/N$, the average atomic weight is:

$$M = \sum_{i=1}^{n} X_i M_i$$

  - If the different types of atoms are isotopes of the same atom, the atomic fractions are called isotopic abundances.

Properties and Structure of Nuclei

Nuclear Radius

- Assume that nuclei are made of “nuclear material” of the same density ρ for all species of nuclei.
- It follows that the mass of the nucleus is given by:

$$m = \rho V = \rho \frac{4\pi R^3}{3}$$

Nuclear Radius (cont)

- The mass of the nucleus is given also by the mass of its constituents (neutrons and protons)

$$m = Nm_n + Zm_p$$

- Because the mass of the proton and the one of the neutron are almost equal to 1 amu, we can write:

$$m = Nm_n + Zm_p \cong N\,amu + Z\,amu = (N+Z)\,amu = A\,amu$$

Nuclear Radius (cont)

- By writing the equality between the two masses, we have:

$$A\,amu = \rho \frac{4\pi R^3}{3}$$

- Solving for R^3, we obtain

$$R^3 = A\left(\frac{3}{4\pi\rho} amu\right)$$

Nuclear Radius (cont)

- Solving for R, by taking the cube root on both hands, we have:

$$R = \sqrt[3]{A}\sqrt[3]{\frac{3}{4\pi\rho}amu}$$

- It turns out that: $\sqrt[3]{\frac{3}{4\pi\rho}amu} = 1.25 \times 10^{-15}\, m$

- So: $R = 1.25 \times 10^{-15} \times \sqrt[3]{A}[m]$ → meters

Binding Energy

- Since particles that constitute the nucleus stay together (held by nuclear interaction forces), the total energy of the nucleus must be lower than the total energy of the particles if they were separated.

$$B = \left[(A - Z) \times E_{neutron} + Z \times E_{proton} \right] - E\left({}^{A}_{Z}X \right)$$

- This is called the **Binding Energy**

Binding Energy (cont.)

- Binding energy from a relativistic perspective

$$E_{neutron} = m^0_{neutron} c^2$$

$$E_{proton} = m^0_{proton} c^2$$

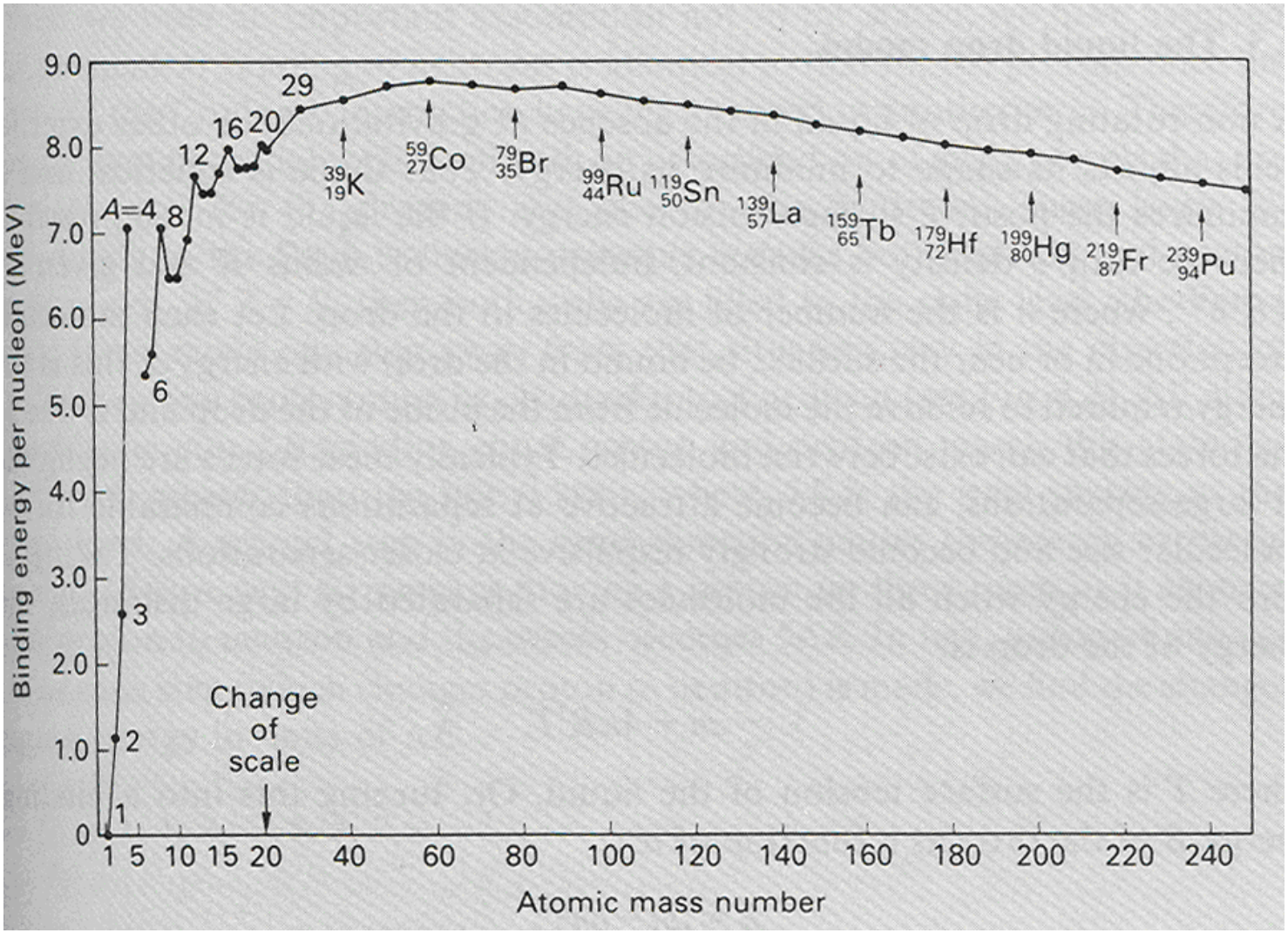
$$E\left({}^A_Z X\right) = M_0\left({}^A_Z X\right) c^2$$

$$B = \left[(A-Z)\times m^0_{neutron} c^2 + Z \times m^0_{proton} c^2\right] - M_0\left({}^A_Z X\right) c^2$$

$$B = c^2 \underbrace{\left\{\left[(A-Z)\times m^0_{neutron} + Z \times m^0_{proton}\right] - M_0\left({}^A_Z X\right)\right\}}_{\Delta}$$

- **The mass of the nucleus is smaller than the sum of the masses of its constituents.**

Binding Energy per Nucleon



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Question Period

- Q: If I climb to the top of the CN tower (approximately 550 m) will my body mass be larger?
- A: Yes, but not enough for people to notice.

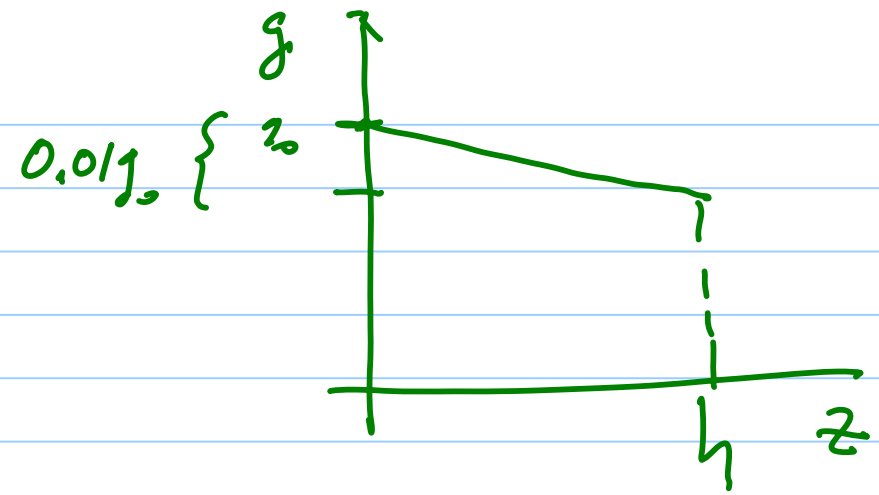
energy at the top = $E_{bottom} + W_{GF}$

$$m_{top} - m_{bottom} = \frac{m_{bottom}c^2 + m_{bottom}gh}{c^2} - m_{bottom} = \cancel{m_{bottom}} + \frac{m_s g s}{c^2} - \cancel{m_s}$$

$$\frac{m_{bottom}gh}{c^2} = \frac{70Kg \times 9.8m/s^2 \times 550m}{\left(3\times10^8 m/s\right)^2} = 4.2\times10^{-12}Kg$$

$$W = \int_0^h g(z)\, m\, dz$$

$$\hookrightarrow g\, m \cdot h$$



$$g(z) = g_0 - \frac{0{,}01\, g_0}{h} \cdot x$$

$$W = \int_0^h \left(g_0 - \frac{0{,}01\, g_0}{h} x \right) dx =$$

$$= g_0 x \Big|_0^h - \frac{0.01}{h} g_0 \frac{x^2}{2} \Big|_0^h =$$

$$= g_0 h - \frac{0.01}{h} g_0 \frac{h^2}{2} =$$

$$= g_0 h - 0.01\, g_0 h = 0.99\, g_0 h$$

$$\Delta m = \frac{W}{c^2} = \frac{m_0 g h}{c^2} = 0.99 \frac{m_0 g h}{c^2} =$$

$$= 0.99 \times 4.2 \times 10^{-12} = 4.16 \times \underline{\underline{10^{-12}}}$$

$$g \sim \frac{1}{r^2}$$

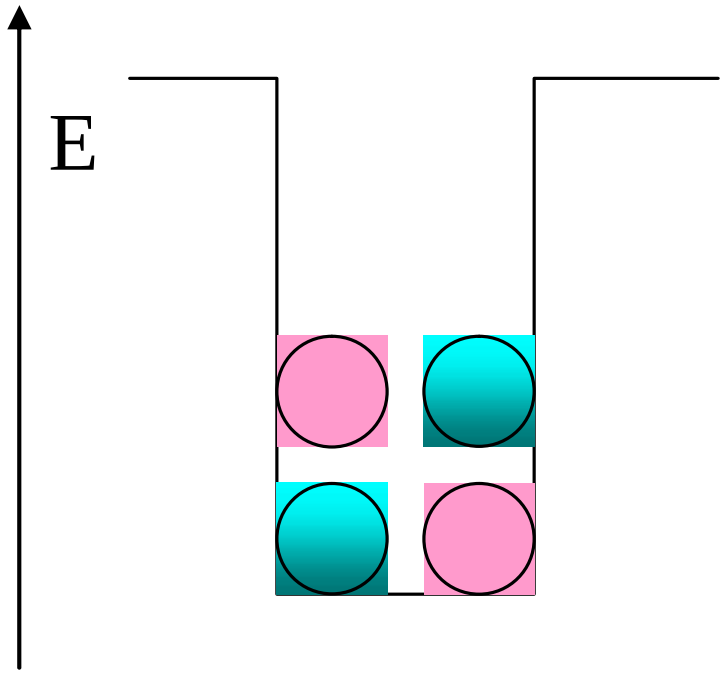
$$\Delta g \sim \frac{1}{(6000000\,m)^2} - \frac{1}{(6000500\,m)^2}$$

Nuclear Models

Shell Model

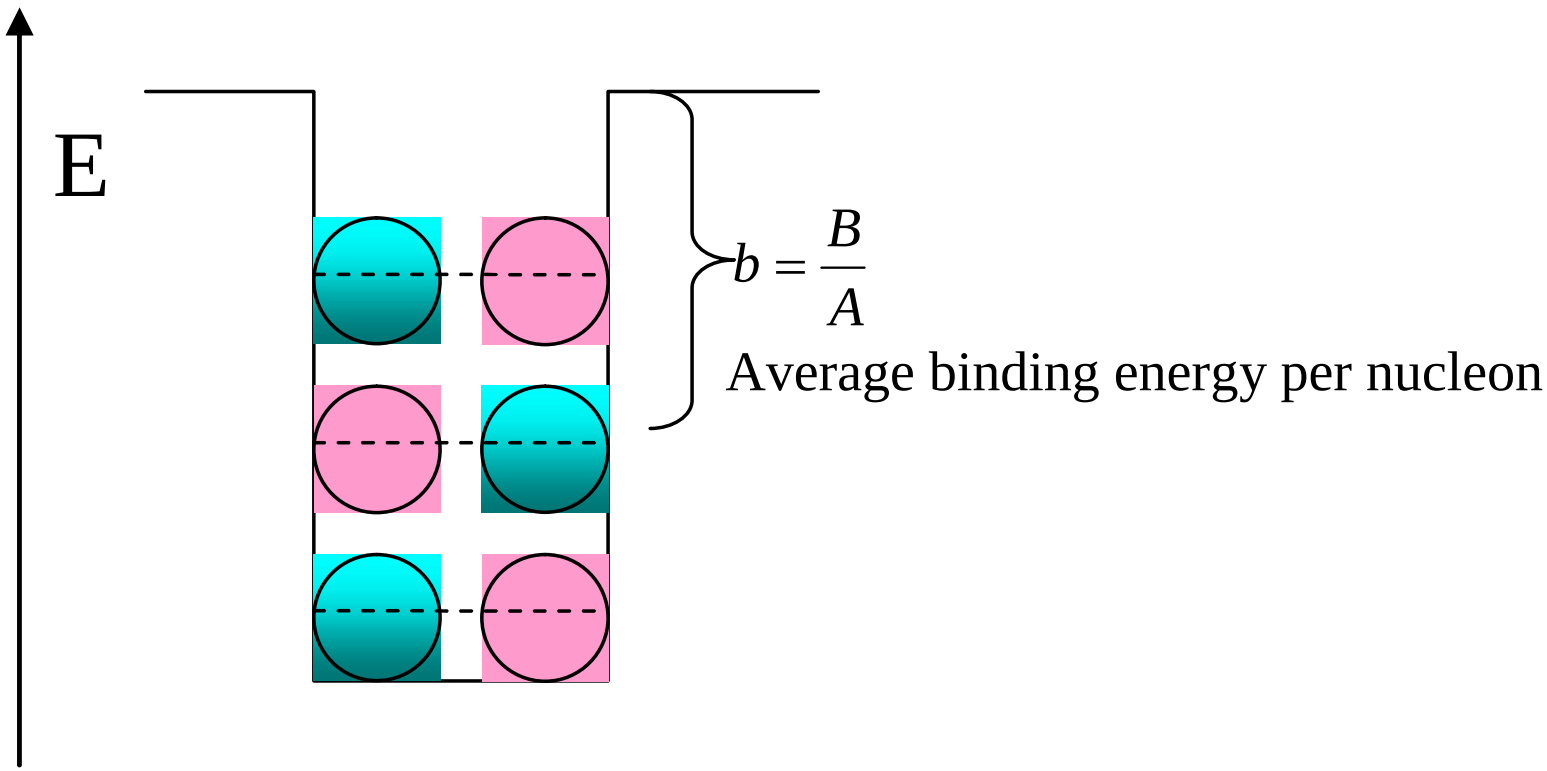
Shell Model - Potential Well

- We can picture the nucleons (protons and neutrons) as “living” in a “potential well” created by the nuclear forces.
- The binding energy is the energy that needs to be communicated to the nucleons to allow all of them to exit the well.



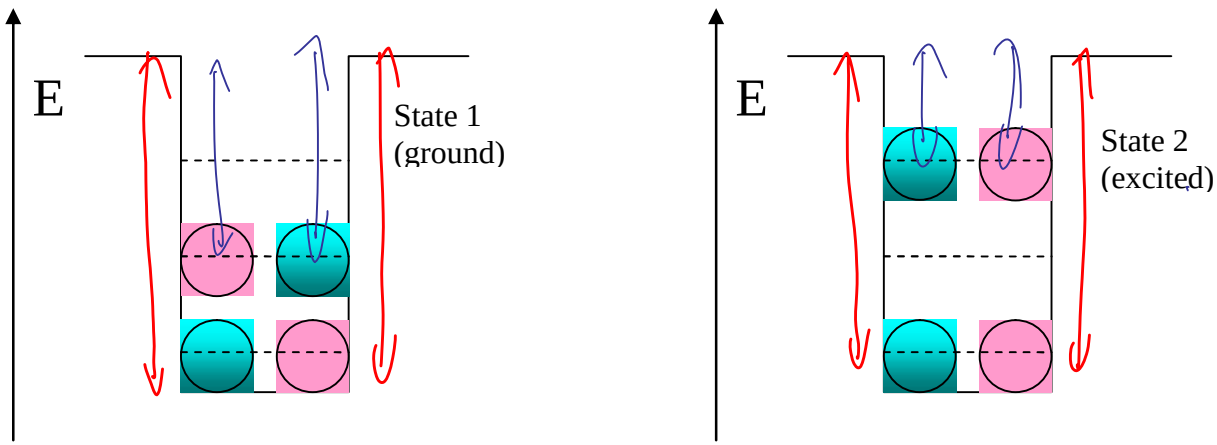
More details on the potential well

- Nucleons can occupy different energy levels in the well, just like electrons can occupy different energy levels in an atom.
- The state of the nucleus is given by the states (energy, spin, parity) of all its nucleons.
- Pauli's exclusion principle applies (No two nucleons can occupy the same state).



More details on the potential well

- Depending on the “arrangement” of nucleons on energy levels inside the well, the nucleus can have different binding energies.
- The lowest energy level of the nucleus (corresponding to the largest binding energy) is called the **ground level**, and the corresponding state is called the **ground state**.
- Higher energy levels are called **excited levels**, and the corresponding states are called **excited states**.



$$E_{excited} > E_{ground}$$

$$B_{ground} > B_{excited}$$

$$c^2 M\left({}^{N}_{Z}X^{excited}\right) > M\left({}^{N}_{Z}X^{ground}\right) c^2$$

Nuclear Energy Levels

- Similar to atomic energy levels

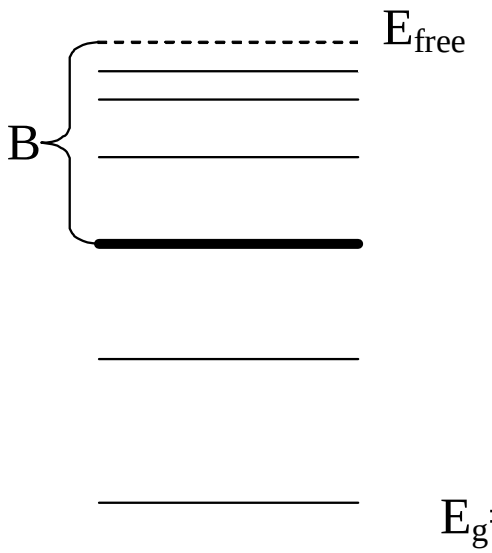
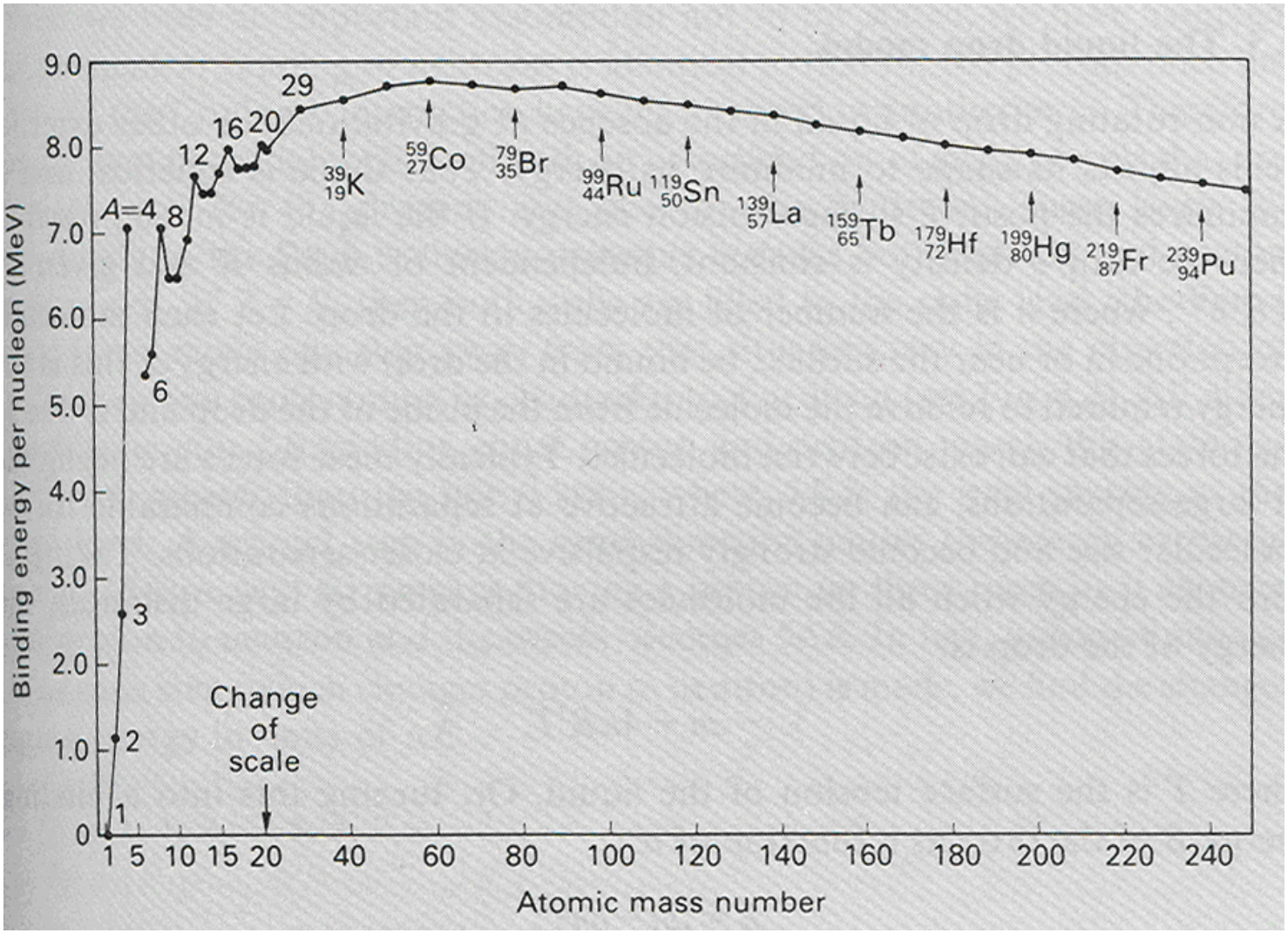


Figure 2.3 The energy levels of carbon 12.

Liquid Drop Model

Binding Energy per Nucleon



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Liquid Drop Nuclear Model

- Attempts to express the binding energy as a function of nuclear characteristics.
- Leads to a semiempirical formula.
  - Shape of formula determined from the model
  - Values of constants determined from measurement

$A - 2Z = N + Z - 2Z = N - Z$

$$B = a_v A - a_s A^{\frac{2}{3}} - a_c \frac{Z(Z-1)}{A^{\frac{1}{3}}} - a_A \frac{(A-2Z)^2}{A^2} + \delta(Z, A)$$

Liquid Drop Model – Meaning of Terms

- **a_v** – Volume effect – proportional to the "volume" of the nucleus, which can be considered to be roughly proportional to A. This term was introduced because it was observed that the binding energy per nucleon is almost constant.
- **a_s** – Surface effect – proportional to the "surface" of the nucleus, roughly proportional to $A^{2/3}$. This negative term was introduced because the nucleons situated close to the surface have fewer neighbors, and hence contribute less to the binding energy.
- **a_c** – Coulomb effect – electrostatic repulsion between protons has a potential energy $\frac{Z(Z-1)e^2}{r} \propto \frac{Z(Z-1)}{A^{\frac{1}{3}}}$.
- **a_A** – Asimmetry effect. It vas observed that nuclei with N=Z are more stable, hence the binding energy is probably smaller if Z and N differ. This term accounts for that effect.
- $\delta(Z,A)$ Pairing term. Introduced because it was found experimentally that two protons or two neutrons are bound stronger than a proton and a neutron. It is zero for odd A, $-a_p \frac{1}{A^{\frac{1}{2}}}$ for both Z and N odd and $+a_p \frac{1}{A^{\frac{1}{2}}}$ for both Z and N even

Liquid Drop Model

- Numerical values of coefficients

a_V	15.7 MeV
a_S	17.8 MeV
a_C	0.71 MeV
a_A	23.6 MeV
a_P	12.0 MeV

Nuclear Reactions

General Expression

$$ {}^{A_{x1}}_{Z_{x1}}X_1 + {}^{A_{x2}}_{Z_{x2}}X_2 \rightarrow {}^{A_{y1}}_{Z_{y1}}Y_1 + {}^{A_{y2}}_{Z_{y2}}Y_2 $$

- Q value

$$ Q = \left[M_0\left({}^{A_{x1}}_{Z_{x1}}X_1\right) + M_0\left({}^{A_{x2}}_{Z_{x2}}X_2\right) - \left(M_0\left({}^{A_{y1}}_{Z_{y1}}Y\right) + M_0\left({}^{A_{y2}}_{Z_{y2}}Y_2\right) \right) \right] c^2 $$

- Q>0 – exothermic reaction (provides energy to the outside)
- Q<0 – endothermic reaction (needs energy from outside in order to proceed)
- The liberated energy is found as kinetic energy of the products, and/or as energy of the emitted particles (photons or other) with zero rest mass

$$X_1 + X_2 \longrightarrow Y_1 + Y_2 + \gamma$$

$$c^2 \Big| \; M(X_1) + M(X_2) = M(Y_1) + M(Y_2) + M(\gamma)$$

$$c^2 M(X_1) + c^2 M(X_2) = c^2 M(Y_1) + c^2 M(Y_2) + \underbrace{E_\gamma}_{= c^2 M_\gamma = h f}$$

Assuming reactants initially at rest:

$$c^2 M_0(X_1) + c^2 M_0(X_2) = c^2 M_0(Y_1) + K_{E_{Y_1}} + c^2 M_0(Y_2) + K_{E_{Y_2}} + E_\gamma$$

$$\underbrace{c^2 M_0(X_1) + c^2 M_0(X_2) - \left[c^2 M_0(Y_1) + c^2 M_0(Y_2)\right]}_{Q} = K_{E_{Y_1}} + K_{E_{Y_2}} + E_\gamma$$

Conservation Laws

- The following quantities are conserved in a nuclear reaction
  - charge
  - number of nucleons
  - energy
  - momentum

Conservation Laws

- Conservation of charge

$$Z_{x1} + Z_{x2} = Z_{y1} + Z_{y2}$$

- Conservation of number of nucleons

$$A_{x1} + A_{x2} = A_{y1} + A_{y2}$$

Conservation Laws

- If additional particles enter or exit the reaction, their charge and number of nucleons need to be accounted for when writing the conservation laws
- Example

$$ {}^{A_{x1}}_{Z_{1x}}X_1 + {}^{A_{x2}}_{Z_{x2}}X_2 \rightarrow {}^{A_{y1}}_{Z_{y1}}Y_1 + {}^{A_{y2}}_{Z_{y2}}Y_2 + e^- $$

$$ Z_{x1} + Z_{x2} = Z_{y1} + Z_{y2} - 1 $$

- We can represent the electron as ${}^{0}_{-1}e$

Conservation Laws

- Conservation of momentum

$$\vec{P}(X_1) + \vec{P}(X_2) = \vec{P}(Y_1) + \vec{P}(Y_2)$$

- Conservation of energy
  - Kinetic + Rest

$$KE_{X1} + KE_{X2} + \left[M_0\left({}^{A_{x1}}_{Z_{1x}}X_1\right) + M_0\left({}^{A_{x2}}_{Z_{x2}}X_2\right)\right]c^2 =$$

$$= KE_{Y1} + KE_{Y2} + \left[M_0\left({}^{A_{y1}}_{Z_{y1}}Y\right) + M_0\left({}^{A_{y2}}_{Z_{y2}}Y_2\right)\right]c^2$$

Fission

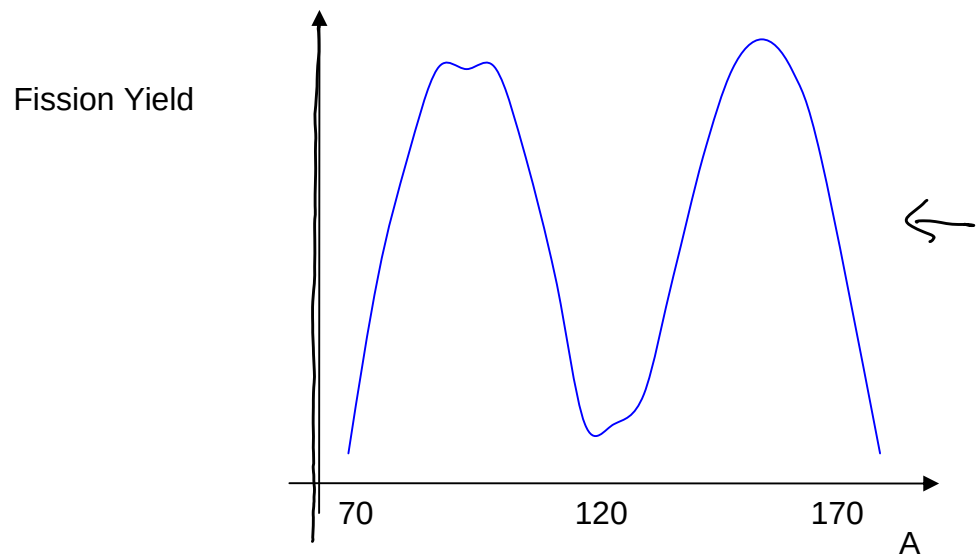
$$ {}^{1}_{0}n + {}^{235}_{92}U \rightarrow X + Y + neutrons $$

- Possible fission reactions

$$ {}^{1}_{0}n + {}^{235}_{92}U \rightarrow {}^{140}_{54}Xe + {}^{94}_{38}Sr + 2n $$

$$ {}^{1}_{0}n + {}^{235}_{92}U \rightarrow {}^{132}_{50}Sn + {}^{101}_{42}Mo + 3n $$

- Distribution of fragments



Radioactive Decay

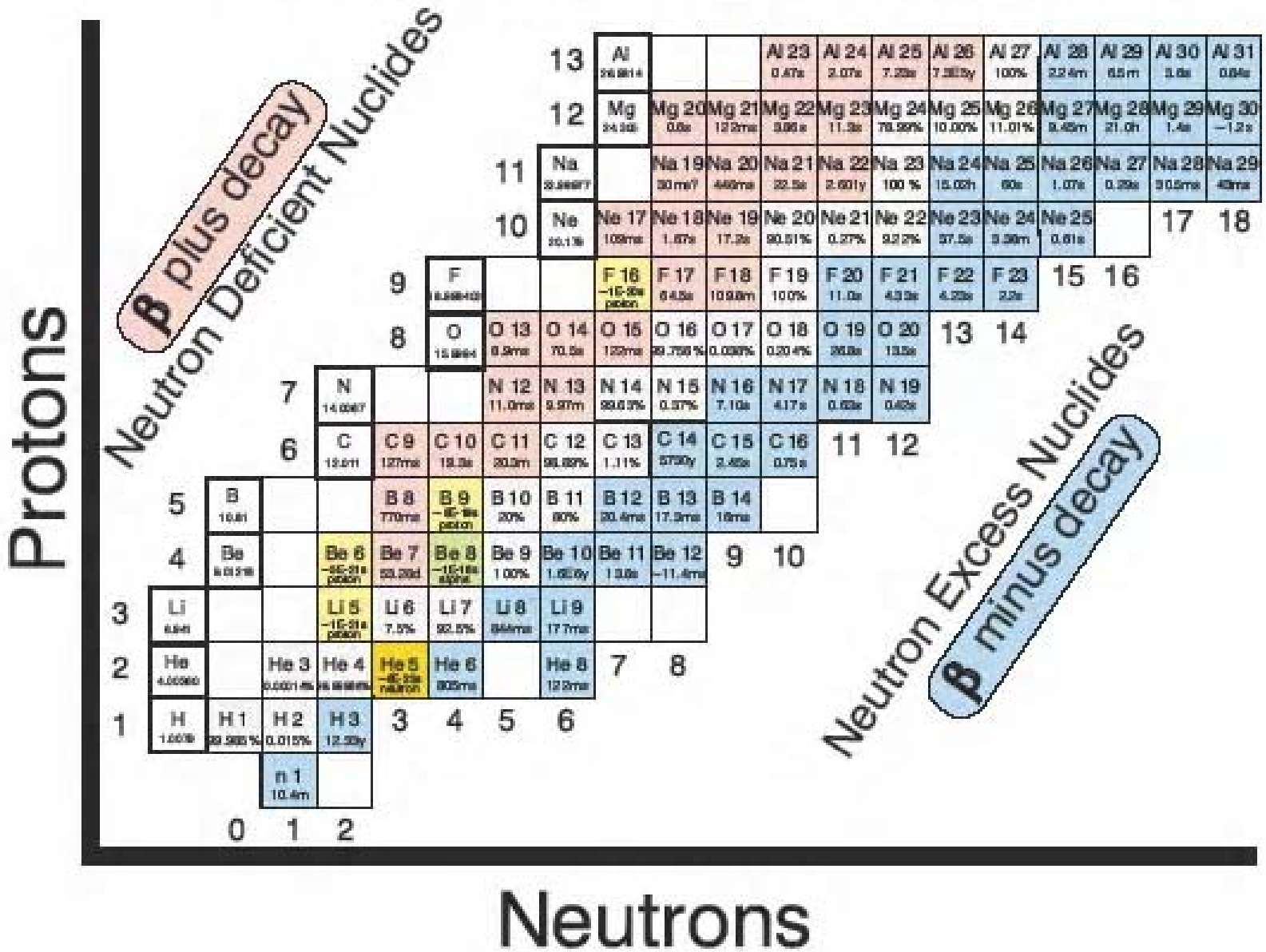
Radioactivity

- Discovered first by Henri Becquerel (1852-1908).
- Becquerel discovered that a mineral containing Uranium would darken a photographic plate even when the latter was wrapped in opaque paper.
- In 1903 Becquerel shared the Physics Nobel Prize with Pierre and Marie Curie, for their discovery and work on radioactivity.

Radioactivity

- Some nuclei are **stable**, while others are **unstable**.
- Unstable nuclei **decay**, by emitting a particle and changing into a different nucleus.
- Most common types of decay (others possible too):
  - Alpha (${}^{4}_{2}\alpha$), Helium nucleus emission
  - Beta (${}^{0}_{-1}\beta$), electron emission
  - Beta plus (${}^{0}_{1}\beta$) positron emission
  - Gamma (${}^{0}_{0}\gamma$), photon emission
  - Electron capture (an electron is “captured” rather than emitted)

CHART OF THE NUCLIDES



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Radioactive Decay

parent nucleus

daughter nucleus

$$ {}^{A}_{Z}X \rightarrow {}^{A-m}_{Z-n}Y + {}^{m}_{n}P $$

emitted particle

- Charge and number of nucleons are conserved.
- For gamma decay, technically the nucleus does not change into a different one. Only its energy state changes.
- Electron capture (still classified as “decay”)

$$ {}^{A}_{Z}X + {}^{0}_{-1}e \rightarrow {}^{A}_{Z-1}Y $$

Alternative Notations

- General decay

$$(Z, A) \rightarrow (Z - n, A - m) + (n, m)$$

- Alpha

$$(Z, A) \rightarrow (Z - 2, A - 4) + {}^{4}_{2}\alpha$$

- Beta minus

$$(Z, A) \rightarrow (Z + 1, A) + {}^{0}_{-1}\beta + \tilde{\upsilon}$$

- Electron capture

$$(Z, A) + e^{-} \rightarrow (Z - 1, A)$$

$$(Z, A) + {}^{0}_{-1}\beta \rightarrow (Z - 1, A)$$

Characteristics of Radioactive Decay

- Nuclei decay randomly.
  - It is impossible to predict which nuclei will decay in a given period of time, and which not.
  - It is impossible to predict when a particular nucleus will decay.
- On average, for large initial numbers of nuclei and for short periods of time Δt, the number of nuclei that decay within Δt is proportional to the time Δt, and to the original number of nuclei present at the beginning of the time interval.

Derivation of the Law of Radioactive Decay

- Let N(t) be the number of X-type nuclei present at time t.
- Let Δt be a short time interval.
- According to the second bullet on the previous slide, we have, on average:

$$\Delta N = -N(t) + N(t+\Delta t) = -\lambda \times N(t) \times \Delta t$$

- λ is called the decay constant, and is measured in s^{-1}.

Derivation (cont)

- The previous can be rewritten as:

$$\frac{\Delta N}{\Delta t} = -\lambda N(t)$$

- which, considering that Δt is small, yields:

$$\frac{dN}{dt} = -\lambda N(t)$$

Derivation (cont)

- Eq. (3) is an ordinary differential equation with constant coefficients. Its solution is of the form:

$$e^{-\lambda t+c} \equiv Ce^{-\lambda t} \; ; \qquad C = e^{c}$$

- The multiplicative constant C can be determined from the number of nuclei present at t=0.

$$\boldsymbol{N(0) \equiv N_0 = C \times e^{-\lambda \times 0} = C}$$

Derivation (cont)

- It follows that the number of X-type nuclei is given at any time t by:

$$N(t) = N_0 \times e^{-\lambda t}$$

- **Law of Radioactive Decay**

Example

- At t=0, a sample of ^{24}Na weights 1.0 mg. How many beta particles are emitted in an hour? (λ =1.2836x10^{-5} s^{-1})
- Solution
  - The number of emitted particles equals the number of decayed nuclei:

$$\Delta N = N_0 - N(t) = N_0 - N_0 \times e^{-\lambda t} = N_0 \times \left(1 - e^{-\lambda t}\right)$$

Example

- The initial number of Na nuclei is:

$$N_0 = \frac{m}{M} = \frac{1.0\times10^{-6}\,Kg}{24amu} = \frac{1.0\times10^{-6}}{24}\times\frac{Kg}{amu} =$$

$$\frac{1.0\times10^{-6}}{24}\times N_A = \frac{1.0\times10^{-6}}{24}\times 6.023\times10^{23} = 2.51\times10^{16}$$

- Hence the number of emitted particles is:

$$\Delta N = 2.51\times10^{16}\times\left(1-e^{-1.2836\times10^{-5}\times3600}\right) = 1.133\times10^{15}$$

Half Life

- Definition
  - The half life, $T_{1/2}$, of a radioactive species is the time after which the initial number of nuclei decreases to one half.
- Expression
  - By definition:

$$N(T_{1/2}) = \frac{N_0}{2}$$

Expression of Half-Life

- This is equivalent to:

$$N_0 \times e^{-\lambda \times T_{1/2}} = \frac{N_0}{2}$$

- Dividing be N_0 we obtain:

$$e^{-\lambda \times T_{1/2}} = \frac{1}{2}$$

Expression of Half-Life

- By taking the natural logarithm of both sides we get:

$$-\lambda \times T_{1/2} = \ln\left(\frac{1}{2}\right) = -\ln(2)$$

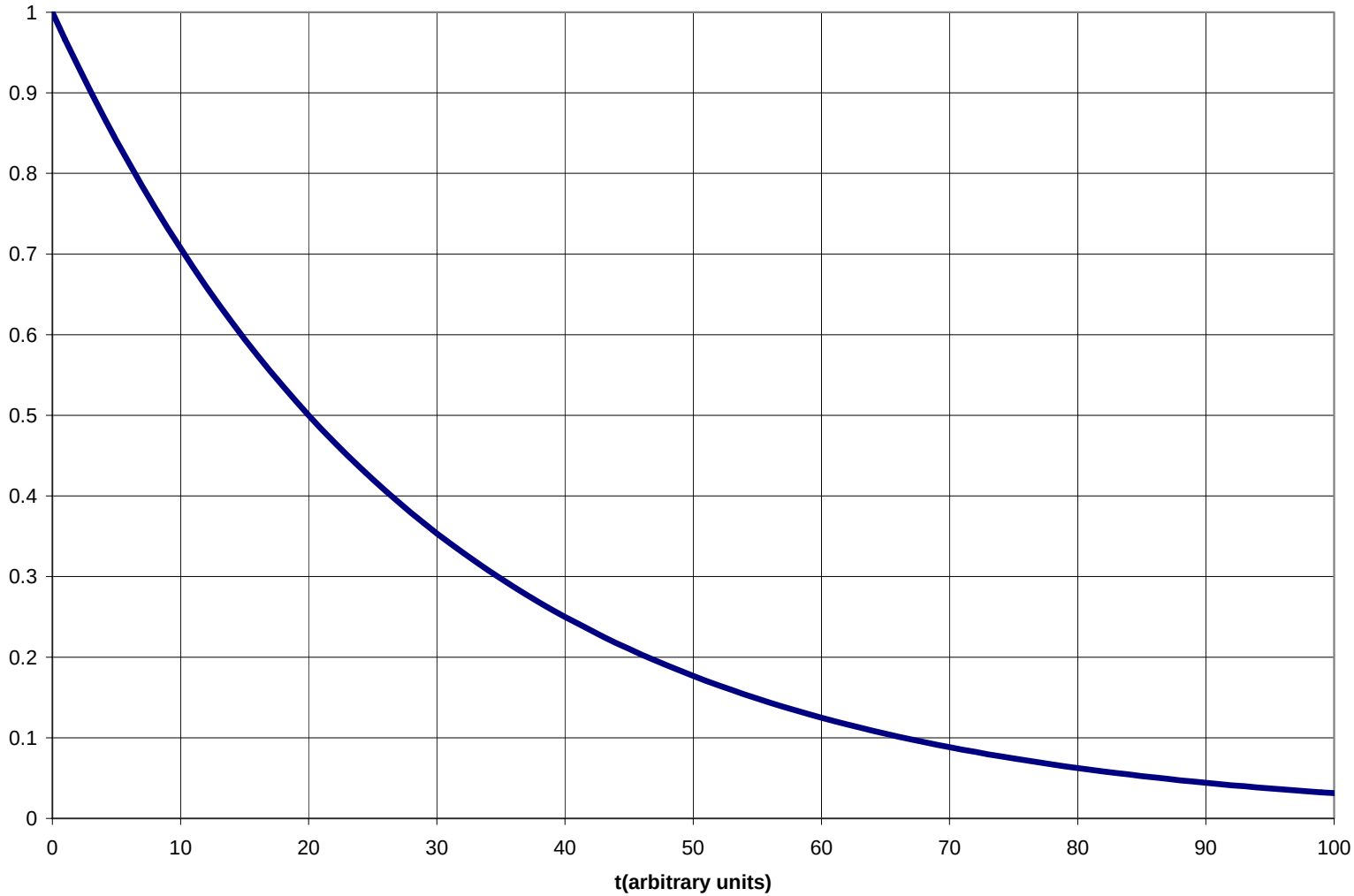
- Finally, we can solve for $T_{1/2}$:

$$T_{1/2} = \frac{\ln(2)}{\lambda}$$

Half Life Important Notes

- Half life can be measured from any moment of time. The number of nuclei left after $T_{1/2}$ elapses will be half of those existent at t_0.
- The number of remaining radioactive nuclei never reaches zero. However, it can become negligibly small.

Exponential Decay



Law of Radioactive Decay – Probabilistic Interpretation

- N(t) out of N_0 nuclei do not decay.
- It cannot be determined *a priori* which nuclei do not decay and which do.
- The ratio $N(t)/N_0$ can be interpreted as the probability of one nucleus **not decaying** after time t.

$$P_{ND} = e^{-\lambda \times t}$$

Law of Radioactive Decay – Probabilistic Interpretation

- Conversely, the probability that a nucleus **does decay** after time t is:

$$P_D = 1 - P_{ND} = 1 - e^{-\lambda \times t}$$

Activity

- The rate at which a radioactive sample decays is called **activity.**

$$\Lambda(t) \equiv -\frac{dN(t)}{dt}$$

- Equivalent definition

$$\Lambda(t) = -\lambda N(t) \Leftarrow -\frac{d}{dt} N_0 e^{-\lambda t} = \lambda N_0 e^{-\lambda t} = \lambda N(t)$$

- <u>Units:</u>
  - <u>1 decay/second = 1 Becquerel (Bq)</u>
  - <u>1 Curie = 3.7×10^{10} Bq</u>

Average Life of a Nucleus

- At t=0 there are N_0 parent nuclei.
- At time t, there are N parent nuclei left.
- At time t, $\Lambda(t)dt = N_0 e^{-\lambda t} dt$ decay in dt
- These nuclei have “lived” t before decaying.

Average Life of a Nucleus

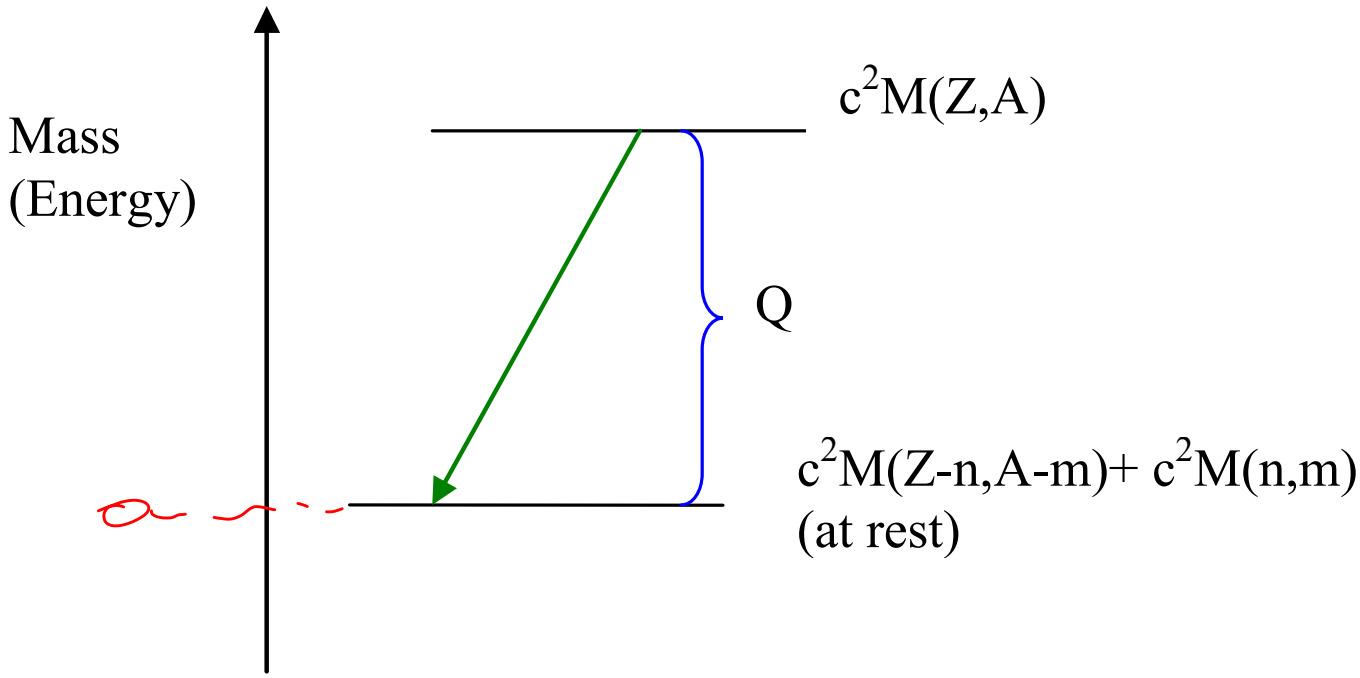
- To get the average life, we need to sum (integrate) over dt and divide by the initial number of nuclei.

$$\tau = \frac{\int_0^\infty t\Lambda(t)dt}{N_0} = \frac{\int_0^\infty t\lambda N_0 e^{-\lambda t}dt}{N_0}$$

$$= \frac{N_0\lambda\int_0^\infty te^{-\lambda t}dt}{N_0} = \lambda\int_0^\infty te^{-\lambda t}dt =$$

$$\lambda\left[\left(te^{-\lambda t}\right)_0^\infty + \int_0^\infty \frac{1}{\lambda}e^{-\lambda t}dt\right] = \int_0^\infty e^{-\lambda t}dt = -\frac{1}{\lambda}\left(e^{-\lambda t}\right)_0^\infty = \frac{1}{\lambda}$$

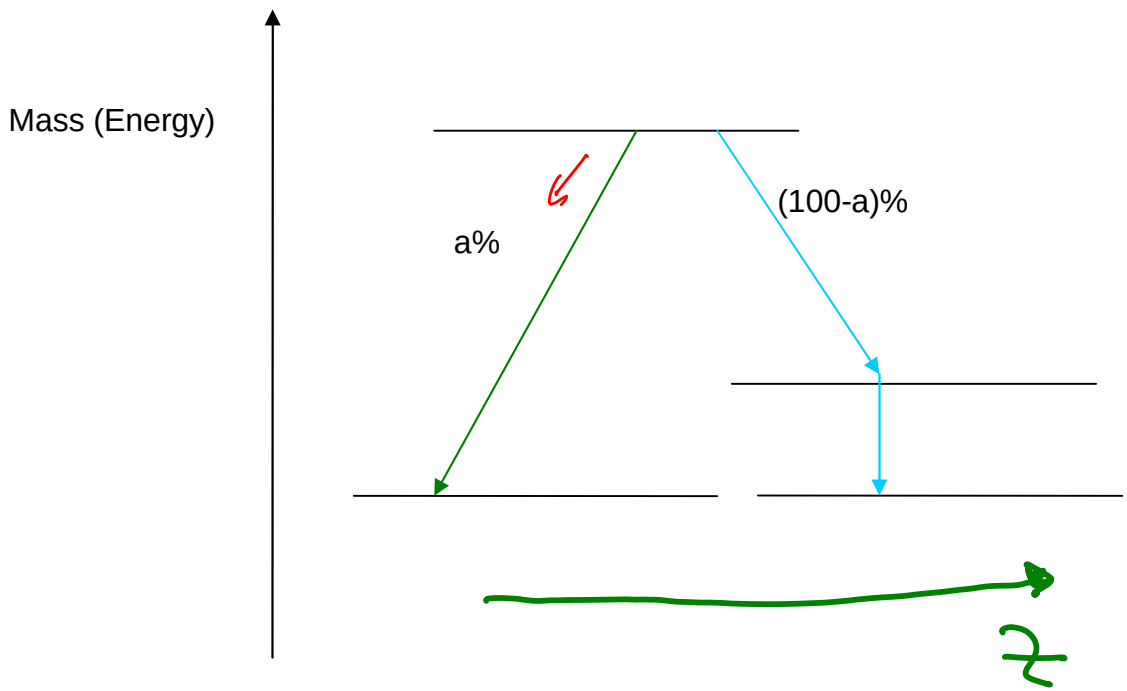
Energy-Level Diagrams for Decay (Decay Scheme)



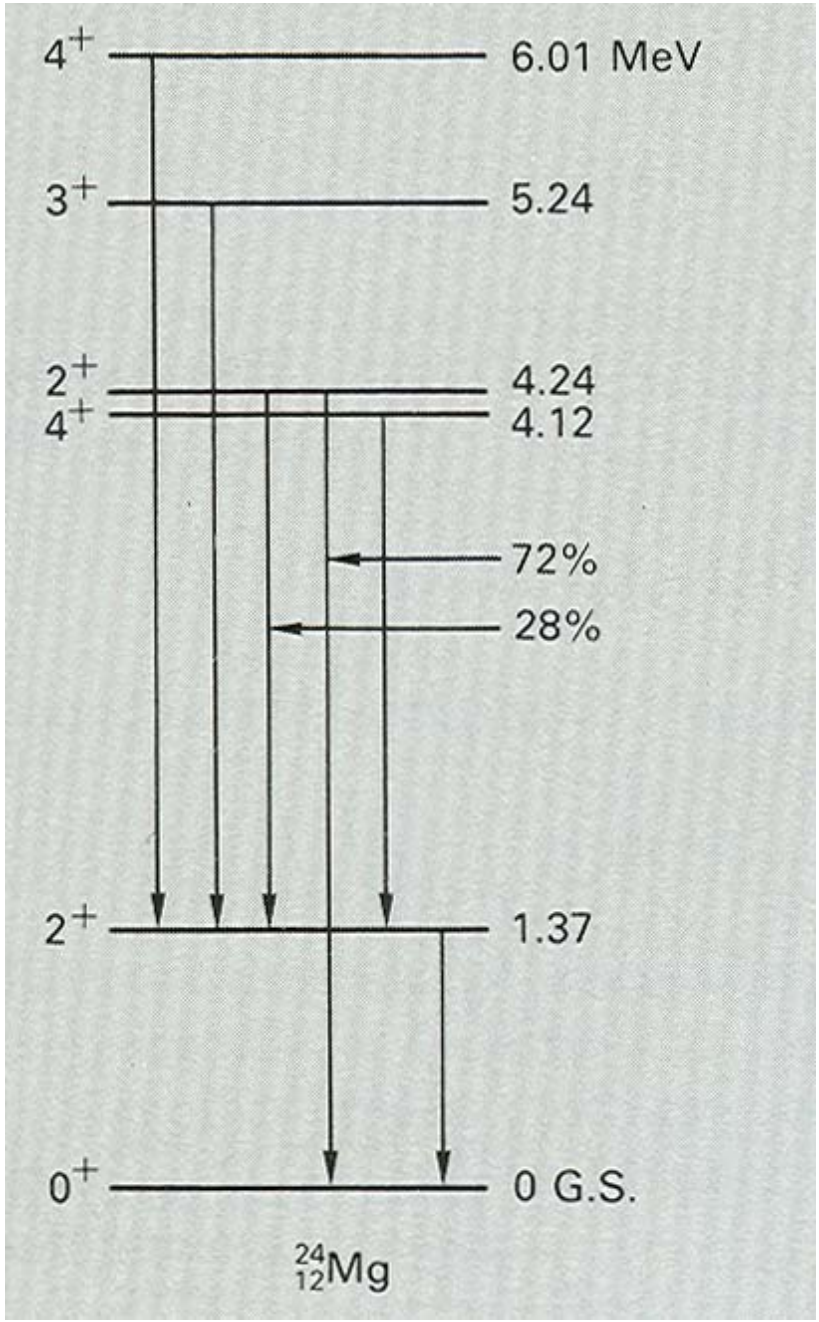
- $Q=[M(Z,A)-M(Z\text{-}n,A\text{-}m)-M(n,m)]c^2$
- $Q>0$ in order for the decay to be energetically possible
- By convention, the lowest energy on this graph is taken to be zero (energy expressed relative to the lowest value).

Multimodal Decay

- Some nuclei can decay in more than one way



Example of Energy Level Diagram with Multimodal Gamma Decay



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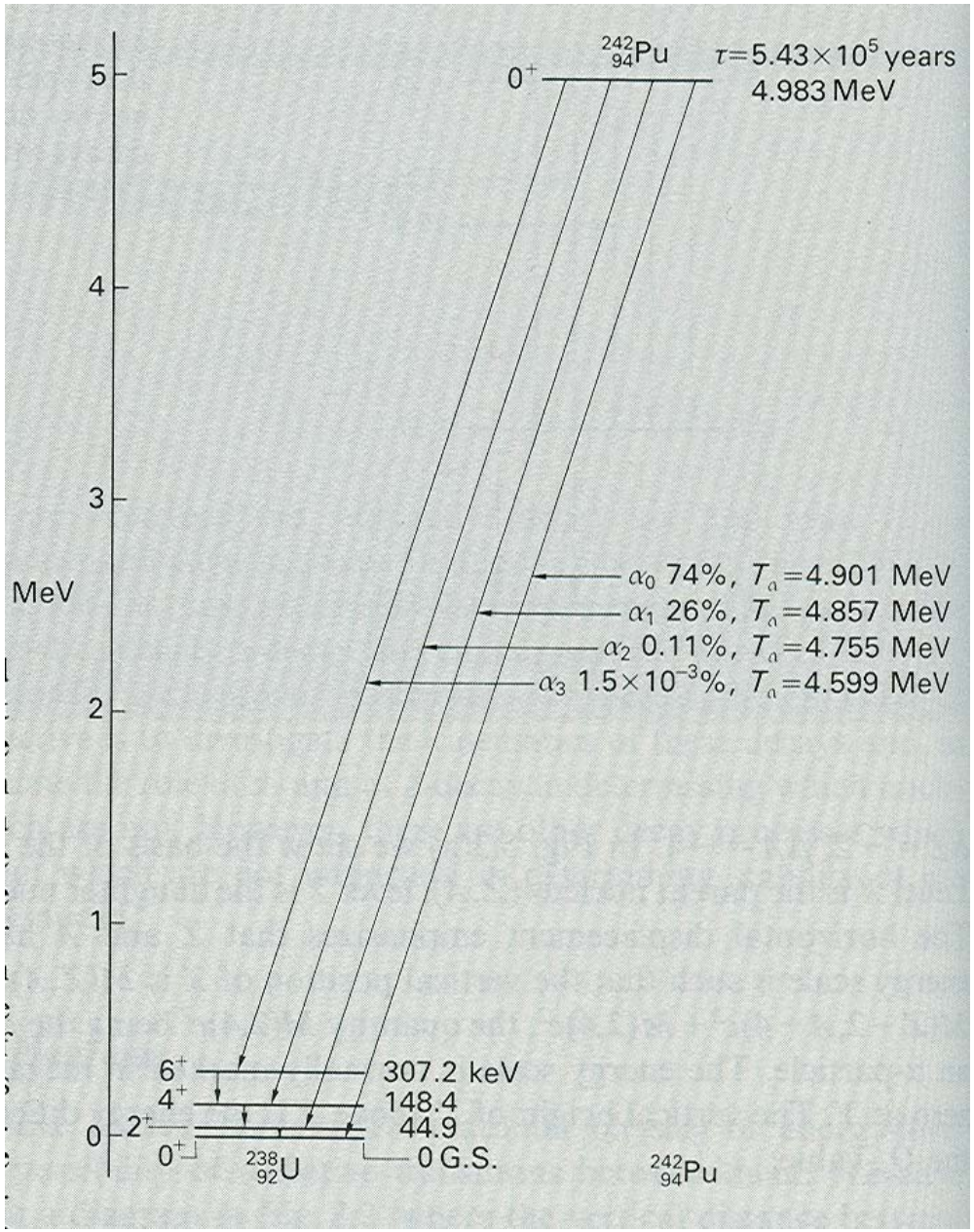
Example of Energy Level Diagram with Multimodal Alpha Decay



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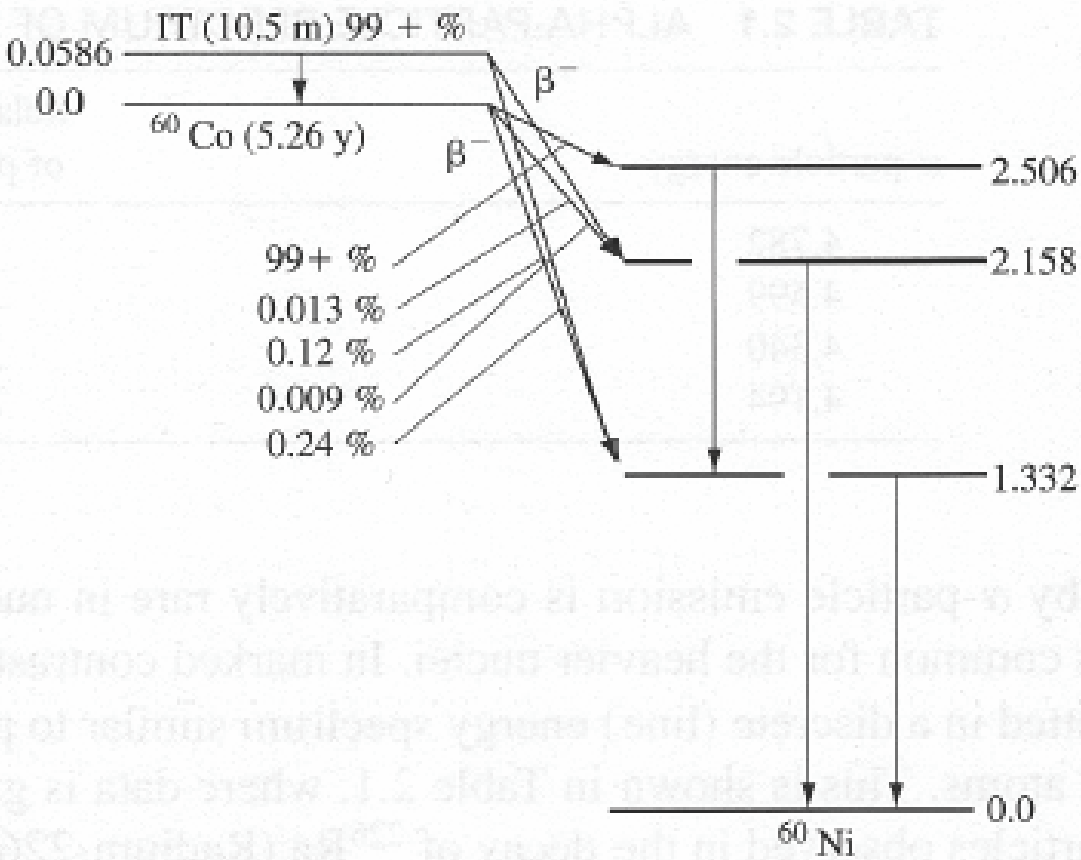
- Example of Energy Level Diagram with Multimodal Beta Decay



Multimodal Decay (cont)

- Branching fractions
  - Fraction of nuclei that decay in a certain mode
  - Have to add up to 100%
- Consider a species of nucleus that can decay by either reaction 1, or reaction 2.
- Let dN be the total number of nuclei that decay in dt. The branching factors are defines as:

$$f_1 = \frac{dN_1}{dN}$$

$$f_2 = \frac{dN_2}{dN}$$

$$f_1 + f_2 = 1$$

Multimodal Decay (cont)

- Partial decay constants

$$\lambda_1 = -\frac{dN_1}{dt}\frac{1}{N} = -\frac{dN}{dt}\frac{dN_1}{dN}\frac{1}{N} = \lambda f_1$$

$$\lambda_2 = -\frac{dN_2}{dt}\frac{1}{N} = -\frac{dN}{dt}\frac{dN_2}{dN}\frac{1}{N} = \lambda f_2$$

$$\lambda_1 + \lambda_2 = \lambda$$

- Partial half-lives (What the half life would be if only that decay mode was present).

$$T_1 = \frac{\ln 2}{\lambda_1}$$

$$T_2 = \frac{\ln 2}{\lambda_2}$$

$$\frac{1}{T_1} + \frac{1}{T_2} = \frac{1}{T}$$

Radioactive Families (Decay Chains)

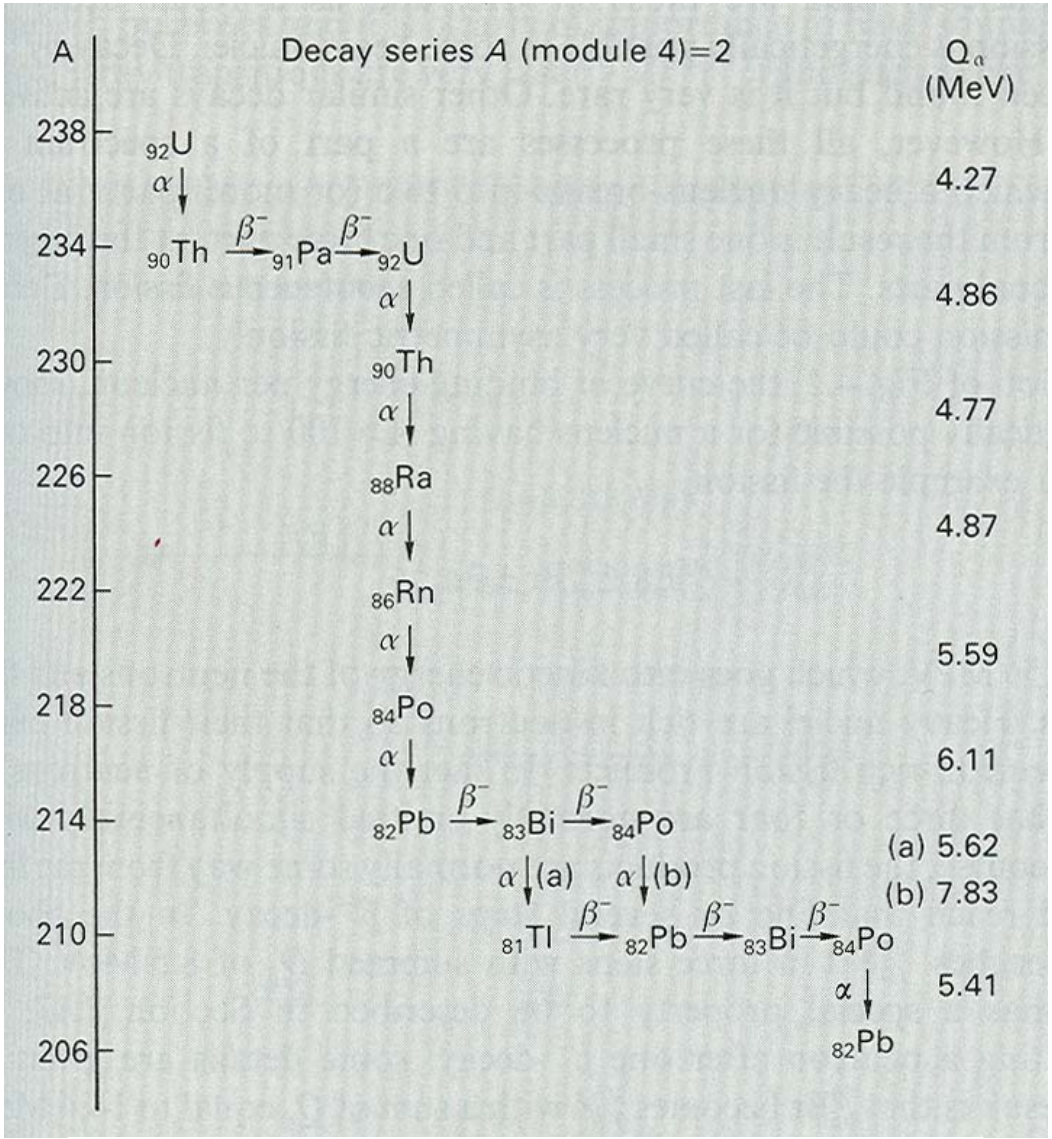
- Consider a nuclide whose daughter is also unstable and decays.

$$ {}^{A}_{Z}Q \rightarrow {}^{A-m}_{Z-n}R + {}^{m}_{n}P $$

$$ {}^{A-m}_{Z-n}R \rightarrow {}^{A-m-m'}_{Z-n-n'}S + {}^{m'}_{n'}P' $$

- This is called a radioactive family, or series.
- Radioactive families can have more than two members.

Example of Radioactive Family



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Atom Density

- Also called ***number density***.
- Is the Number of Atoms per Unit Volume
- Connection with (mass) density
  - n = # of atoms in volume V
  - M = **mass** of each atom (dimensions of mass)
  - $\mathbf{M}$ = atomic **weight** (dimensionless)
  - N = Atom density

$$N = \frac{n}{V}$$

$$\rho = \frac{m}{V}$$

$$\rho = \frac{m}{V} = \frac{nM}{V} = M\frac{n}{V} = MN =$$

$$= \mathbf{M} \times amu \times N = \mathbf{M}\frac{1g}{N_A}N = \mathbf{M}\frac{N}{N_A}(g) \cong$$

$$\cong A\frac{N}{N_A}(g)$$

Interaction of Radiation with Matter

Heavy Charged Particles e.g. alpha particles

- Interact mostly with electrons (there are usually much more electrons than nuclei) via Coulombic force
- Are much heavier than electrons
- Lose little energy in each individual interaction with any one electron
- Eventually do slow down as a consequence of the many interactions
- Have straight-line trajectories

i.e. not heavy

Fast Light Charged Particles (e.g. electrons)

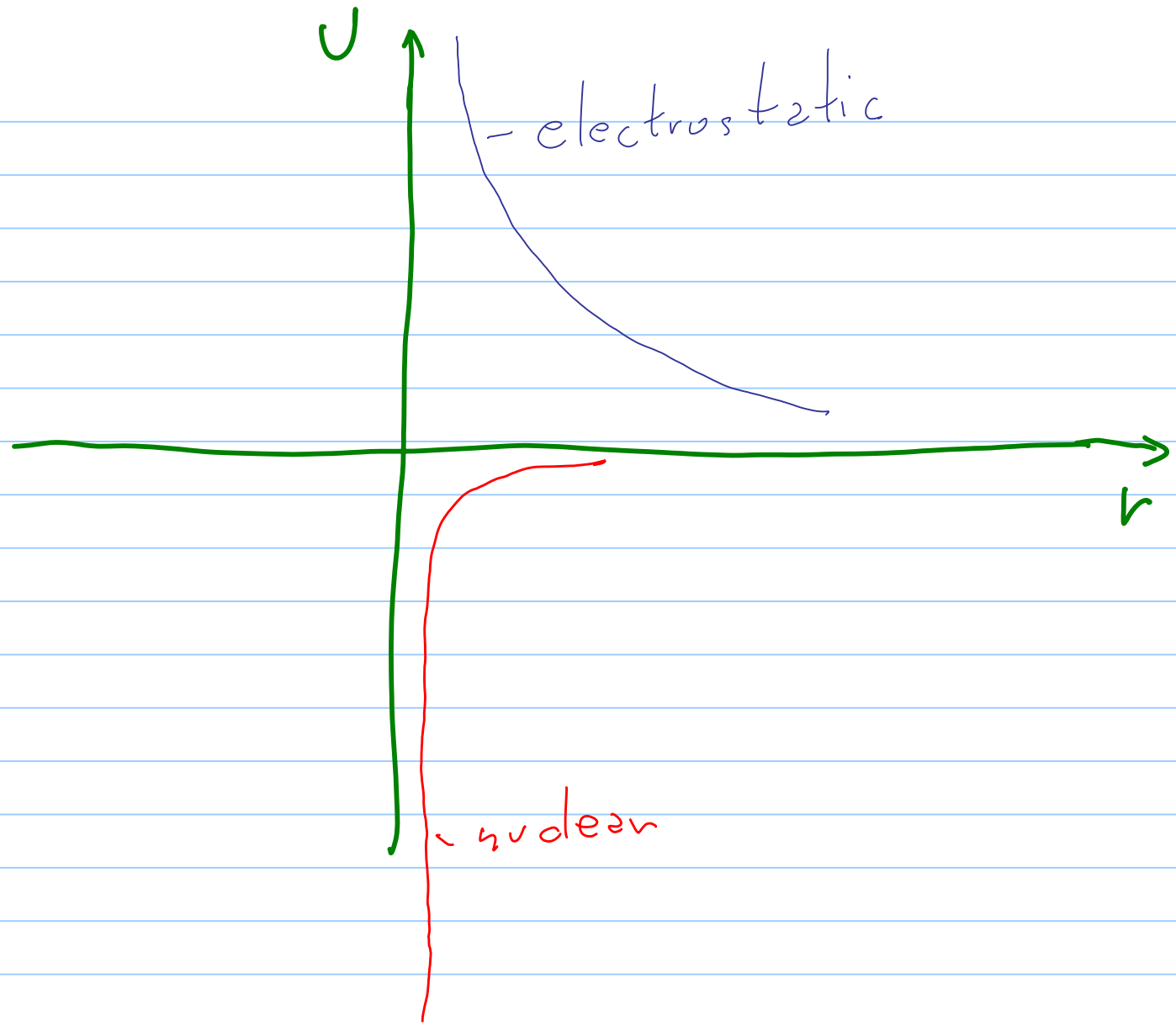
- Interact mostly with electrons (there are usually much more electrons than nuclei) via Coulombic force
- Are much of the same mass as electrons
- Have broken-line trajectories
- Slow down quickly, after only few collisions.
- Lose energy by two mechanisms
  - collisions (Can lose a lot of energy in each individual interaction with any one electron)
  - Radiation (When accelerated, incident electrons produce bremsstrahlung - electromagnetic radiation – photons)

Photons

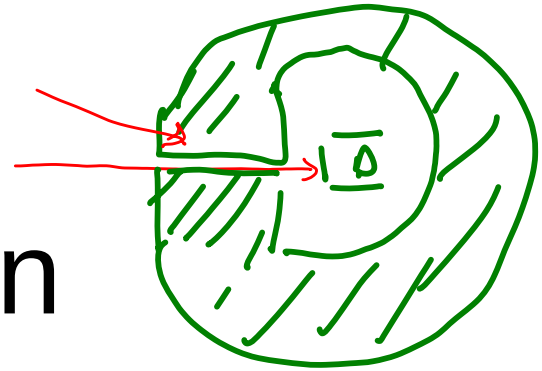
- Can have several types of interactions (all depend on energy)
  - Photoelectric effect
  - Compton Scattering
  - Pair production

Neutrons

- Interact with nuclei via nuclear forces, since they have no charge, hence they cannot interact electrostatically with electrons
- Possible reactions
  - Elastic scattering
  - Inelastic Scattering
  - radiative capture (absorption)
  - (n, 2n)
  - fission
  - n, charged particle


U
electrostatic
r
nuclear

Photon Attenuation

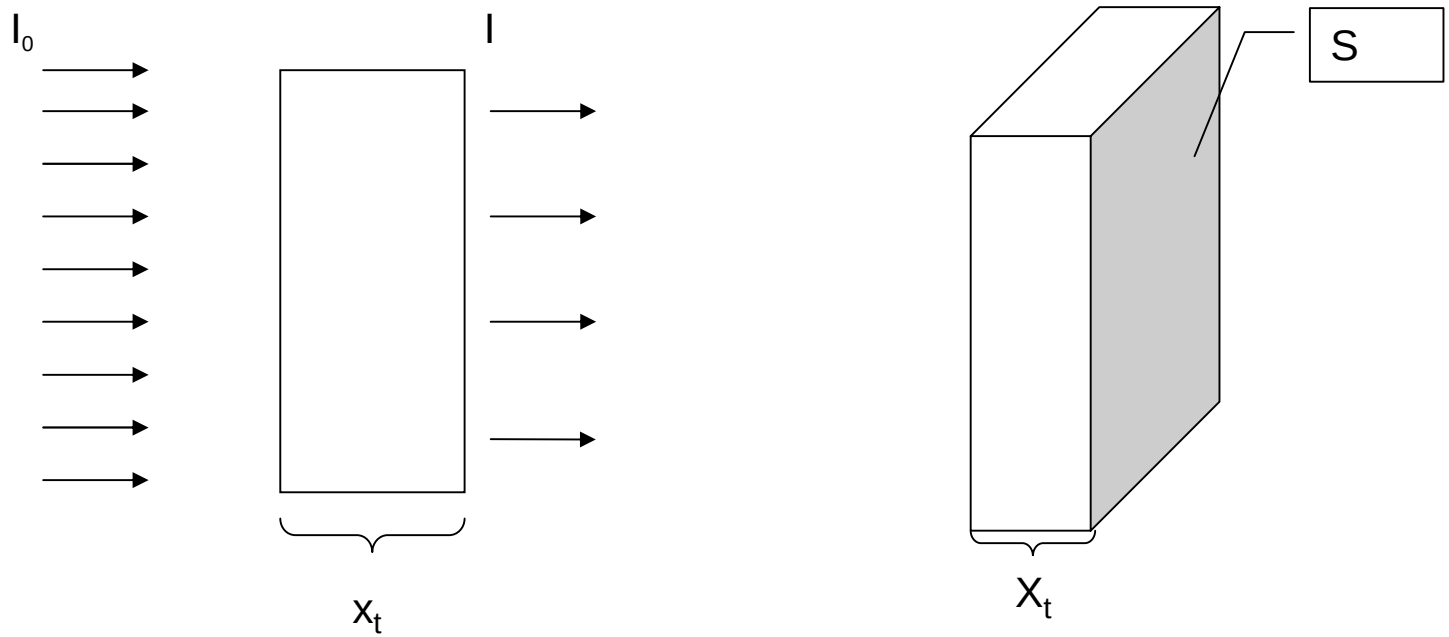


- Attenuation of a collimated beam
  - Consider a beam of photons of intensity I_0 that hits a target of thickness x_t, and a collimated detector that measures the intensity of the beam emerging from the target. The fact that the detector is collimated means that only the particles that have not interacted in any way are detected.
  - The intensity is defined as the number of photons that pass through a surface S per unit time per unit area.

$$I = \frac{N_p}{S \times t}$$

Photons

- Attenuation of a collimated beam



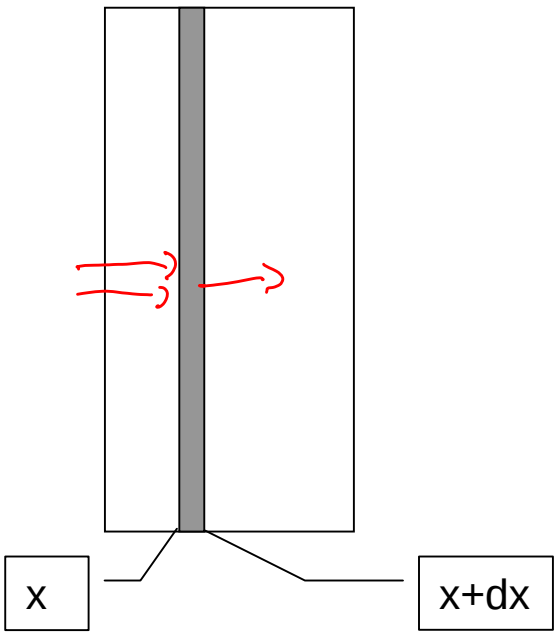
- The atom (number) density of atoms in the slab is N_a. (number of atoms per unit volume)
- The area of the material surface perpendicular to the beam is denoted by S.

Photons
Attenuation of Collimated Beam

- Consider a thin "slice" of material, of thickness dx situated at depth x in the material.
- Consider each atom can be represented as a hard ball of radius, r_a , and with a corresponding cross-section area $\sigma = \pi r_a^2$
  - Also called "microscopic cross section"
- The number of atoms in the slice is $dN = N_a S dx$
  - where N_a is the atom density
- Consider the photons to be infinitely small (points)

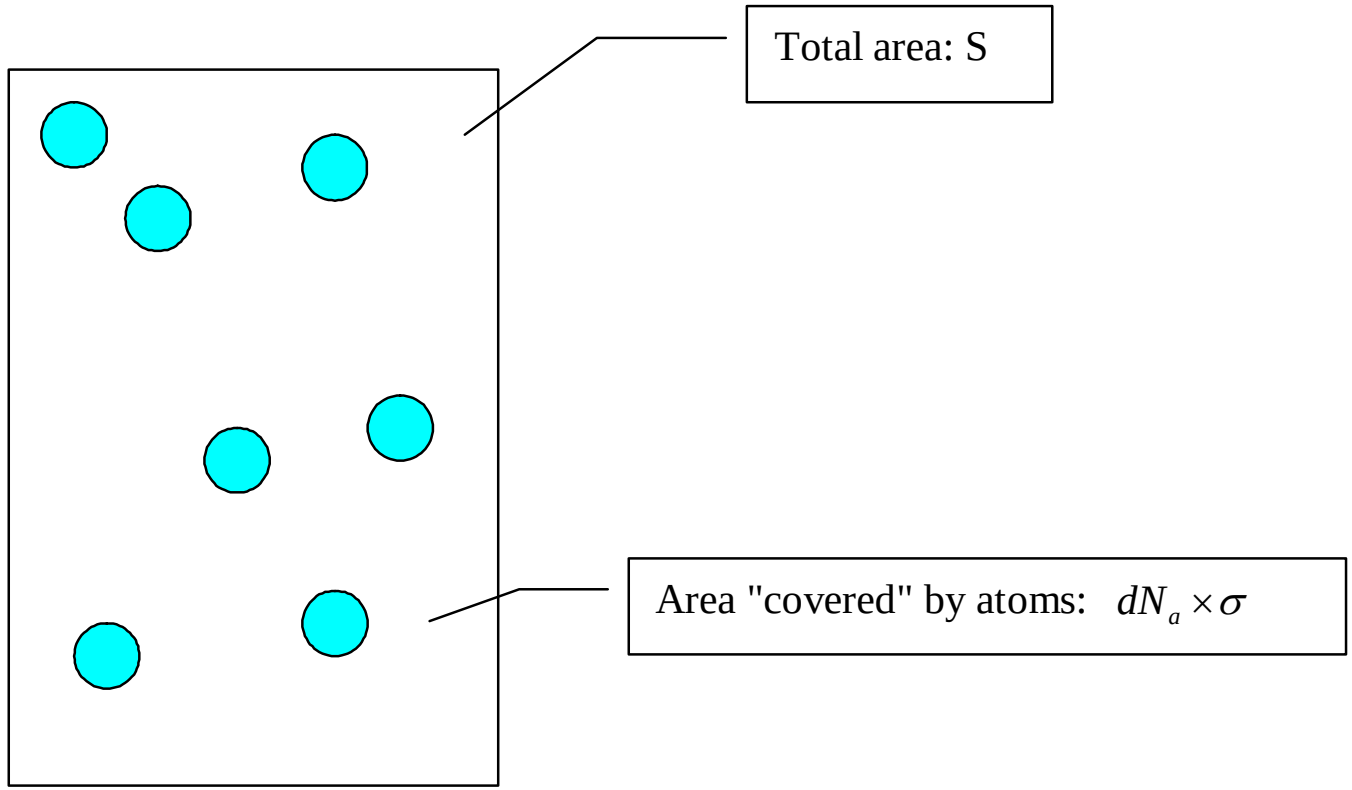
Photons
Attenuation of Collimated Beam

- Thin slice of material



Photons
Attenuation of Collimated Beam

- View of the dx slice from the photons' perspective



Photons
Attenuation of Collimated Beam

- The probability that a photon “hits” an atom equals the ratio between the area “covered” by atoms and the total area of the slice.
- Let $N_p(x)$ be the total number of photons that enter the slice over a time t

$$N_p(x) = I(x)St$$

- Let $N_p(x+dx)$ be the total number of photons that exit the slice dx over a time t

$$N_p(x+dx) = I(x+dx)St$$

Photons
Attenuation of Collimated Beam

- The probability that a photon interacts with an atom

$$dP_{coll} = \frac{dN_a \times \sigma}{S} = \frac{N_a \times S \times dx \times \sigma}{S} = N_a \times \sigma \times dx = \mu \times dx$$

- Attenuation coefficient

$$\mu = N_a \times \sigma$$

$\frac{\mu}{\rho}$

- Number of photons that interact and are therefore removed from the beam

$$dN_p = N_p \times dP_{coll} = N_p \times \mu \times dx$$

Photons
Attenuation of Collimated Beam

- Setting up the differential equation
- Account for the fact that the number of photons that interact represent the change in the number of photons that exit the slice, with a negative sign

$$-dN_p(x) = N_p(x) \times \mu \times dx$$

$$-\frac{dN_p}{dx} = N_p \cdot \mu$$

- Solution

$$N_p(x) = N_{p_0} e^{-\mu x}$$

- N_{p0} is the number of photons entering the material at x=0 over time t

Photons
Attenuation of Collimated Beam

- Given that

$$I = \frac{N_p}{S \times t} = N_{p0} e^{-\mu t}$$

- We also have

$$I(x) = \frac{N_p(x)}{S \times t} = I_0 e^{-\mu x}$$

$$\frac{I_0}{I} = e^{\mu x}$$

$$\ln\left(\frac{I_0}{I}\right) = \mu x$$

$$\mu = \frac{\ln(I_0/I)}{x}$$

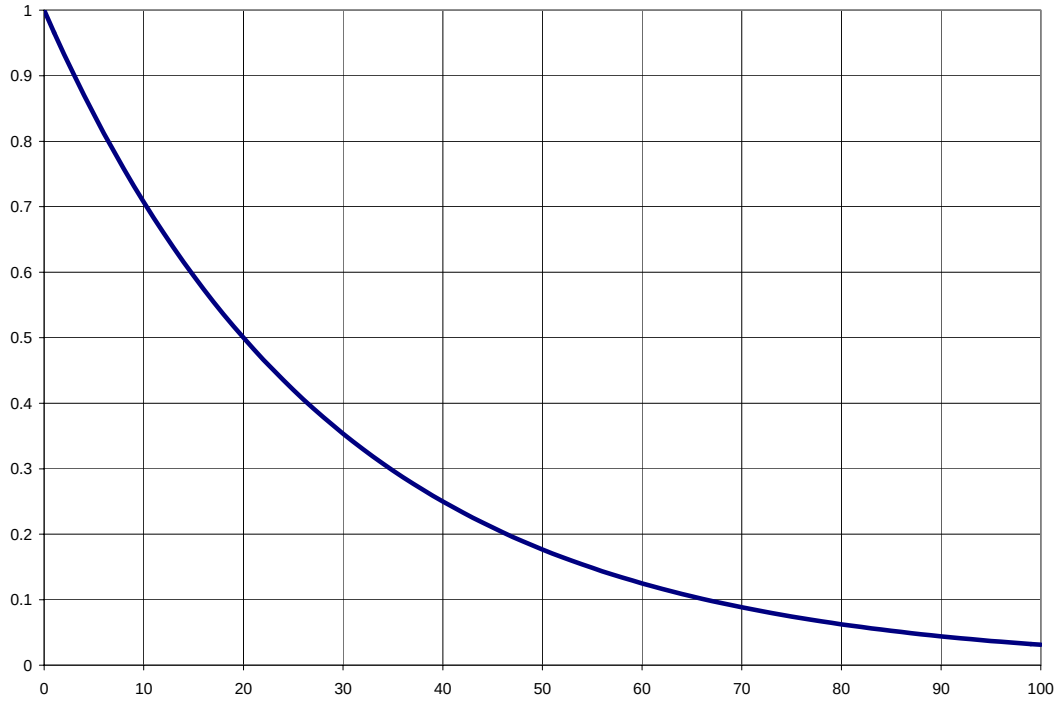
Photon Attenuation

- Exponential attenuation

$N_p(x)$

or

$I(x)$



Reaction (Collision) Rate Density

- For a thin slice of thickness dx:

$$R \equiv F = \frac{\text{number of collisions}}{\text{time} \times \text{volume}} =$$

$$= \frac{dN_p}{t \times S \times dx} = \frac{N_p(x) \times \mu \times dx}{t \times S \times dx} =$$

$$= \frac{N_p(x)}{t \times S}\mu = \frac{I(x) \times S \times \Delta t}{t \times S}\mu = I(x)\mu$$

Neutron Attenuation

- Same reasoning as for photons, but with a few specific features
  - Instead of the density of atoms previously denoted by N_a we talk about the density of nuclei, denoted simply by N. That is because neutrons interact with nuclei and not with atoms as a whole.
- The product $N\sigma$ is called ***macroscopic cross section*** and denoted by Σ

$$\Sigma = N\sigma$$

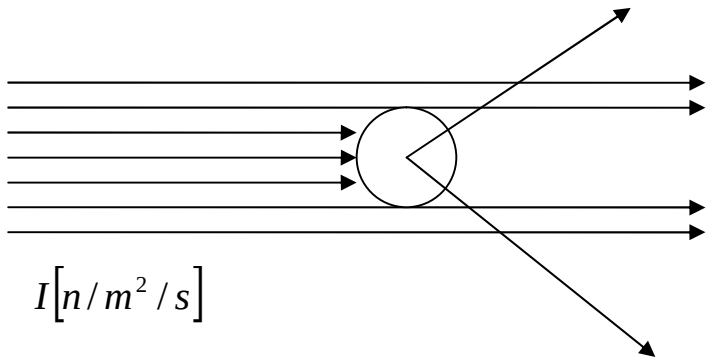
Neutron Reaction (Collision) Rate Density

- Same as for photons, but with different notations

$$R \equiv F = I\Sigma$$

Microscopic Cross Sections

Consider a single nucleus in a parallel beam of monoenergetic neutrons



Assume (for now) that scattering and absorption are the only possible reactions.

Reaction Rates (for a single nucleus)

$$R_t = R_s + R_a$$

Probability of a Certain Reaction Type

$$P_s = \frac{R_s}{R_t}$$

$$P_a = \frac{R_a}{R_t}$$

$$P_a + P_s = 1$$

Microscopic Cross sections for Individual Reactions

$$R_t = I\sigma_t$$

$$R_s = R_t P_s = I\sigma_t P_s = I\sigma_s \Rightarrow \sigma_s \equiv \sigma_t P_s$$

$$R_a = R_t P_a = I\sigma_t P_a = I\sigma_a \Rightarrow \sigma_a \equiv \sigma_t P_a$$

We can write:

$$\sigma_t = \frac{R_t}{I}$$

$$\sigma_a = \frac{R_a}{I}$$

$$\sigma_s = \frac{R_s}{I}$$

- The microscopic cross sections can hence be interpreted as the probability of interaction (with one particular nucleus), per unit incident intensity.

$$\Sigma = N \cdot \sigma$$

$$[\Sigma] = [N] \cdot [\sigma]$$

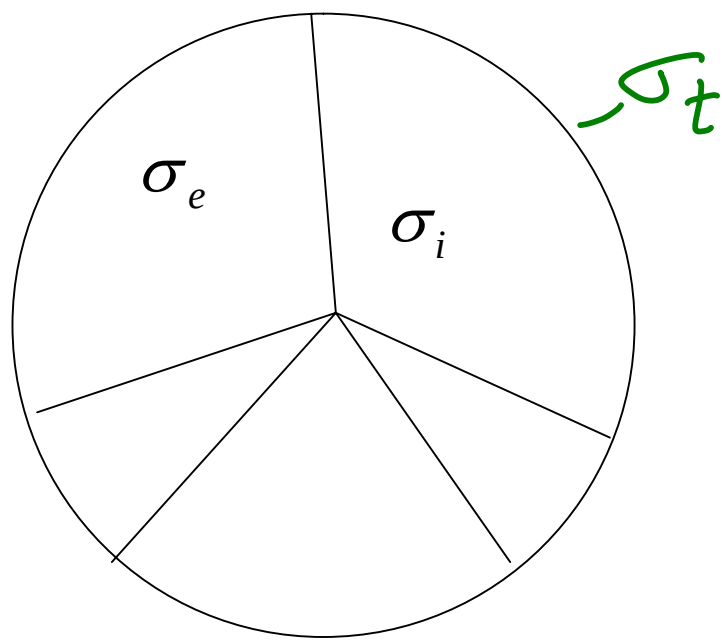
$$[\Sigma] = cm^{-3} \cdot cm^{2} = cm^{-1}$$

The sum of individual microscopic cross sections equals the total macroscopic cross section. In our simplified case

$$\sigma_t = \sigma_a + \sigma_s$$

For the general case:

$$\sigma_t = \sigma_e + \sigma_i + \sigma_\gamma + \sigma_f + \ldots$$



Energy Dependence of Microscopic Cross Sections

- So far we have assumed the beam to be monoenergetic
- In reality, neutrons can have different energies.
- The microscopic cross sections depend on the energy of the incident neutrons. ***The nucleus appears larger or smaller depending on how fast the incoming neutron is moving!***

$$\sigma_t = \sigma_t(E)$$

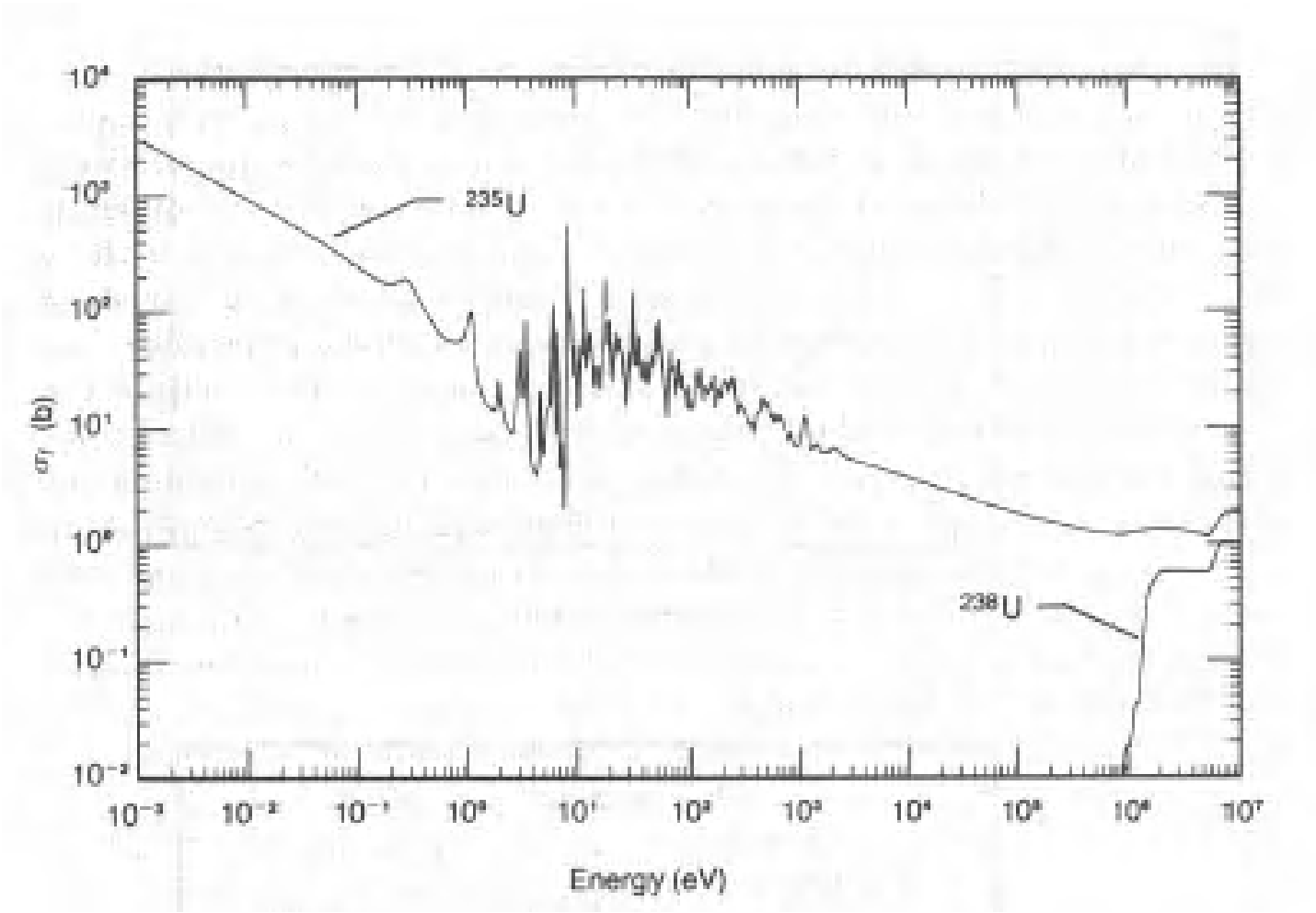
$$\sigma_a = \sigma_a(E)$$

$$\sigma_s = \sigma_s(E)$$

Reaction rate per nucleus

$$R(E) = I\sigma(E)$$

Energy Dependence of Microscopic Cross Section



$$\frac{N_{nuclei}}{V} = N$$

Volumetric Reaction (Collision) Rate (Collision Density)

$$F = \frac{R_{\text{single-nucleus}} \times N_{nuclei}}{V} = R_{\text{single-nucleus}} \times N$$

– total # of nuclei in volume V

Where N is the number density of nuclei.

$$F = I\sigma \times N = I\Sigma$$

We have thus recovered the formula obtained previously by using the attenuation of a collimated beam.

Dependence of the energy of the incident neutrons

$$F(E) = I\sigma(E) \times N = I\Sigma(E)$$

Macroscopic Cross Sections for Mixtures

Consider a mixture of nuclei with number densities N_i.bombarded by a parallel beam of monoenergetic neutrons of intensity I.

The volumetric reaction rate density for each nucleus type i is:

$$F_i = I\sigma_i \times N_i = I\Sigma_i$$

cm^{-3}

The total reaction rate density is:

$$F = \sum_i F_i = \sum_i I\sigma_i \times N_i = I\underbrace{\sum_i \overbrace{\sigma_i \times N_i}^{\Sigma_i}}_{\Sigma} =$$

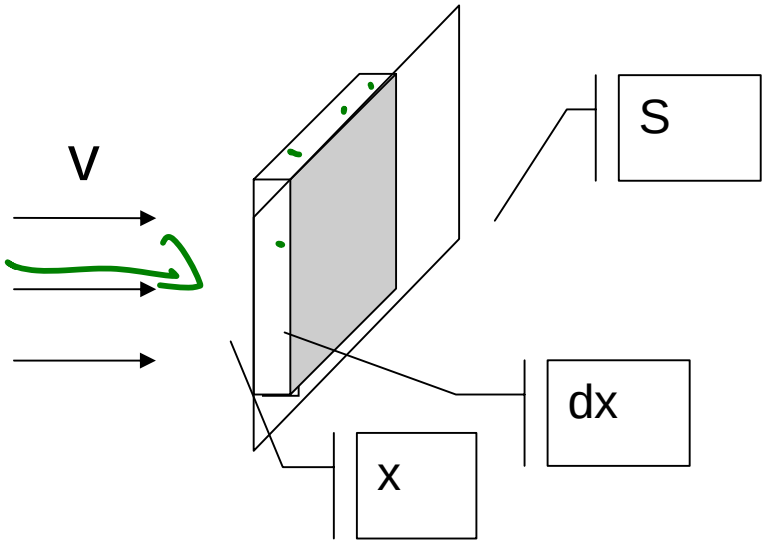
$$= I\sum_i \Sigma_i = I\Sigma$$

The total macroscopic cross section equals the sum of the (partial) macroscopic cross sections for each nuclear species

$$\Sigma = \sum_i \Sigma_i = \sum_i N_i \sigma_i$$

Neutron Beam Intensity

- Let n(x) be the neutron density (neutrons/cc)
- Consider monoenergetic neutrons (All have the same speed)
- Let v be the speed of neutrons.



$dt = \frac{dx}{v}$

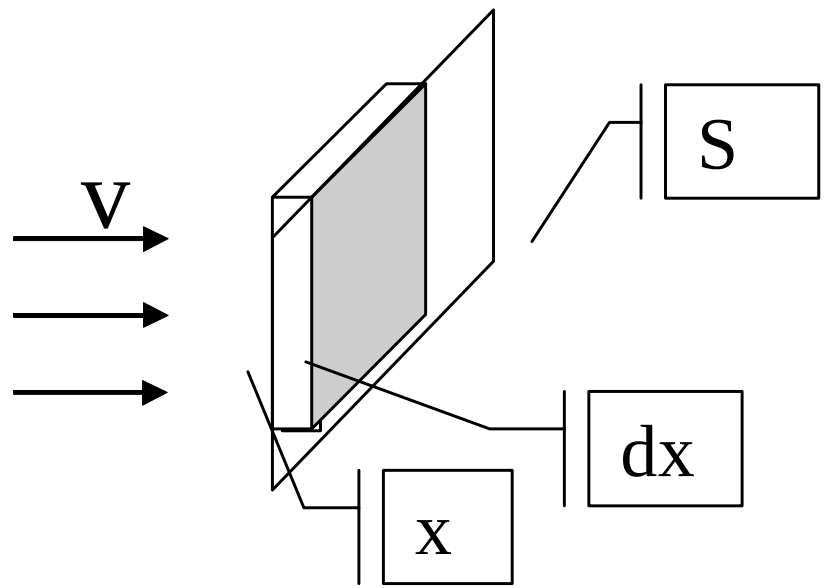
Neutron Beam Intensity

- Consider a thin “slice” of beam, of thickness dx, that crosses surface S.
- There are $dN_n = nSdx$ neutrons in this slice.

Neutron Beam Intensity

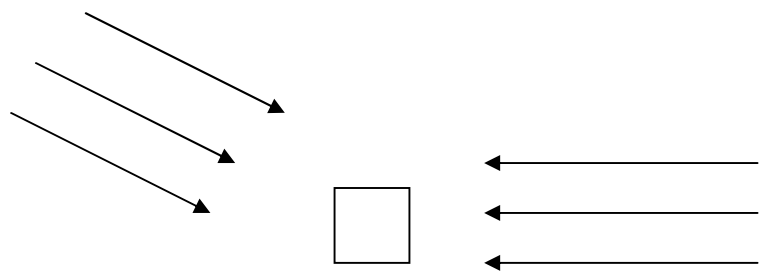
- It takes the neutrons in the slice time $dt = \frac{dx}{\mathrm{v}}$ to cross surface S.
- The beam intensity is therefore:

$$I = \frac{dN_n}{Sdt} = \frac{nSdx}{S\frac{dx}{\mathrm{v}}} = n\mathrm{v}$$



Neutron Flux

What happens if we have a small piece of material bombarded by two beams of monoenergetic neutrons (both having the same energy)?



Reaction (collision) Rate

$$F = \underbrace{I_1\sigma \times N}_{F_1} + \overbrace{I_2\sigma \times N}^{F_2} = \left(I_1 + I_2\right)\Sigma =$$

$$= \left(n_1 \mathrm{v} + n_2 \mathrm{v}\right)\Sigma = n\mathrm{v}\Sigma = \Phi\Sigma$$

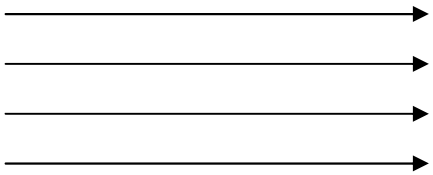
$n = n_1 + n_2$

Neutron flux for monoenergetic neutrons:

$$\Phi = n\mathrm{v}$$

Neutron Current

Consider a beam of monoenergetic neutrons



The intensity is given by:

$$I = n\mathrm{v}$$

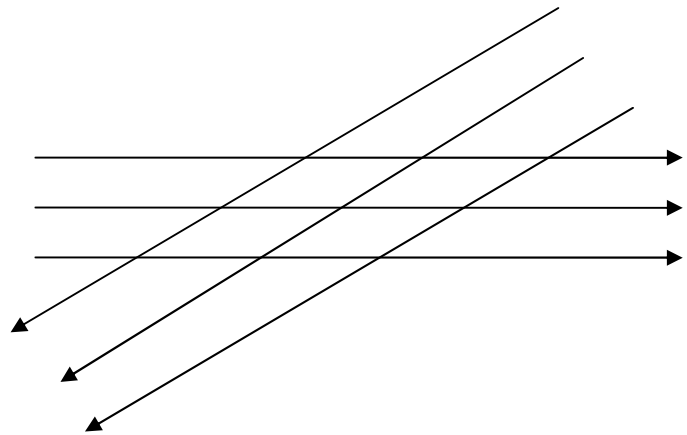
The flux is given by

$$\Phi = n\mathrm{v}$$

The current is a vectorial quantity:

$$\vec{J} = n\vec{\mathrm{v}}$$

For two intersecting beams of different-energy neutrons:



Neutron Flux

$$\Phi = n_1 \mathrm{v}_1 + n_2 \mathrm{v}_2 = \Phi_1 + \Phi_2$$

Neutron Current

$$\vec{J} = n_1 \vec{\mathrm{v}}_1 + n_2 \vec{\mathrm{v}}_2 = \vec{J}_1 + \vec{J}_2$$

$\vec{J}_1$ $\vec{J}_2$

For many intersecting beams:

$$\Phi = \sum_i n_i \mathrm{v}_\mathrm{i} = \sum_i \Phi_i$$

$$\vec{J} = \sum_i n_i \vec{\mathrm{v}}_\mathrm{i} = \sum_i \vec{J}_i$$

Usefulness of Neutron Flux

Consider a small sample of material placed at the intersection of several beams of neutrons.

The total collision density in the sample is equal to the sum of the collision densities due to the neutrons in each beam.

$$F = \sum_i F_i$$

We rewrite the expression for the total collision density

$$F = \sum_i F_i = \sum_i \Sigma \times \Phi_i =$$

$$= \Sigma \sum_i \Phi_i = \Sigma \Phi$$

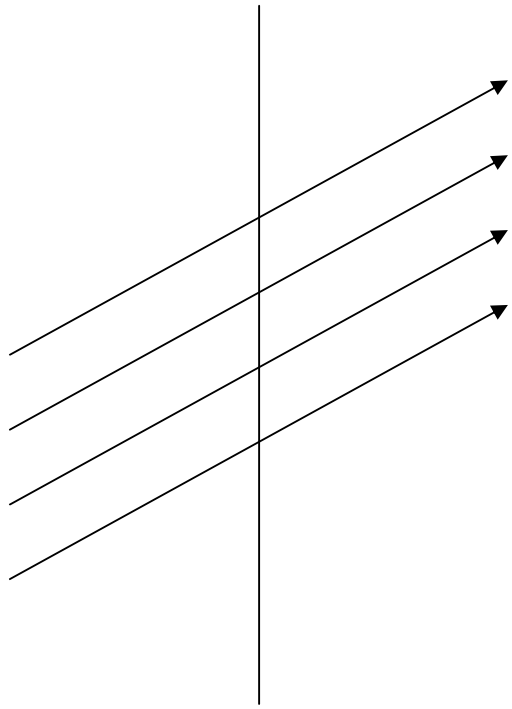
So:

$$F = \Sigma \Phi$$

Regardless of how many beams we have (one or more).

Usefulness of Neutron Current

Consider a monoenergetic neutron beam that intersects a plane surface.

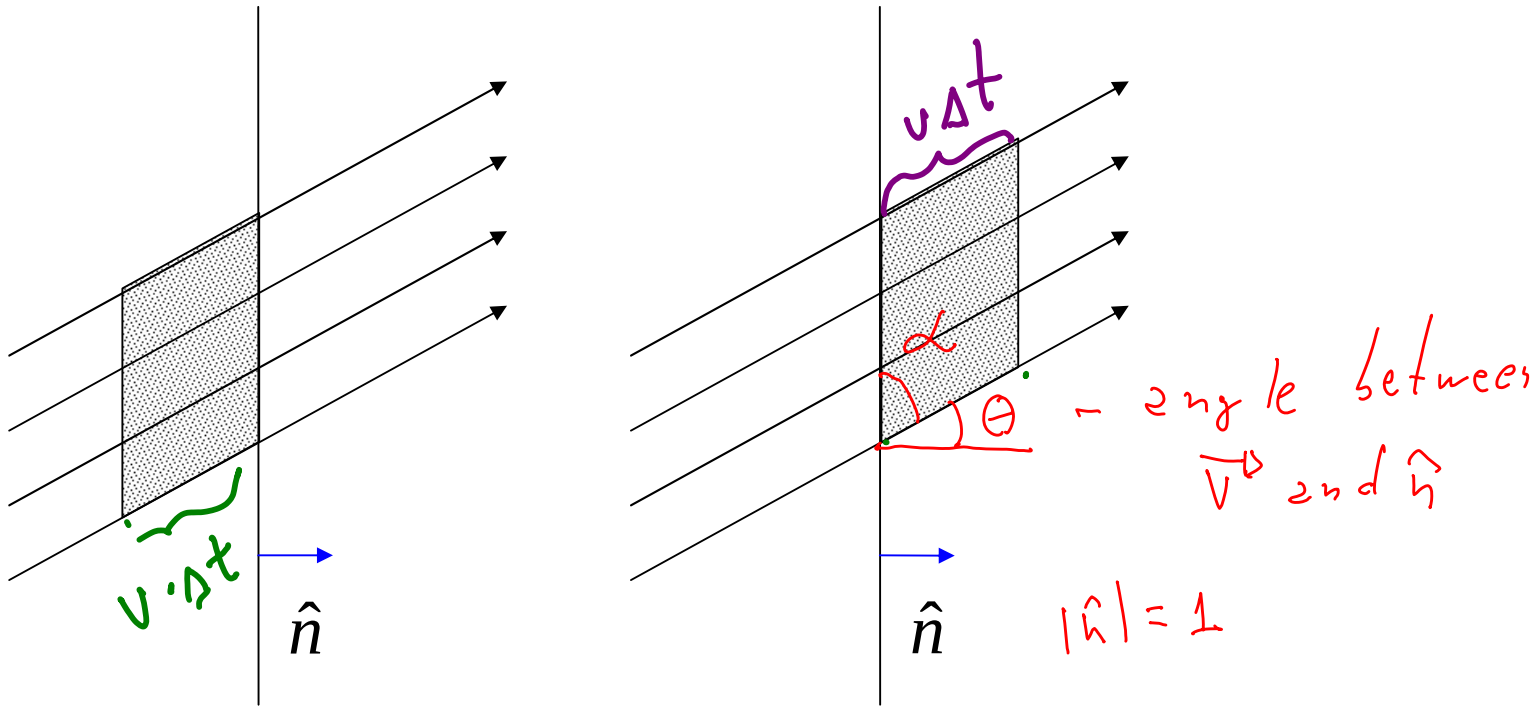


We want to determine the rate at which neutrons cross this surface. Per unit area.

$$R = \frac{\Delta N}{S \Delta t}$$

Before

After (Δt elapsed)



This is a side view. Imagine the figure to be 1 cm thick.

The rate at which neutrons cross the surface in Δt is given by the neutrons in the marked region.

$\sin\alpha$

$$R = \frac{\Delta N}{S\Delta t} = \frac{n\Delta V}{S\Delta t} = \frac{nS\mathrm{v}\Delta t\cos\theta}{S\Delta t} =$$

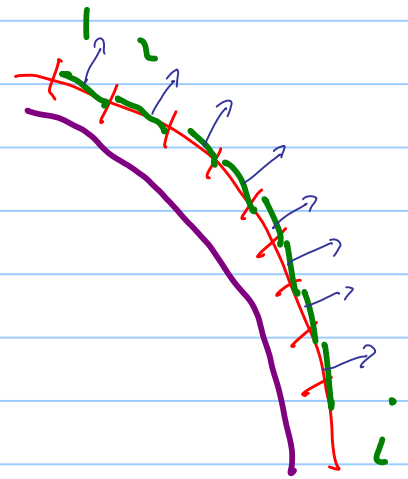
$$= n\mathrm{v}\cos\theta = n\vec{\mathrm{v}}\cdot\vec{n} = (n\vec{\mathrm{v}})\cdot\vec{n} = \vec{J}\cdot\vec{n}$$

$\vec{a}\cdot\vec{b} = a\cdot b\cdot\cos\theta$

$b=1$

$a\cdot\cos\theta$

Rate at which neutrons cross the entire surfaces



$$R_c = \sum_i \vec{J}_i \cdot \hat{n}_i \cdot dS_i$$

$$R = \int_S \vec{J}\, \vec{n}\, ds = \int_S \vec{J}\, d\vec{s}$$

$$\vec{n}\, ds \equiv d\vec{s}$$

Multiple Beams

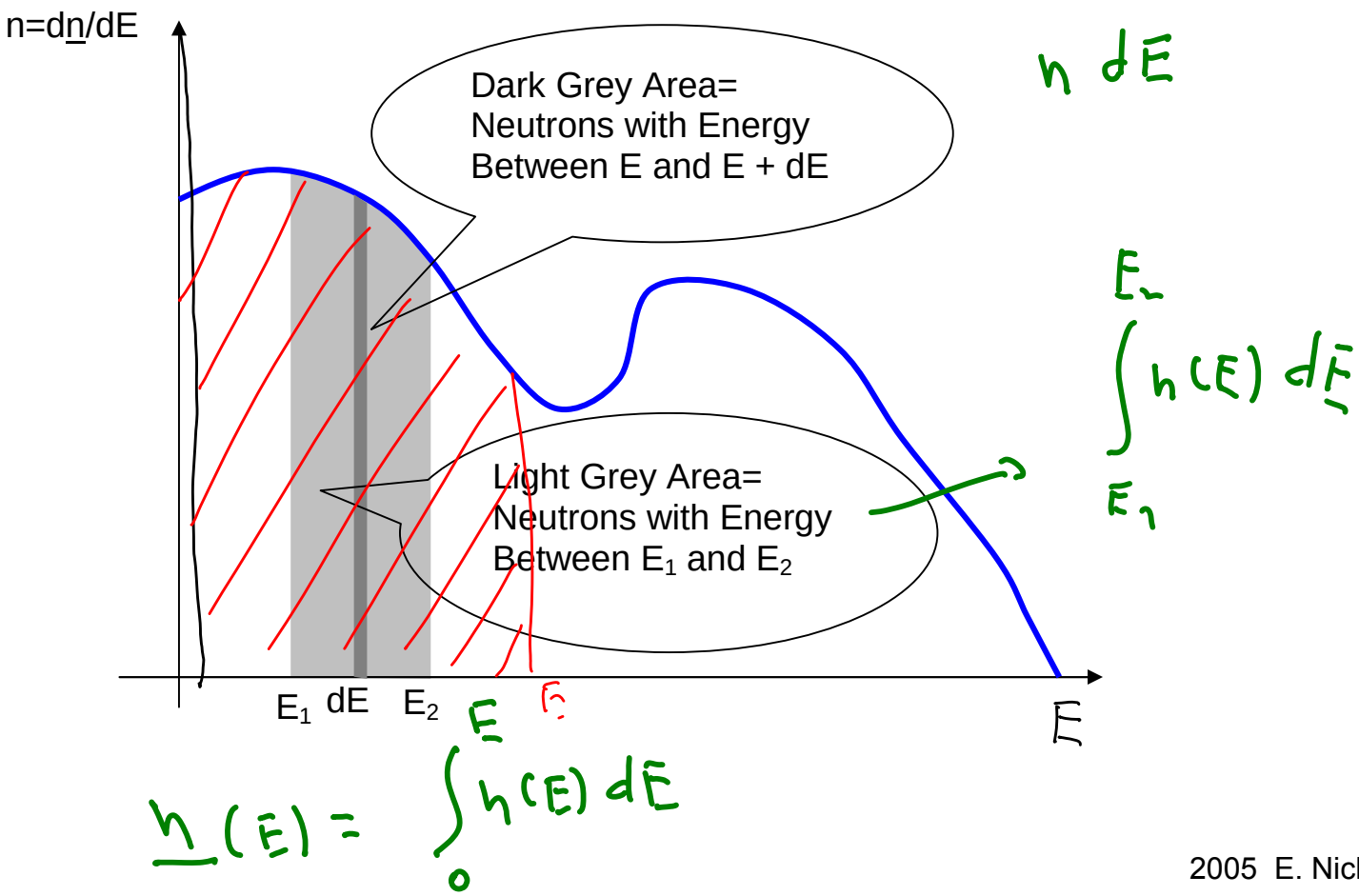
The number of neutrons crossing the surface per unit time per unit area is the sum of the neutrons in each beam that cross the surface per unit time per unit area.

$$R = \sum_i R_i = \sum_i \vec{J}_i \vec{n} =$$

$$= \left(\sum_i \vec{J}_i \right) \vec{n} = \vec{J} \vec{n}$$

Energy Dependence of Neutron Density, Flux and Current

Consider now a parallel beam that has neutrons of different energies (speeds).



(volumetric) density of neutrons with energy less or equal to E.

$$\underline{n}(E)$$

Neutron density spectrum (Energy-dependent neutron density)

$$n(E) = \frac{d\underline{n}(E)}{dE}$$

(volumetric) density of neutrons with energy between E and E+dE.

$$n(E)dE$$

$$n(E) = \frac{\#\text{ neutrons in } dV \text{ with energy between } E, \text{ and } E+dE}{dV\, dE}$$

Beam intensity for neutrons with energy between E and E+dE

$$dI(E) = n(E)\mathrm{v}(E)dE$$

Energy-dependent beam intensity (Beam Intensity Spectrum)

$$I(E) = n(E)\mathrm{v}(E)$$

Energy-dependent Flux (Flux spectrum)

$$\Phi(E) = n(E)\mathrm{v}(E)$$

Energy Dependent Current (Current Spectrum)

$$\vec{J}(E) = n(E)\vec{\mathrm{v}}(E)$$

Total Reaction Rate for Reaction x

$$R_x = \int n(E)\mathrm{v}(E)\Sigma_x(E)dE = \int_0^\infty \Phi(E)\Sigma_x(E)dE$$

Subscript x can stand for total collisions, or just absorption, or elastic scattering, etc.

Elastic

Energy Loss in Scattering Collisions

Read section from textbook

Important Results

$$(E')_{\min} = \left(\frac{A-1}{A+1}\right)^2 E = \alpha E$$

The energy of the scattered neutron is always lower than the energy of the incident neutron.

The slowing down of neutrons by elastic collisions is called **moderation**.

The lower the mass number of the target nucleus, the lower the minimum energy of the scattered neutron.

For ^{235}U, we have:

$$(E')_{\min} = \left(\frac{A-1}{A+1}\right)^2 E = \left(\frac{234}{236}\right)^2 E = 0.98E$$

Not an effective moderator.

For ^{1}H we have:

$$(E')_{\min} = \left(\frac{A-1}{A+1}\right)^2 E = \left(\frac{0}{2}\right)^2 E = 0$$

Effective moderator. A neutron can lose all its energy in an elastic collision with a Hydrogen nucleus (proton).

Hydrogen is present in water. Water is used as moderator.

The bonds with O are very weak compared to the forces entailed in the elastic collision. The H nucleus can be considered free.

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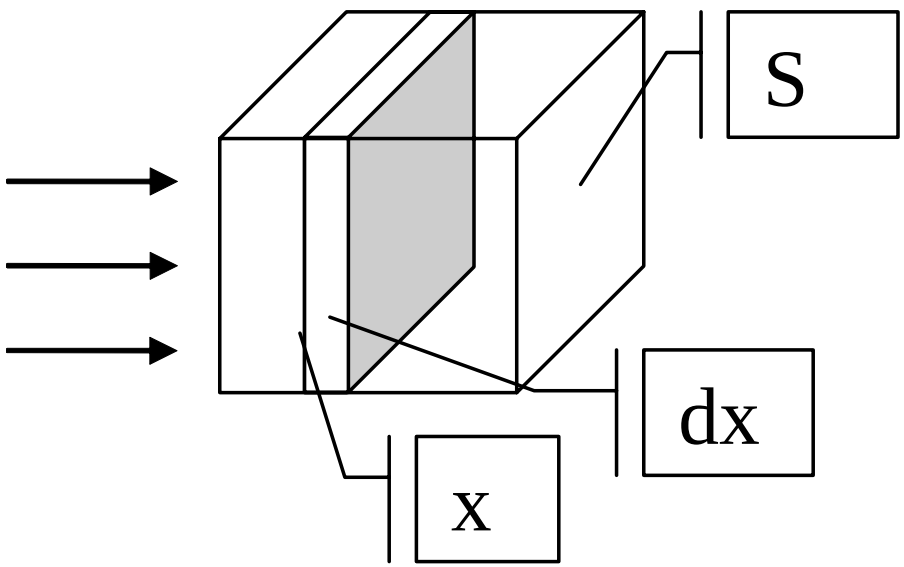
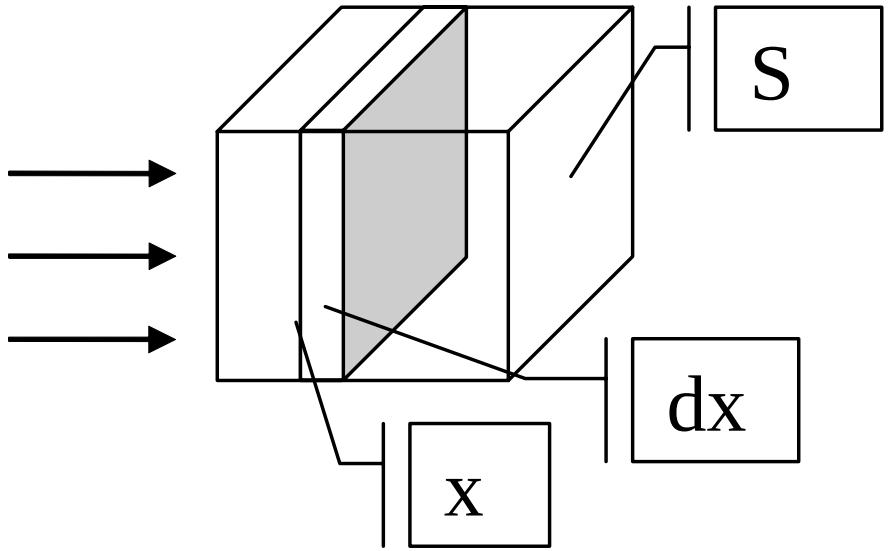
Neutron Attenuation Revisited

Parallel beam of monoenergetic neutrons

For such a beam

$$I \equiv \Phi \equiv J \quad = nv$$





Neutron balance in the volume of thickness dx

$$\underbrace{I(x)S}_{\text{neutrons entering the volume}} - \underbrace{I(x+dx)S}_{\text{neutrons exiting the volume}} = \underbrace{\Sigma(x)\Phi(x)Sdx}_{\text{neutrons colliding (reacting) in the volume}}$$

The neutron balance equation can be rewritten:

$$I(x)S - I(x+dx)S = \Sigma(x)I(x)Sdx$$

Dividing by Sdx on both sides we obtain

$$\frac{I(x) - I(x+dx)}{dx} = \Sigma(x)I(x)$$

Equivalent to:

$$-\frac{dI(x)}{dx} = \Sigma(x)I(x) \Leftrightarrow \frac{dI(x)}{dx} = -\Sigma(x)I(x)$$

If the macroscopic cross section is constant, then:

$$\frac{dI(x)}{dx} = -\Sigma I(x)$$

Which can be integrated to obtain:

$$I(x) = I(0)e^{-\Sigma x}$$

$$I(x) \quad I_0 e^{-\int_0^x \Sigma(x')dx'}$$

Exactly what we obtained before, by using a different kind of reasoning.

Moral: If assumptions are right and reasoning correct, the results are the same regardless of the method used.

Mean Free Path

Neutrons that react (collide) between x and x+dx have had a "free path" of length x.

To find the mean free path, we need to average over all the neutrons that interact from $x=0$ to $x=\infty$.

rate of interaction at x

$$\lambda = \frac{\int_0^\infty x I(x)\Sigma dx}{\int_0^\infty I(x)\Sigma dx} = \frac{\int_0^\infty x I(0)e^{-\Sigma x}\Sigma dx}{\int_0^\infty I(0)e^{-\Sigma x}\Sigma dx} =$$

$$\frac{I(0)\Sigma\int_0^\infty x e^{-\Sigma x}dx}{I(0)\Sigma\int_0^\infty e^{-\Sigma x}dx} = \frac{\int_0^\infty x e^{-\Sigma x}dx}{\int_0^\infty e^{-\Sigma x}dx}$$

$\int f g' dx = f g - \int f' g dx$

The numerator is integrated by parts to give

$$\int_0^\infty \overset{f}{x} \overset{g'}{e^{-\Sigma x}} dx = \int_0^\infty x \left(-\frac{e^{-\Sigma x}}{\Sigma} \right)' dx =$$

$$= x \left(-\frac{e^{-\Sigma x}}{\Sigma} \right) \Bigg|_0^\infty - \int_0^\infty \left(-\frac{e^{-\Sigma x}}{\Sigma} \right) dx =$$

$$= \left(-\frac{e^{-\Sigma x}}{\Sigma^2} \right) \Bigg|_0^\infty = \frac{1}{\Sigma^2}$$

The denominator integrates as:

$$\int_0^\infty e^{-\Sigma x} dx = \left(-\frac{e^{-\Sigma x}}{\Sigma} \right) \Bigg|_0^\infty = \frac{1}{\Sigma}$$

It follows that:

$$\lambda = \frac{\frac{1}{\Sigma^2}}{\frac{1}{\Sigma}} = \frac{1}{\Sigma}$$

Fission

Fission

$$ {}_{0}^{1}n + {}_{Z_X}^{A_X}X \rightarrow {}_{Z_A}^{A_A}A + {}_{Z_B}^{A_B}B + \tilde{\nu}\,{}_{0}^{1}n + \tilde{\mu}\,{}_{-1}^{0}e + \gamma $$

$$ \tilde{\nu} = 2 or 3 $$

A & B = Fission Products (Fission Fragments)

Conservation Laws

Number of nucleons

$$A_X + A_n = A_A + A_B + \tilde{\nu}$$

$$A_X + 1 = A_A + A_B + \tilde{\nu}$$

Charge

$$Z_X = Z_A + Z_B - \tilde{\mu}$$

Energy

$$c^2\left[M(n)+M(X)\right]=c^2\left[M(A)+M(B)\right]+$$
$$+\tilde{\nu}c^2M(n)+c^2\tilde{\mu}M(e)+E_\gamma$$

M is the (total) relativistic mass

Using the rest mass and kinetic energy E, we have:

$$c^2\left[M_0(n)+M_0(X)\right]+E_n=$$
$$=c^2\left[M_0(A)+M_0(B)\right]+$$
$$+\tilde{\nu}c^2M_0(n)+c^2\tilde{\mu}M_0(e)+$$
$$+E_\gamma+E_A+E_B+E_{\tilde{\nu}n}+E_{\tilde{\mu}e}$$

M_0 is the rest mass

The above can be rewritten using the Q value:

$$c^2\left[M_0(n)+M_0(X)\right]+E_n=$$
$$=c^2\left[M_0(A)+M_0(B)\right]+$$
$$+\tilde{\nu}c^2M_0(n)+c^2\tilde{\mu}M_0(e)+Q+E_n$$

For fission, Q is approximately 200 MeV

Distribution of Fission Energy

Form	Emitted Energy, Me V	Recoverable Energy, Me V
Fission fragments	168	168
Fission-product decay		
β-rays	8	8
γ-rays	7	7
neutrinos	12	—
Prompt γ-rays	7	7
Fission neutrons (kinetic energy)	5	5
Capture γ-rays	—	3–12
Total	207	198–207

Most energy is taken by fission fragments and deposited locally.

Fission Mechanism (simplified)

Fission occurs through the formatic f a compound nucleus which, in turn, can decay very rapidly in several different ways.

$$ {}_{0}^{1}n + {}_{Z_X}^{A_X}X \rightarrow {}_{Z_X}^{A_X+1}X $$

$$ {}_{Z_X}^{A_X+1}X \begin{cases} \rightarrow A' + B' + \gamma & (\text{mode } 1 - \text{prompt } \gamma) \\ \rightarrow A' + B' + \tilde{\nu}_p n & (\text{mode } 2 - \text{prompt n}) \end{cases} $$

$$ \nu_p = 2 - 3 $$

Both A' and B' can be stable or further decay in several possible modes:

$$A' \begin{cases} \longrightarrow A & (A' \text{ was stable}) \\ \xrightarrow{\lambda} A + \gamma & (\text{mode } 1) \\ \xrightarrow{\lambda} A + {}^{0}_{-1}\beta_{HE} & (\text{mode } 3) \\ \xrightarrow{\lambda} A'' + {}^{0}_{-1}\beta_{LE} & (\text{mode } 4) \\ \quad \downarrow \text{ (fast)} & \\ \quad A + n & (\text{delayed neutron}) \end{cases}$$

If A' decays according to mode 4, then it is called a *precursor* and A'' is called an *emitter*.

We cannot predict in advance which nuclei will be precursors, but we can predict, on the average how many will do so. This number is equal to the number of delayed neutrons emitted, called the *delayed neutron yield*.

$$\nu_d = \frac{\#\text{ of delayed neutrons}}{\#\text{ of fissions}}$$

We cannot predict how many prompt neutrons will be emitted in each reaction either. But we can predict how many will be produced on the average. This is called the *prompt neutron yield*.

$$\nu_p = \frac{\#\text{ of prompt neutrons}}{\#\text{ of fissions}}$$

On the average, the fission reaction can be written:

$$n + X \rightarrow A + B + \nu_p n_p + \nu_d n_d + \mu e + \gamma$$

The *total neutron yield* is defined as:

$$\nu = \nu_d + \nu_p$$

The delayed neutron fraction is:

$$\beta = \frac{\nu_d}{\nu}$$

Delayed Neutrons

Are emitted by emitters which result from the beta decay of precursors.

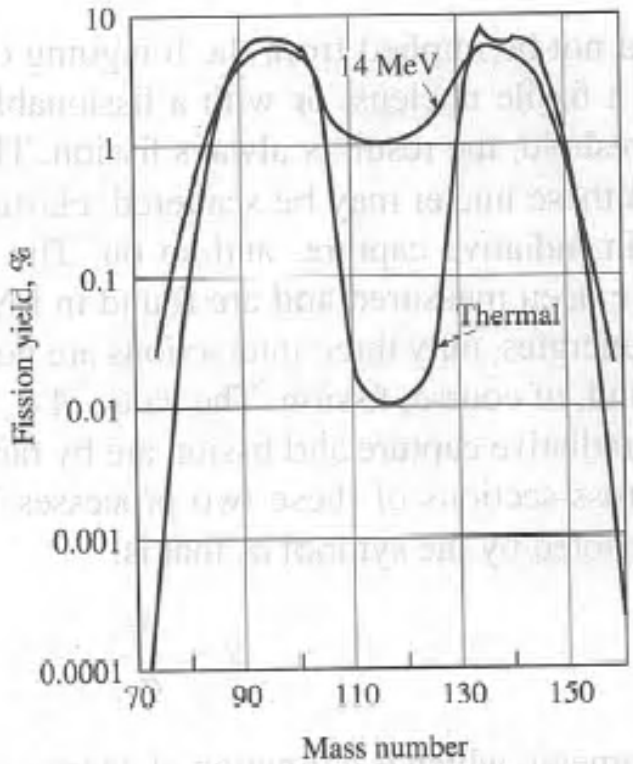
There are 6 precursor (delayed neutron) groups, based on their half-life.

TABLE 3.5 DELAYED NEUTRON DATA FOR THERMAL FISSION IN ^{235}U*

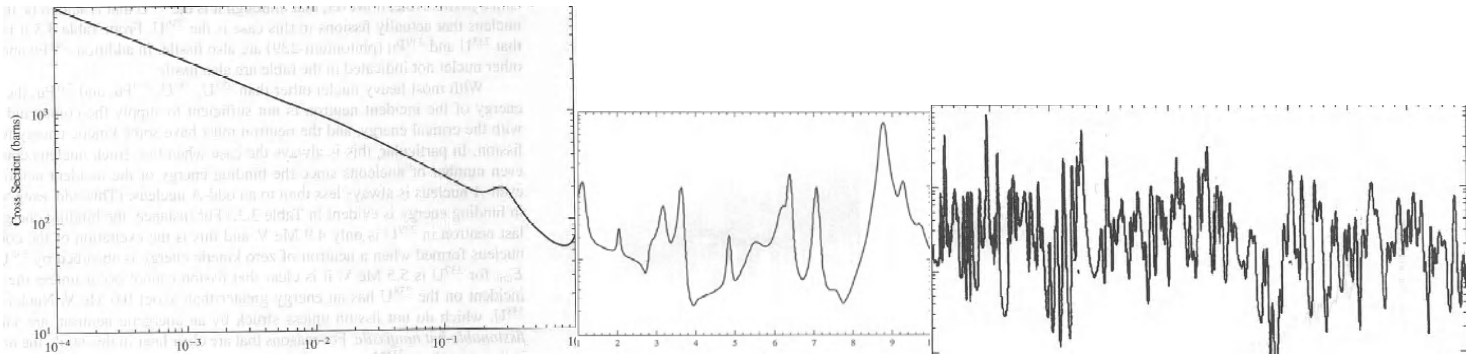
Group	Half-Life (sec)	Decay Constant (l_i, sec^{-1})	Energy (ke V)	Yield, Neutrons per Fission	Fraction (β_i)
1	55.72	0.0124	250	0.00052	0.000215
2	22.72	0.0305	560	0.00346	0.001424
3	6.22	0.111	405	0.00310	0.001274
4	2.30	0.301	450	0.00624	0.002568
5	0.610	1.14	—	0.00182	0.000748
6	0.230	3.01	—	0.00066	0.000273
				Total yield: 0.0158	
				Total delayed fraction (β): 0.0065	

Fission Products (Heavy Nuclei)

Mass is distributed asymmetrically.

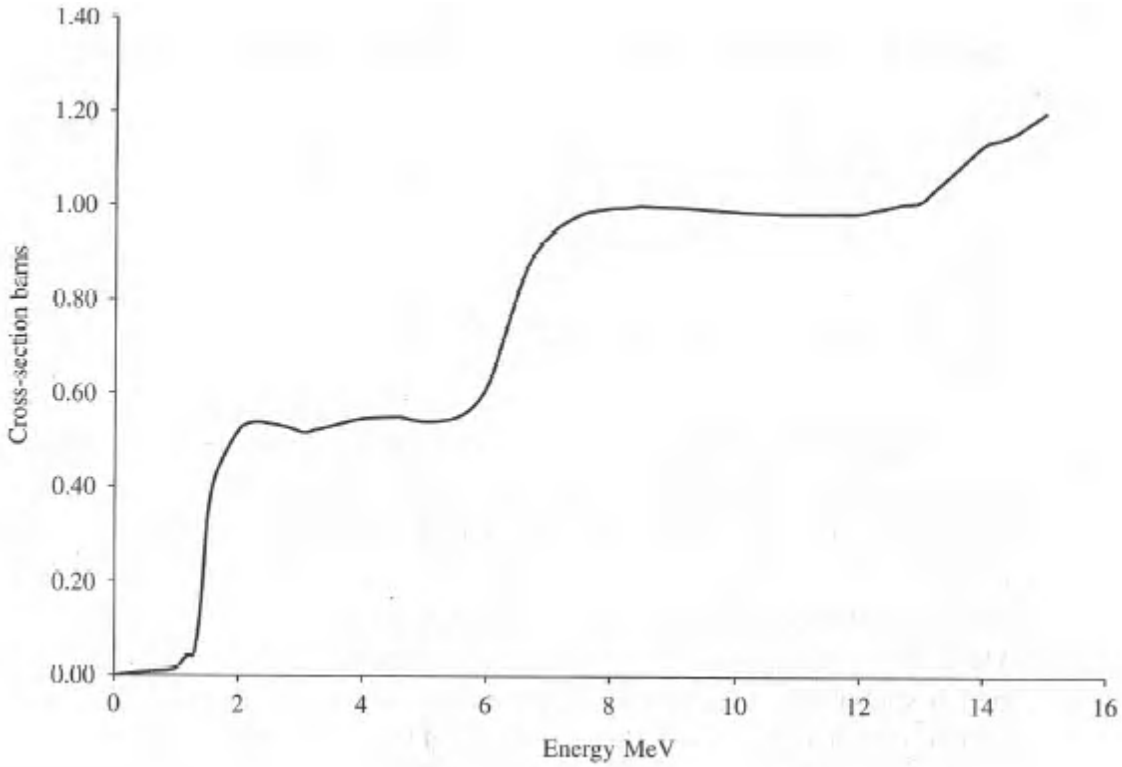


Energy Dependence of Fission Cross Section for ^{235}U



^{235}U is *fissile*, i.e. undergoes fission with near-zero energy neutrons with high probability.

Energy Dependence of Fission Cross Section for ^{238}U



^{238}U is *fissionable*, but not fissile, i.e. it can undergo fission, but with higher energy neutrons and with lower probability.

Energy Spectrum of Fission Neutrons

Normalized Energy Spectrum

$$\chi(E) \equiv \frac{n(E)}{n_t} = \frac{n(E)}{\int_0^\infty n(E)dE}$$

It follows that:

$$\int_0^\infty \chi(E)dE = \int_0^\infty \frac{n(E)}{\int_0^\infty n(E')dE'}dE = \frac{1}{\int_0^\infty n(E')dE'}\int_0^\infty n(E)dE = 1$$

Energy Spectrum of Fission Neutrons

Prompt-neutron spectrum (E_{avg}=2MeV)

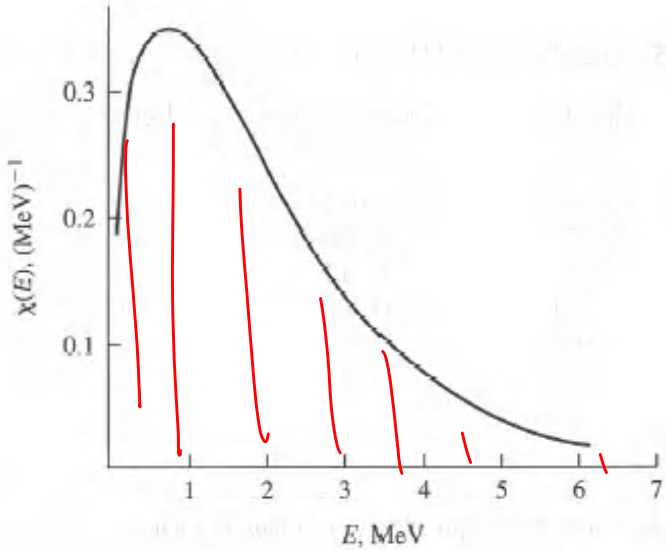


Figure 3.14 The prompt neutron spectrum.

Delayed-neutron energies are slightly lower.

Facts

- Fission neutron energies are much higher than thermal energies (0.025 eV), so they are not appropriate for efficient fission in fissile materials.
- To achieve fission efficiently, the neutrons need to be slowed down (their energy needs to be reduced). This process is called *moderation*. It is achieved by elastic collision with light nuclei (usually Hydrogen or Deuterium)
- Reactors that use thermal neutrons for fission are called *Thermal Reactors*.

Fission-Related Parameters

Capture-to-fission ratio

$$\alpha = \frac{\sigma_\gamma}{\sigma_f}$$

Number of neutrons released per absorbed neutron.

$$\eta = \nu \frac{\sigma_f}{\sigma_a}$$

For mixtures of fissile **and** non-fissile elements:

$$\eta = \frac{1}{\Sigma_a} \sum_i \nu_i \Sigma_{fi}$$

Nuclear Fission Reactors

Thermal Reactor Components

- ***Fuel*** **-** consists of nuclei that fission liberating energy
- ***Moderator*** **-** slows down fast neutrons resulting from fission to thermal energies so they can fission fuel nuclei
- ***Coolant*** **-** removes the heat

The three can be:

- mixed together $\rightarrow$ Homogeneous Reactor
- separated $\rightarrow$ Heterogeneous Reactor

Most reactors are **heterogeneous**.

Power Reactors

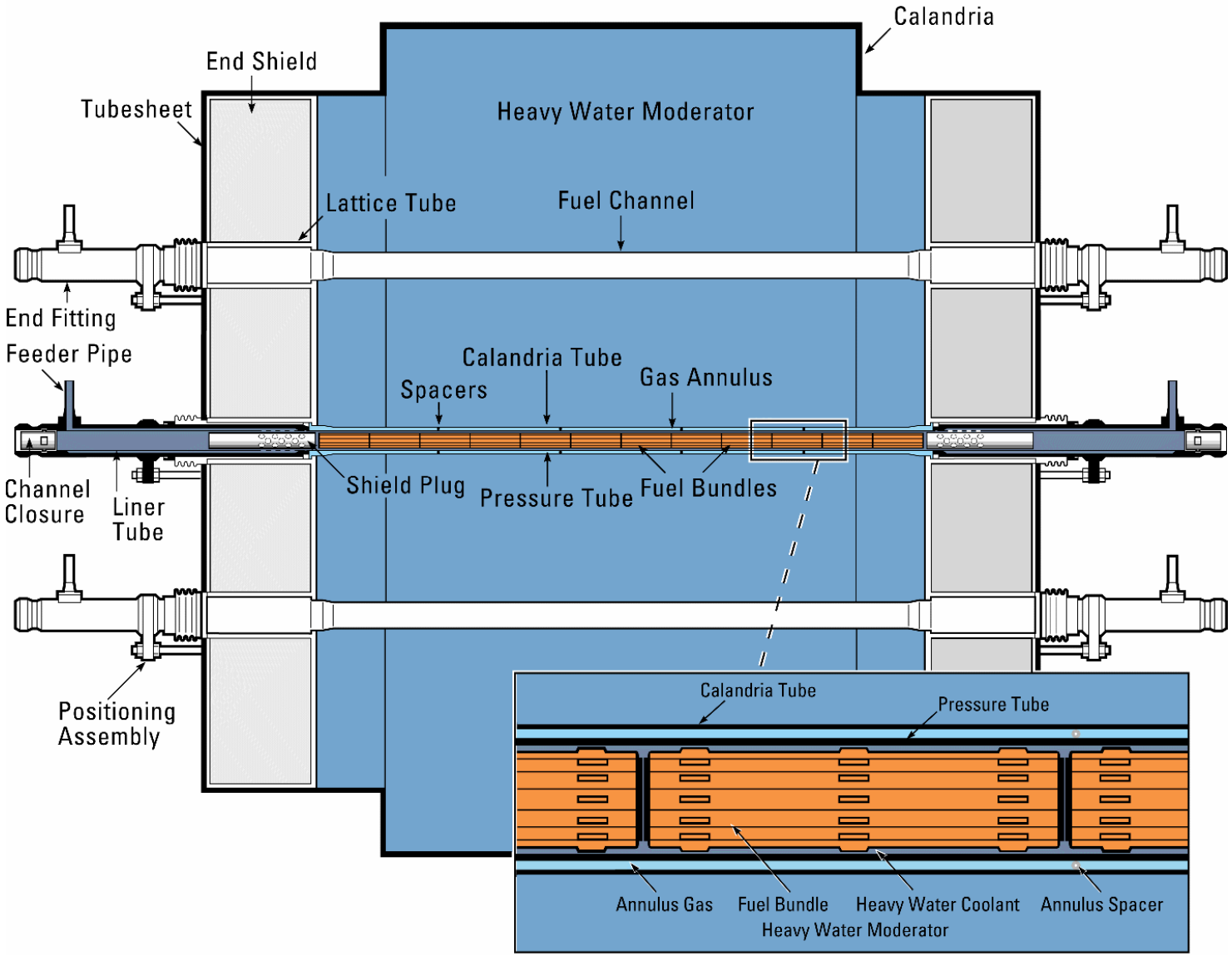
- **Pressurized Water Reactors**
- ***Pressurized Heavy-Water Reactors (CANDU)***
- Gas-Cooled Reactors
- Other

CANDU Reactors

- Heterogeneous

- *Fuel:* Natural Uranium Oxide
  - (UO_2 0.7% ^{235}U, 99.3% ^{238}U)
- *Coolant:* Heavy Water (D_2O)
- *Moderator:* Heavy Water (D_2O)

CANDU Reactor Schematic



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CANDU Reactor - How it Works

- Fissions take place in the fuel
- Most energy from fissions is taken up by fission fragments which stop in less that one micron.
- In stopping, the fission fragments' kinetic energy becomes heat, which raises the fuel temperature.
- The fuel is cooled by the coolant, which takes the heat from the fuel to the steam generators.
- Neutrons are also produced from fission.
- Fission neutrons are slowed-down by elastic collisions in the moderator and, to a smaller extent, in coolant.
- Once they become thermal, neutrons can induce new fissions, keeping the chain-reaction going.

CANDU Reactor - How it Works (cont.)

- Part of the neutrons get absorbed by radiative capture or "leak" out of the reactor. These do not induce fissions.
- On the average, only one neutron per each fission succeeds in inducing a new fission, so there is a uniform rate of fissions and not an avalanche of fissions.

Neutron Diffusion and Moderation

General Nomenclature

Consider a quantity, say the number of collisions N_{coll}:

We call *rate*, the ratio between the amount of that quantity that is found or produced between time t and time t+dt and dt. (i.e. the collision rate is the ratio between the number of collisions that occur between t and t+dt divided by dt):

$$R_{coll} = \frac{dN_{coll}}{dt}$$

We call (energy) spectrum the ratio between the amount of that quantity that is found or produced between energy E and E+dE and dE (i.e. the collision spectrum is the ratio between the number of collisions suffered by neutrons with energies between E and E+de and dE):

$$N_{coll}(E) = \frac{dN_{coll}}{dE}$$

We call the *normalized* spectrum the ratio between the spectrum and it's integral over energy.

$$N_{normalized}(E) = \frac{N_{coll}(E)}{\int_0^\infty N_{coll}(E)dE} = \frac{N_{coll}(E)}{N_{coll}}$$

We call (volumetric) density, the ratio between the total quantity existing or produced in volume dV and dV (i.e. the collision density is the ratio between the number of collisions suffered by neutrons in volume dV and dV)

$$n_{coll} = \frac{dN_{coll}}{dV}$$

We can have names that imply double ratios, e.g.

Collision density spectrum. - the ratio between the number of collisions suffered by neutrons in dV with energies between E and E+dE and dVdE

$$n_{coll}(E) = \frac{dN_{coll}}{dVdE}$$

Also called ***energy-dependent collision density*** or ***collision density per unit energy***.

Collision density rate:

$$r(t) = \frac{dN_{coll}}{dVdt}$$

Oftentimes, when talking about double ratios people omit to name one of them, so you must pay attention to the context.

For example, one will often refer to the collision rate or collision density, when, in fact, meaning collision density rate, or even collision density rate spectrum (same as energy-dependent collision density rate)

The same letter is sometimes used to denote different quantities.

Always look at the context.

Recapitulation of Basic Concepts

Volumetric total reaction (collision) rate density for monoenergetic neutrons

$$F = \Sigma_t \phi$$

or

$$F = \Sigma_t n \mathrm{v}$$

Reaction rate for neutrons with energies between E and E+dE:

$$\frac{mv^2}{2} = E$$

$$v = \sqrt{\frac{2E}{m}}$$

$$dF = \Sigma(E) \times n(E)dE \times \mathrm{v(E)}$$

(Total) Reaction rate for neutrons of all energies:

$$F = \int_0^\infty \Sigma_t(E)n(E)\mathrm{v(E)dE} = \int_0^\infty \Sigma_\mathrm{t}(E)\phi(E)dE$$

where

$$\phi(E) = n(E)\mathrm{v(E)}$$

Reaction Rates for Individual Reactions

Scattering reaction rate density:

$$F_s = \int_0^\infty \Sigma_s(E)\phi(E)dE$$

Absorption reaction rate density (number of neutrons absorbed per cm^3 per s):

$$F_a = \int_0^\infty \Sigma_a(E)\phi(E)dE$$

Fick's Law (Diffusion Law)

- Will accept it without proof.
- Valid far from interfaces.
- Valid for materials with relatively low absorption.

Gives the neutron current as a function of the neutron flux

Assume monoenergetic neutrons
If the flux only varies along the x axis:

$$J_x = -D\frac{d\phi}{dx}$$

D = Diffusion Coefficient

In three dimensions (and monoenergetic neutrons):

$$\vec{J} = -D\,grad\,\phi = -D\nabla\phi$$

Definition of gradient: (cartesian coordinates)

$$\nabla f(x, y, z) = \begin{bmatrix} \dfrac{\partial f}{dx} \\ \dfrac{\partial f}{dy} \\ \dfrac{\partial f}{dz} \end{bmatrix}$$

Number of particles crossing a surface of orientation $\hat{n}$ per unit time per unit area:

$$J_n = \vec{J} \cdot \hat{\vec{n}}$$

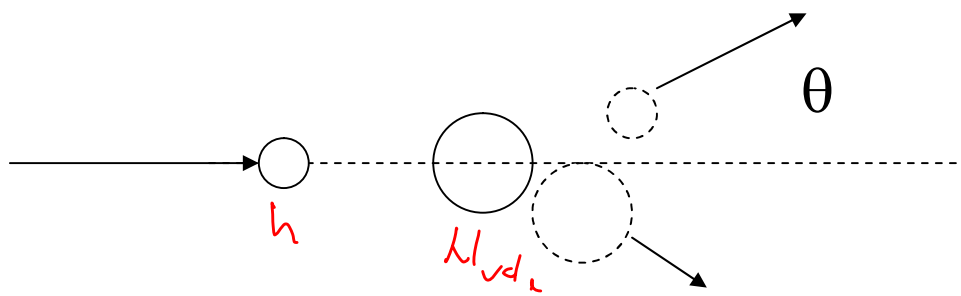
$$D = \frac{\lambda_{tr}}{3}$$

Transport mean free path

$$\lambda_{tr} = \frac{1}{\Sigma_{tr}} = \frac{1}{\Sigma_s (1 - \overline{\mu})}$$

Average of the cosine of the scattering angle

$$\bar{\mu} = \overline{\cos\theta}$$

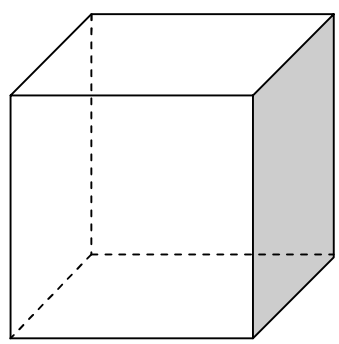


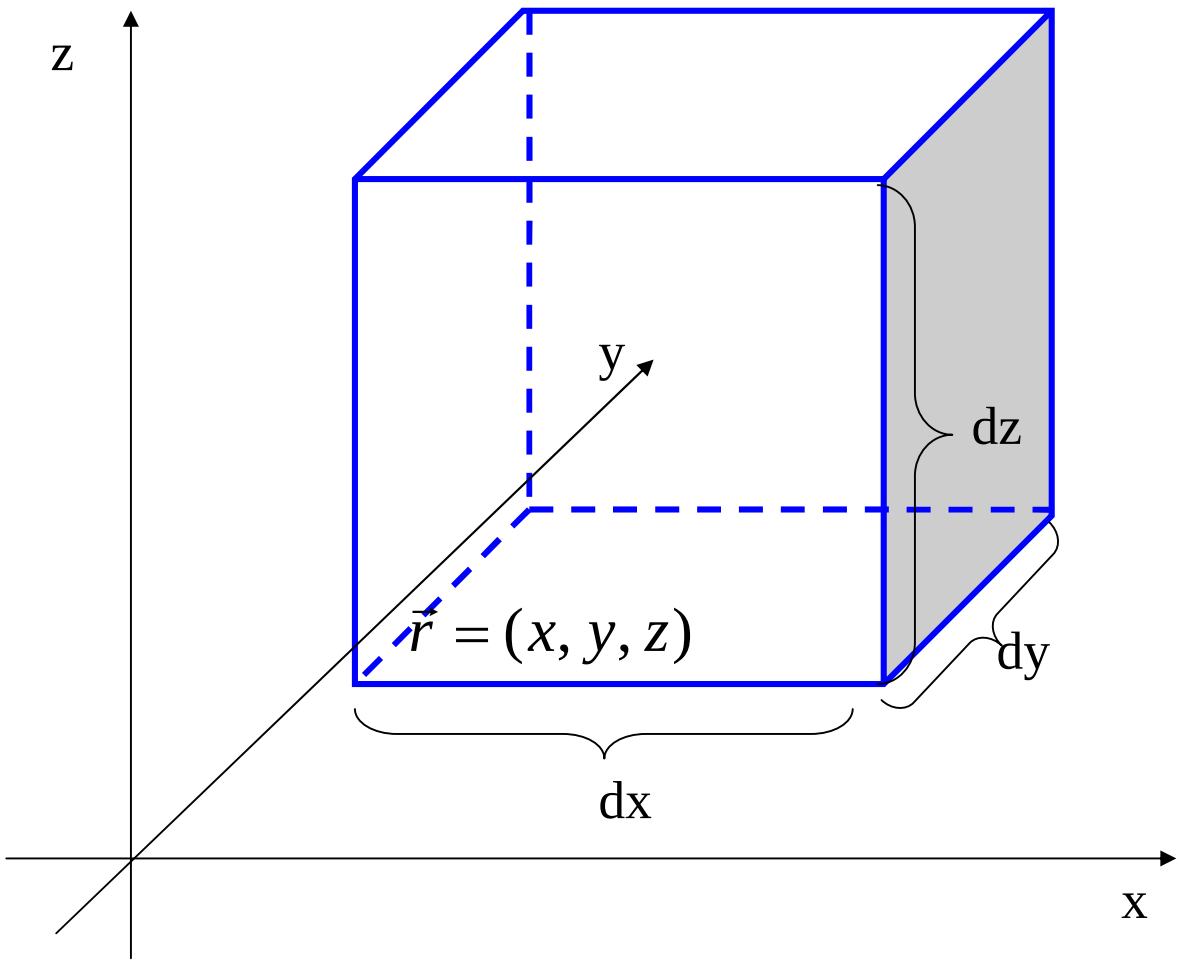
$$\bar{\mu} = \frac{2}{3A}$$

Neutron Balance Equation (equation of Continuity) for Monoenergetic Neutrons

Expresses the conservation of neutrons

$$\left[\text{Rate of change in neutron number in a small volume } dV\right] =$$
$$= \left[\text{Rate of neutron production in volume } dV\right] -$$
$$- \left[\text{Rate of neutron absorption in volume } dV\right] -$$
$$\left[\text{Rate of neutron leakage from } dV\right]$$

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dy
dx
$\vec{r} = (x, y, z)$

Infinitesimally small element means that:

- whenever we integrate over its volume, we can approximate the neutron density, neutron flux and neutron current to be constant throughout the volume. dV
- whenever we integrate over a face, we can assume the neutron density, neutron flux and neutron current to be constant over that face
- whenever we integrate over an edge, we can assume the neutron density, neutron flux and neutron current to be constant along that edge

We cannot make the same approximation when we take differences of quantities (neutron density, flux, current) at two points in the volume. That is because the difference is already a very small number, comparable to the quantity's variation from point to point.

Number of neutrons in dV

$$ndV$$

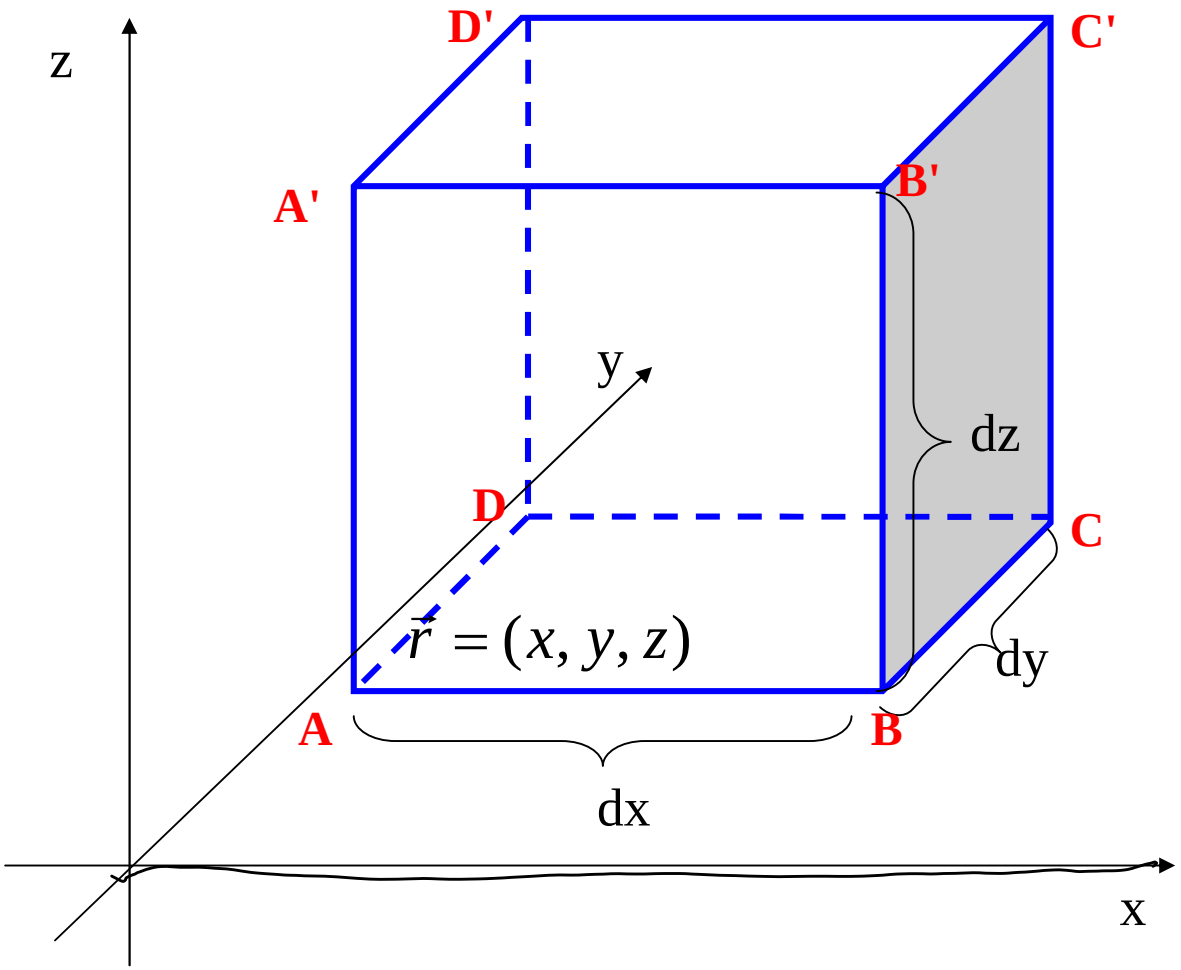
Production rate

$$R_p = sdV$$

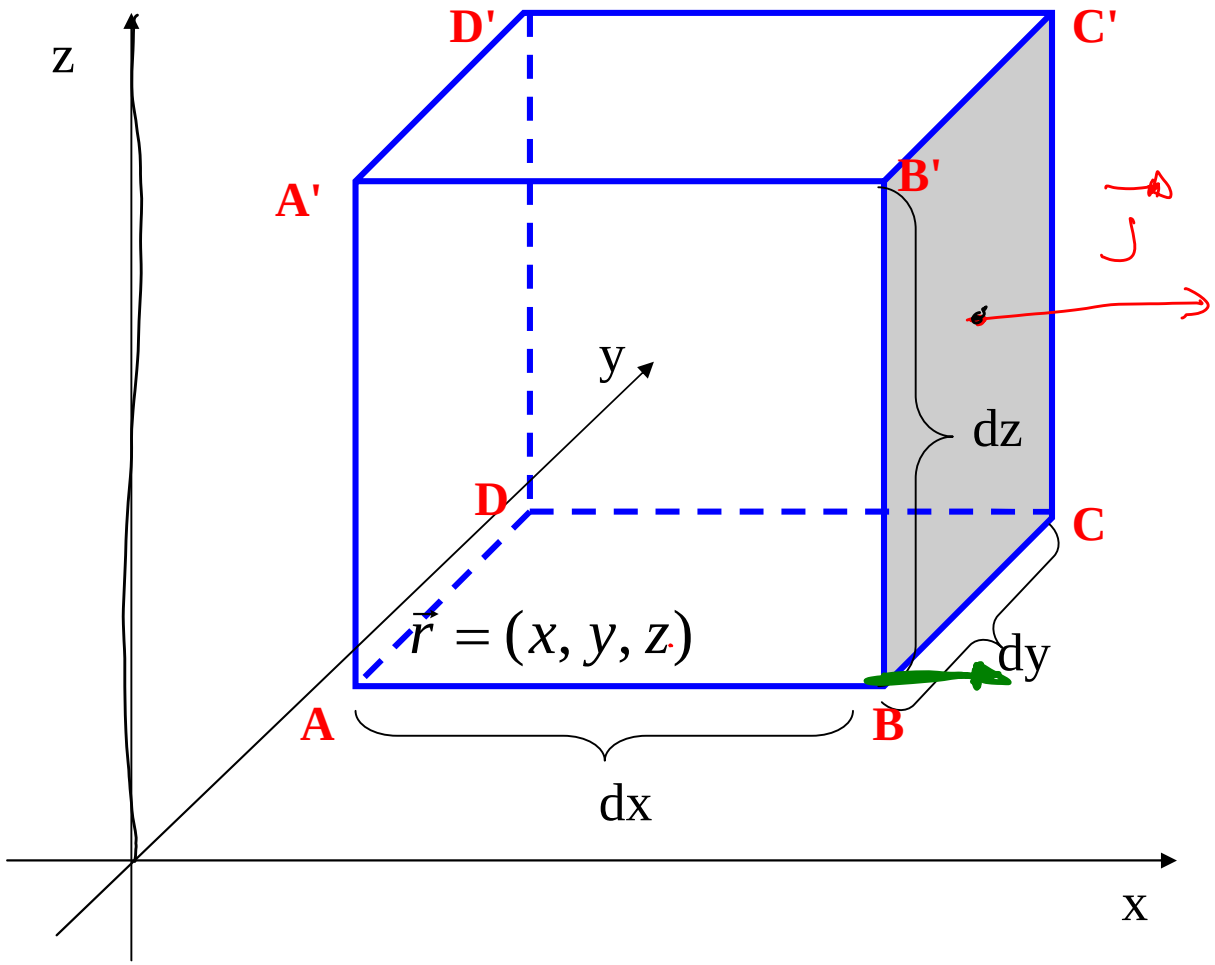
Absorption Rate

$$R_a = \Sigma_a \phi dV$$

Infinitesimal (i.e. very small) Volume



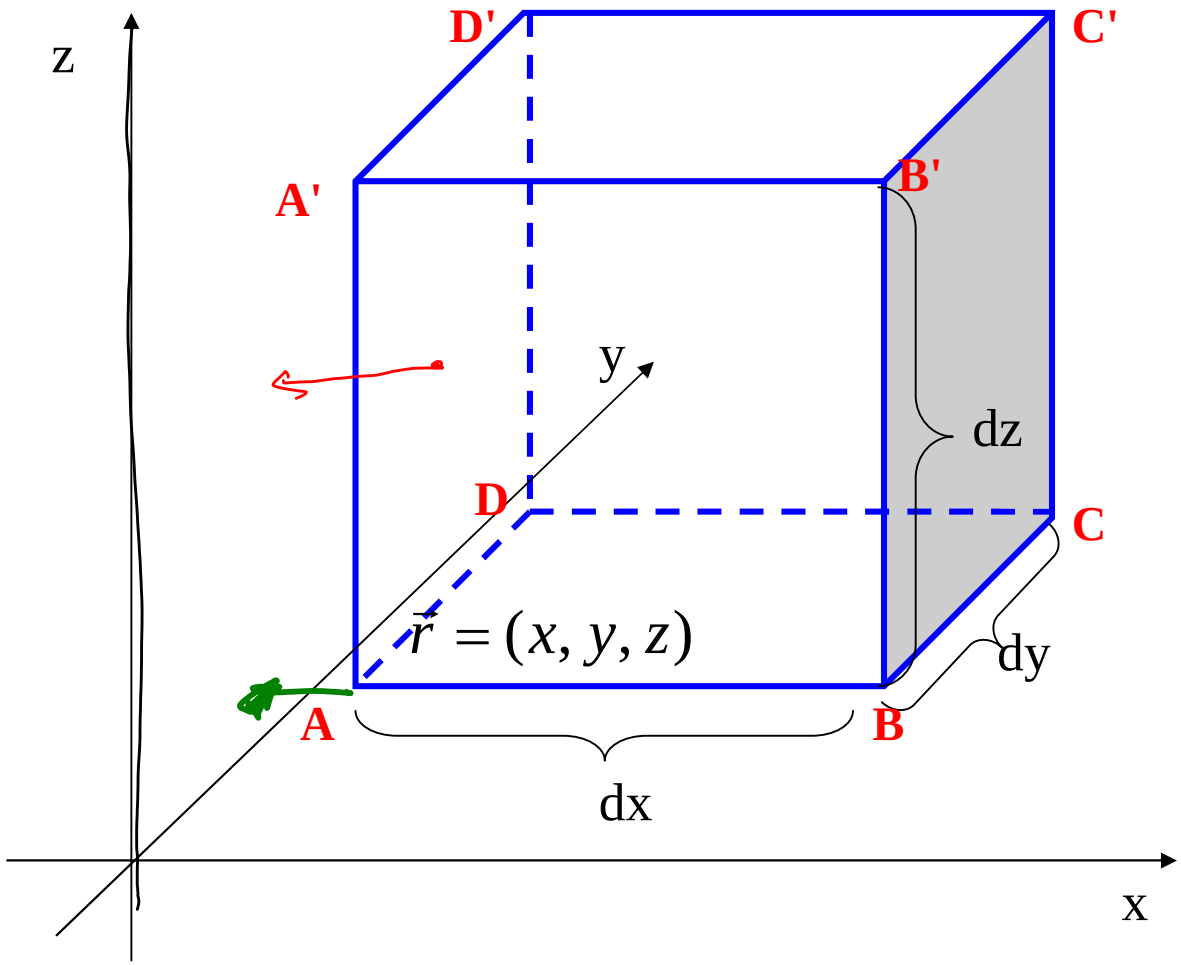
Leakage Through Face BCC'B'



$$LK_{x+} = LK_{BCC'B'} \cong \vec{J}\left(x + dx, y + \frac{dy}{2}, z + \frac{dz}{2}\right) \cdot \hat{u}_x dydz$$

$$LK_{x+} = LK_{BCC'B'} \cong J_x(x + dx, y, z)dydz$$

Leakage Through Face ADD'A'



$$LK_{x-} = LK_{ADD'A'} \cong \vec{J}\left(x, y + \frac{dy}{2}, z + \frac{dz}{2}\right) \cdot \left(-\hat{u}_x\right) dydz$$

$$LK_{x-} = LK_{ADD'A'} \cong -J_x\left(x, y, z\right) dydz$$

Net Leakage Along X Axis

$$LK_x = LK_{x+} + LK_{x-} = LK_{BCC'B'} + LK_{ADD'A'} \cong$$
$$\cong J_x(x+dx, y, z) - J_x(x, y, z)dydz$$

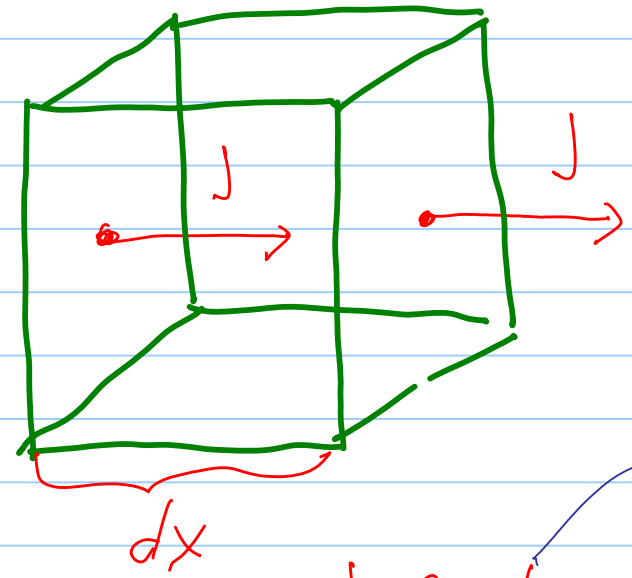
Let's remember that:

$$\frac{\partial J_x}{\partial x}(x, y, z) = \frac{J_x(x+dx, y, z) - J_x(x, y, z)}{dx}$$

Hence:

$$J_x(x+dx, y, z) - J_x(x, y, z) = \frac{\partial J_x}{\partial x}(x, y, z)dx$$

$$LK_x \cong J_x(x+dx, y, z) - J_x(x, y, z)dydz =$$
$$= \frac{\partial J_x}{\partial x}(x, y, z)dxdydz$$



$$LK_x = \frac{\partial J_x}{\partial x}\, dx\, dy\, dz \approx \frac{J_x(x+dx, y, z) - J_x(x, y, z)}{dx}\, dx\, dy\, dz =$$

$$= \frac{\partial J_x}{\partial x} = 0$$

Total Leakage out of dV

$$LK = LK_x + LK_y + LK_z =$$

$$= \frac{\partial J_x}{\partial x}(x,y,z)dxdydz + \frac{\partial J_y}{\partial y}(x,y,z)dxdydz + \frac{\partial J_z}{\partial z}(x,y,z)dxdydz =$$

$$= \left(\frac{\partial J_x}{\partial x} + \frac{\partial J_y}{\partial y} + \frac{\partial J_z}{\partial z} \right)\Bigg|_{x,y,z} dxdydz = \left(div\vec{J} \right) dxdydz = \left(\nabla \cdot \vec{J} \right) dxdydz$$

Definition of divergence for a vector function $\vec{f}(x,y,z)$:

$$div\vec{f} \equiv \nabla \cdot \vec{f} = \left(\frac{\partial f_x}{dx} + \frac{\partial f_y}{dy} + \frac{\partial f_z}{dz} \right)$$

Rate of Change of Number of Neutrons in dV

$$R_{change} = \frac{\#neutrons(t+dt) - \#neutrons(t)}{dt} = \frac{n(t+dt)dV - n(t)dV}{dt} =$$

$$= \frac{n(t+dt) - n(t)}{dt} dV = \frac{\partial n}{\partial t} dxdydz$$

Neutron Balance Equation for ~~DV~~ dV

$$\frac{\partial n}{\partial t} dxdydz = sdxdydz - \Sigma_a \Phi dxdydz - \nabla \cdot \vec{J} dxdydz$$

Dividing by the volume $dV = dxdydz$ we obtain:

$$\frac{\partial n}{\partial t} = s - \Sigma_a \phi - \nabla \cdot \vec{J}$$

Valid regardless of whether Fick's law holds true or not

Neutron Balance in the Diffusion Approximation

Assume Fick's Law to be true:

$$\vec{J} = -D\nabla\Phi$$

Substitute into the neutron balance eq:

$$\frac{\partial n}{\partial t} = -\nabla \cdot \left(-D\nabla\phi\right) - \Sigma_a \phi + s$$

This is the *time-dependent diffusion equation for monoenergetic neutrons.*

It is important because by solving it we find the flux and the flux allows us to calculate all reaction rates, including fission rate - which is really what we are after, by using $R = \Sigma\phi$.

If the diffusion coefficient is constant:

$$\frac{\partial n}{\partial t} = D\nabla \cdot \left(\nabla \phi\right) - \Sigma_a \phi + s$$

Remember the definition of the Laplacian:

$$\Delta f(x, y, z) \equiv \nabla^2 f = \nabla \cdot \left(\nabla f\right) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

The diffusion eq. can then be rewritten:

$$\frac{\partial n}{\partial t} = D\nabla^2 \phi - \Sigma_a \phi + s$$

$$f(x, y, z)$$

$$(\nabla f)_x = \frac{\partial f}{\partial x}$$

$$(\nabla f)_y = \frac{\partial f}{\partial y}$$

$$(\nabla f)_z = \frac{\partial f}{\partial z}$$

$$\nabla \cdot (\nabla f) = \frac{\partial}{\partial x}(\nabla f)_x + \frac{\partial}{\partial y}(\nabla f)_y + \frac{\partial}{\partial z}(\nabla f)_z =$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial z}\right) =$$

$$= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

If we keep in mind that

$$\phi = n\mathrm{v} \Rightarrow n = \frac{\phi}{\mathrm{v}}$$

We obtain:

$$\frac{1}{\mathrm{v}}\frac{\partial \phi}{\partial t} = D\nabla^2\phi - \Sigma_a\phi + s$$

Steady-State Situation (no time dependence)

$$D\nabla^2\phi - \Sigma_a\phi + s = 0$$

Dividing by D:

$$\nabla^2\phi - \frac{\Sigma_a}{D}\phi + \frac{s}{D} = 0$$

Introducing notation (Diffusion Length):

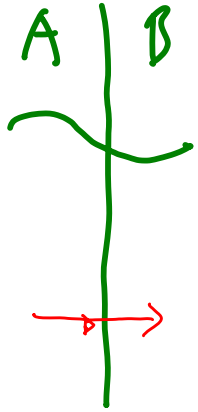
$$L^2 = \frac{D}{\Sigma_a}$$

$$-\nabla^2\phi + \frac{1}{L^2}\phi = \frac{s}{D}$$

Interface Conditions for the Diffusion equation:

$$\phi_A = \phi_B$$

$$J_{A\perp} = J_{B\perp}$$

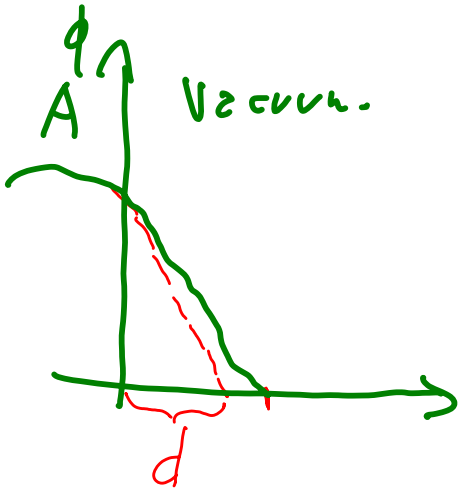


Vacuum Interface

$$\phi(d) = 0$$

$$d = 0.71\lambda_{tr}$$

$$d = 2.13D$$



The Concept of Infinite Homogeneous Medium

Medium is the same at any point
Hence, there is no reason why the flux would be different an any particular point

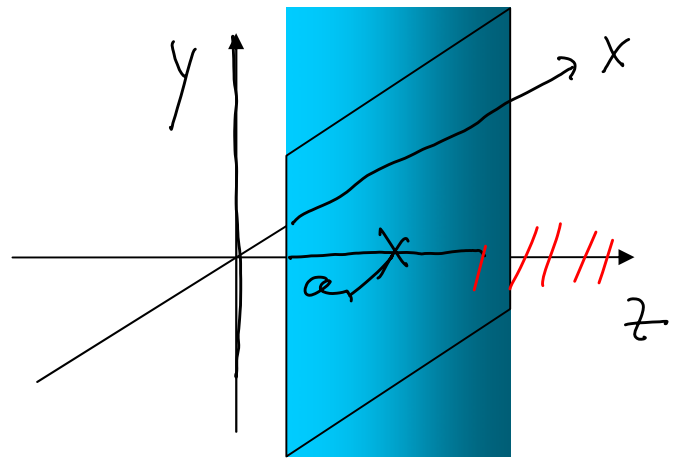
$$\Phi(x, y, z) = \Phi = ct$$

The current is given by Fick's Law

$$\vec{J} = \nabla\Phi = \begin{bmatrix} \dfrac{\partial\Phi}{\partial x} \\ \dfrac{\partial\Phi}{\partial y} \\ \dfrac{\partial\Phi}{\partial z} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0$$

The current is zero in an infinite homogeneous medium

The Concept of Homogeneous Half Space

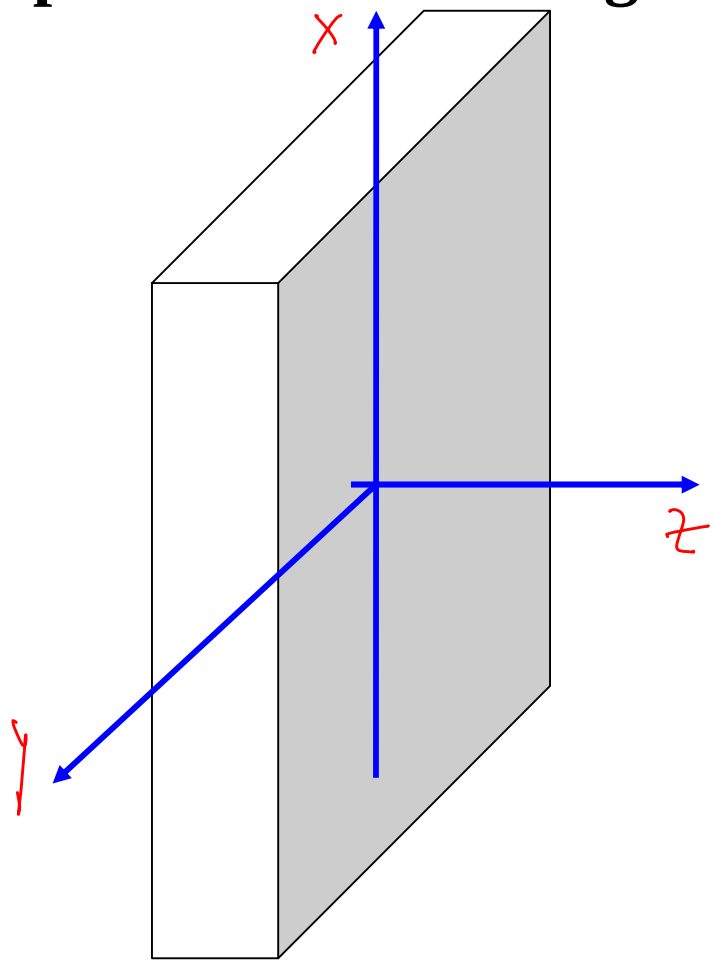


$$x \in (-\infty, \infty)$$
$$y \in (-\infty, \infty)$$
$$z \in (a, \infty)$$

In such a configuration, since for the same z all points are identical, there is no variation in the flux with x or y

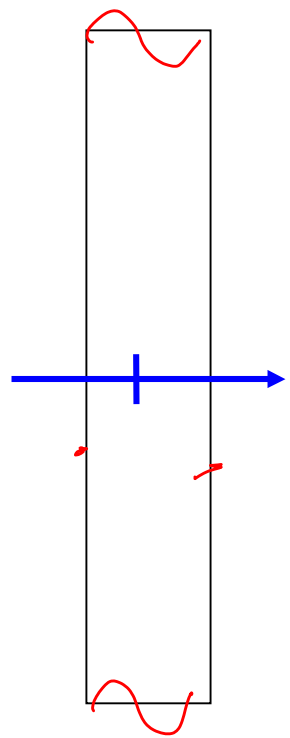
$$\Phi(x, y, z) = \Phi(z)$$

The Concept of Infinite Homogeneous Slab



$$x \in (-\infty, \infty)$$
$$y \in (-\infty, \infty)$$
$$z \in (-a, a)$$

Looking at it from one side:



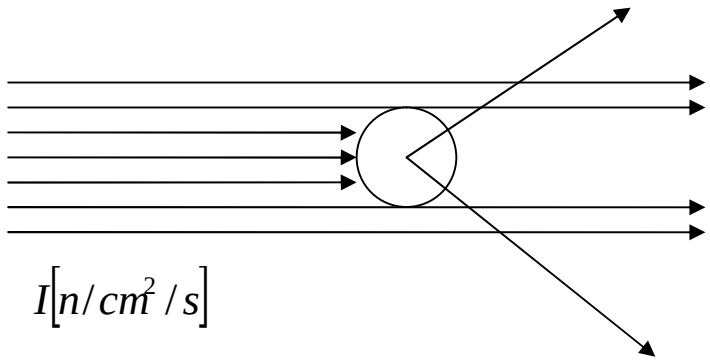
In such a configuration, since for the same z all points are identical, there is no variation in the flux with x or y

$$\Phi(x, y, z) = \Phi(z)$$

Differential Microscopic Scattering Cross Sections

Beam of monoenergetic neutrons

$$I = n\mathrm{v}(E)$$



Scattering rate:

$$R_s = I\sigma_s(E)$$

Equivalently, we can write (using only macroscopic quantities that can be measured):

$$\sigma_s(E) = \frac{R_s}{I}$$

By scattering, neutrons lose energy.

Let $dR_s(E')$ be the rate at which neutrons are scattered in energy range E', E'+dE'

We have:

$$\int_0^\infty dR_s(E') = R_s$$

Definition of the differential scattering microscopic cross section

$$\sigma_s(E \to E') \equiv \frac{dR_s(E')}{IdE'}$$

Equivalently, we can write:

$$\sigma_s(E \to E') \equiv \frac{dR_s(E')}{IdE'}\frac{R_s}{R_s} = \frac{R_s}{I}\frac{dR_s(E')}{R_s dE'} = \sigma_s(E)\frac{dR_s(E')}{R_s dE'}$$

Scattering Kernel

$$k(E \to E') = \frac{dR_s(E')}{R_s dE'} = \frac{dP(E, E')}{dE'}$$

$$\sigma_s(E \to E') = \sigma_s k(E \to E')$$

The scattering kernel can be interpreted as the probability density function for a neutron of energy E to be scattered such that its final energy is between E' and E'+dE'.

The differential and total scattering cross section satisfy:

$$\sigma_s(E) = \int_0^\infty \sigma_s(E \to E') dE'$$

Differential Macroscopic Scattering Cross Sections

$$\Sigma_s(E \to E') = N\sigma_s(E \to E')$$

or, using the scattering kernel:

$$\Sigma_s(E \to E') = N\sigma_s(E)k(E \to E') =$$
$$= \Sigma_s(E)k(E \to E')$$

Volumetric reaction rate at which neutrons scatter within energy range (E, E+dE)

$$R_s(E \to E') = I\Sigma_s(E \to E')$$

Energy-Dependent Neutron Balance Equation

Balance Equation for Neutrons with Energy Between E and E+dE

[rate of change of number of neutrons in volume dV with energy within range (E, E+dE)] =
[rate of production in volume dV of neutrons with energy within range (E, E+dE)] +
[rate of scattering of neutrons in dV into energy range (E, E+dE)] -
[rate of absorption in dV of neutrons with energy in range (E, E+dE)] -
-[rate of scattering of neutrons in dV outside of energy range (E, E+dE)] -
[rate of leakage out of dV of neutrons with energy within range (E, E+dE)]

[rate of change of number of neutrons in volume dV with energy within range (E, E+dE)]

$$R_{change}(E) = \frac{n(E,t+dt)dEdV - n(E,t)dEdV}{dt} =$$

$$= \frac{n(E,t+dt) - n(E,t)}{dt} dV = \frac{\partial n(E)}{\partial t} dEdV$$

$$\frac{n(t+dt) - n(t)}{dt}$$ = forward derivative

$\neq$

why not $$\frac{n(t) - n(t-dt)}{dt}$$ = backward derivative

lim $dt \to 0$

[rate of production in volume dV of neutrons with energy within range (E, E+dE)]

$$R_p(E) = s(E)\,dE\,dV$$

s(E) = number of neutrons produced inside dV with energies between E and E+dE, divided by dEdV.

[rate of scattering of neutrons in dV into energy range (E, E+dE)]

Rate at which neutrons with energy within $(E';E'+dE)$ scatter such that their energy is within $(E;E+dE)$

$$R_s(E' \rightarrow E) = \Phi(E')dE' \times \Sigma_s(E' \rightarrow E)dEdV$$

Rate at which all neutrons scatter such that their energy is within $(E;E+dE)$

$$R_s(\rightarrow E) = \left[\int_0^\infty \Phi(E') \times \Sigma_s(E' \rightarrow E)dE'\right]dEdV$$

[rate of absorption in dV of neutrons with energy in range (E, E+dE)]

$$R_a(E) = \Phi(E)dE \times \Sigma_a(E)dV = \Phi(E) \times \Sigma_a(E)dEdV$$

[rate of scattering of neutrons in dV outside of energy range (E, E+dE)]

$$R_s(E \rightarrow) = \Phi(E)dE \times \Sigma_s(E)dV = \Phi(E) \times \Sigma_s(E)dEdV$$

Note that:

$$\Sigma_s(E) = \int_0^\infty \Sigma_s(E \rightarrow E')dE'$$

[rate of leakage out of dV of neutrons with energy within range (E, E+dE)]

$$LK(E) = \nabla \cdot \vec{J}(E) dE dV$$

Balance Equation for Neutrons with Energy Between E and E+dE

$$R_{change}(E) = R_p(E) + R_s(\rightarrow E) - R_a(E) - R_s(E \rightarrow) - LK(E)$$

$$\frac{\partial n(E)}{\partial t} dEdV = s(E)dEdV + \left[\int_0^\infty \Phi(E') \times \Sigma_s(E' \rightarrow E) dE' \right] dEdV -$$

$$- \Phi(E) \times \Sigma_a(E) dEdV - \Phi(E) \times \Sigma_s(E) dEdV - \nabla \cdot \vec{J}(E) dEdV$$

Dividing by dEdV we obtain the energy-dependent neutron balance equation (continuity equation):

$$\frac{\partial n(E)}{\partial t} = s(E) + \int_0^\infty \Phi(E') \times \Sigma_s(E' \rightarrow E) dE' -$$

$$- \Phi(E) \times \Sigma_a(E) - \Phi(E) \times \Sigma_s(E) - \nabla \cdot \vec{J}(E)$$

We can show the dependence on time explicitly:

$$\frac{\partial n(E,t)}{\partial t} = s(E,t) + \int_0^\infty \Phi(E',t) \times \Sigma_s(E' \to E) dE' -$$

$$- \Phi(E,t) \times \Sigma_a(E) - \Phi(E,t) \times \Sigma_s(E) - \nabla \cdot \vec{J}(E,t)$$

Definition of energy-dependent flux:

$$\Phi(E) = n(E)\mathrm{v}(\mathrm{E}) \Rightarrow n(E) = \frac{\Phi(E)}{\mathrm{v}(\mathrm{E})}$$

Substituting the expression for the energy-dependent neutron density, we obtain:

$$\frac{1}{\mathrm{v}(E)} \frac{\partial \Phi(E,t)}{\partial t} = s(E,t) + \int_0^\infty \Phi(E',t) \times \Sigma_s(E' \to E) dE' -$$

$$- \Phi(E,t) \times \Sigma_a(E) - \Phi(E,t) \times \Sigma_s(E) - \nabla \cdot \vec{J}(E,t)$$

Energy-Dependent Steady-State Neutron Balance Equation

$$0 = s(E,\cancel{t}) + \int_0^\infty \Phi(E',\cancel{t}) \times \Sigma_s(E' \to E)dE' -$$

$$-\Phi(E,\cancel{t}) \times \Sigma_a(E) - \Phi(E,\cancel{t}) \times \Sigma_s(E) - \nabla \cdot \vec{J}(E,\cancel{t})$$

Diffusion Approximation (use Fick's Law)

$$\vec{J}(E) = -D(E)\nabla\Phi(E)$$

$$0 = s(E,\cancel{t}) + \int_0^\infty \Phi(E',\cancel{t}) \times \Sigma_s(E' \to E)dE' -$$

$$-\Phi(E,\cancel{t}) \times \Sigma_a(E) - \Phi(E,\cancel{t}) \times \Sigma_s(E) + \nabla \cdot \left(D(E)\nabla\Phi(E)\right)$$

For position-independent diffusion coefficient:

$$0 = s(E,\cancel{t}) + \int_0^\infty \Phi(E',\cancel{t}) \times \Sigma_s(E' \to E)dE' -$$

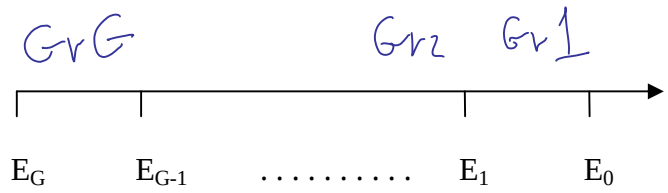
$$-\Phi(E,\cancel{t}) \times \Sigma_a(E) - \Phi(E,\cancel{t}) \times \Sigma_s(E) + D(E)\nabla^2\Phi(E)$$

Multigroup Formalism

Approximate treatment of the energy-dependent diffusion equation.

Energy Groups

Divide the energy domain $(0, E_{\max})$ into intervals called *groups*



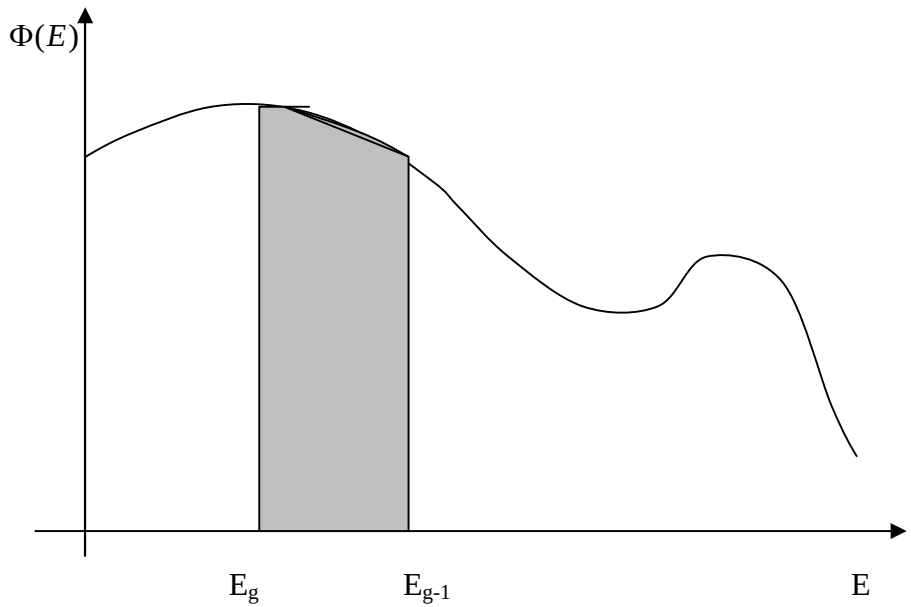
$$0 = E_G < E_{G-1} \ldots\ldots < E_2 < E_1 < E_0 = E_{\max}$$

Group neutron density

$$n_g \equiv \int_{E_g}^{E_{g-1}} n(E)dE$$

(Energy) Group Flux

$$\Phi_g \equiv \int_{E_g}^{E_{g-1}} \Phi(E)dE$$



Can depend on parameters such as position and/or time

$$\Phi_g(\vec{r}) \equiv \int_{E_g}^{E_{g-1}} \Phi(\vec{r},E)dE$$

Group Current

$$\vec{J}_g \equiv \int_{E_g}^{E_{g-1}} \vec{J}(E)dE$$

Can depend on parameters such as position and/or time

$$\vec{J}_g(\vec{r}) \equiv \int_{E_g}^{E_{g-1}} \vec{J}(\vec{r},E)dE$$

Group Source

$$s_g = \int_{E_g}^{E_{g-1}} s(E)dE$$

Group Reaction Rates

Reaction Rate for a single Nucleus

$$R_g^{\text{single nucleus}} \equiv \int_{E_g}^{E_{g-1}} R(E)dE = \int_{E_g}^{E_{g-1}} \Phi(E)\sigma(E)dE$$

Reaction Rate Density for a Material

$$R_g \equiv \int_{E_g}^{E_{g-1}} R(E)dE = \int_{E_g}^{E_{g-1}} \Phi(E)\Sigma(E)dE$$

Can depend on parameters such as position and/or time

$$R_g(\vec{r}) \equiv \int_{E_g}^{E_{g-1}} R(\vec{r},E)dE = \int_{E_g}^{E_{g-1}} \Phi(\vec{r},E)\Sigma(\vec{r},E)dE$$

Group Cross Sections

Microscopic Group Cross Sections

$$\sigma_g \equiv \frac{\int_{E_g}^{E_{g-1}} \Phi(E)\sigma(E)dE}{\int_{E_g}^{E_{g-1}} \Phi(E)dE} = \frac{R_g^{\text{single nucleus}}}{\Phi_g} \; ; \; R_g^{\text{nucleus}} = \phi_g \cdot \sigma_g$$

Macroscopic Group Cross Sections

$$\Sigma_g \equiv \frac{\int_{E_g}^{E_{g-1}} \Phi(E)\Sigma(E)dE}{\int_{E_g}^{E_{g-1}} \Phi(E)dE} = \frac{R_g}{\Phi_g} \qquad R_g = \phi_g \Sigma_g$$

Inter-Group Scattering (Transfer) Cross Sections

Microscopic

$$\sigma_{g\to g'} = \frac{\int_{E_g}^{E_{g-1}} \left[\int_{E_{g'}}^{E_{g'-1}} \Phi(E)\sigma_s(E \to E')dE' \right] dE}{\int_{E_g}^{E_{g-1}} \Phi(E)dE}$$

nucleus

$$\frac{R_{g\to g'}}{\phi_g}$$

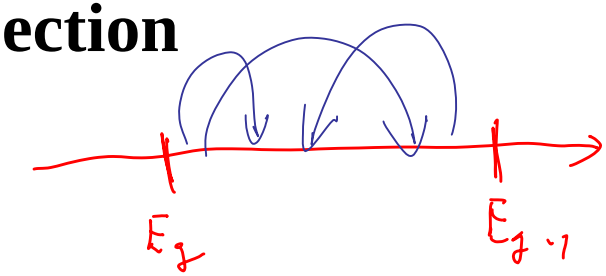
Macroscopic

$$\Sigma_{g\to g'} = \frac{\int_{E_g}^{E_{g-1}} \left[\int_{E_{g'}}^{E_{g'-1}} \Phi(E)\Sigma_s(E \to E')dE' \right] dE}{\int_{E_g}^{E_{g-1}} \Phi(E)dE}$$

$$\cancel{g \neq g'} \quad \frac{R_{g\to g'}}{\phi_g}$$

$$R_{g-g'} = \phi_g \Sigma_{g-g'}$$

Intra-Group Scattering Cross Section



Microscopic

$$\sigma_{g\to g} = \frac{\int_{E_g}^{E_{g-1}} \left[\int_{E_g}^{E_{g-1}} \Phi(E)\sigma_s(E \to E')dE' \right] dE}{\int_{E_g}^{E_{g-1}} \Phi(E)dE}$$

$$\frac{R_{g\to g}}{\phi_g}$$

Macroscopic

$$\Sigma_{g\to g} = \frac{\int_{E_g}^{E_{g-1}} \left[\int_{E_g}^{E_{g-1}} \Phi(E)\Sigma_s(E \to E')dE' \right] dE}{\int_{E_g}^{E_{g-1}} \Phi(E)dE} = \frac{R_{g\to g}}{\phi_g}$$

$$R_{g-g} = \phi_g \Sigma_{g-g}$$

Multigroup Neutron balance Equation

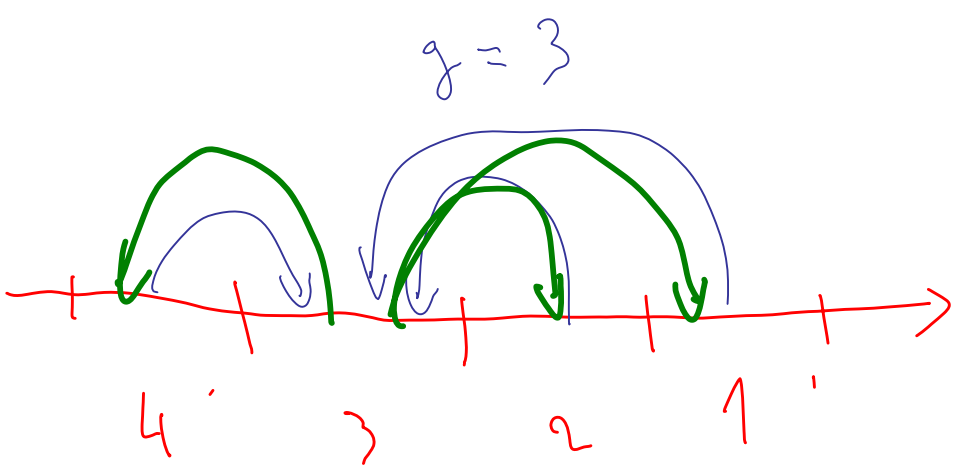
[rate of change of number of neutrons in volume dV with energy within group g] =
[rate of production in volume dV of neutrons with energy within group g] +
[rate of scattering of neutrons in dV into energy group g] -
[rate of absorption in dV of neutrons with energy in group g] -
-[rate of scattering of neutrons in dV outside of energy group g] -
[rate of leakage out of dV of neutrons with energy within group g]

Multigroup Neutron balance Equation

$$\frac{\partial}{\partial t} n_g dV = s_g dV + \sum_{g'=1, g' \neq g}^{G} \Sigma_{sg' \to g} \phi_{g'} dV - \Sigma_{ag} \phi_g dV - \sum_{g'=1, g' \neq g}^{G} \Sigma_{sg \to g'} \phi_g dV - \nabla \cdot \vec{J}_g dV$$

Dividing by dV we obtain:

$$\frac{\partial}{\partial t} n_g = s_g + \sum_{g'=1, g' \neq g}^{G} \Sigma_{sg' \to g} \phi_{g'} - \Sigma_{ag} \phi_g - \sum_{g'=1, g' \neq g}^{G} \Sigma_{sg \to g'} \phi_g - \nabla \cdot \vec{J}_g$$



Multigroup Neutron balance Equation

Multigroup Fick's Law:

$$\vec{J}_g = -D_g \nabla \phi_g$$

Multigroup Diffusion Equation

$$\frac{\partial}{\partial t} n_g = s_g + \sum_{g'=1, g' \neq g}^{G} \Sigma_{sg' \to g} \phi_{g'} - \Sigma_{ag} \phi_g - \sum_{g'=1, g' \neq g}^{G} \Sigma_{sg \to g'} \phi_g + \nabla \cdot \left(D_g \nabla \phi_g \right)$$

For constant diffusion coefficient:

$$\frac{\partial}{\partial t} n_g = s_g + \sum_{g'=1, g' \neq g}^{G} \Sigma_{sg' \to g'} \phi_g - \Sigma_{ag} \phi_g - \sum_{g'=1, g' \neq g}^{G} \Sigma_{sg \to g'} \phi_g + D_g \nabla^2 \phi_g$$

Steady-state (no time dependence)

$$s_g + \sum_{g'=1, g' \neq g}^{G} \Sigma_{sg' \to g} \phi_{g'} - \Sigma_{ag} \phi_g - \sum_{g'=1, g' \neq g}^{G} \Sigma_{sg \to g'} \phi_g + D_g \nabla^2 \phi_g = 0$$

Finally, changing the order of the terms, we can write:

$$-D_g \nabla^2 \phi_g - \sum_{g'=1, g' \neq g}^{G} \Sigma_{sg' \to g} \phi_{g'} + \Sigma_{ag} \phi_g + \sum_{g'=1, g' \neq g}^{G} \Sigma_{sg \to g'} \phi_g = s_g$$

A Different Way of Deriving the Multigroup Diffusion Equation

Integrating the energy-dependent diffusion equation over energy group g.

$$\frac{\partial}{\partial t} n_g = s_g + \sum_{g'=1}^{G} \Sigma_{sg' \to g} \phi_{g'} - \Sigma_{ag} \phi_g - \Sigma_{sg} \phi_g + \nabla \cdot \left(D_g \nabla \phi_g \right)$$

For constant diffusion coefficient:

$$\frac{\partial}{\partial t} n_g = s_g + \sum_{g'=1}^{G} \Sigma_{sg' \to g} \phi_{g'} - \Sigma_{ag} \phi_g - \Sigma_{sg} \phi_g + D_g \nabla^2 \phi_g$$

Steady-state (no time dependence)

$$s_g + \sum_{g'=1}^{G} \Sigma_{sg' \to g} \phi_{g'} - \Sigma_{ag} \phi_g - \Sigma_{sg} \phi_g + D_g \nabla^2 \phi_g = 0$$

The above can be rearranged to:

$$-D_g \nabla^2 \phi_g - \sum_{g'=1}^{G} \Sigma_{sg' \to g} \phi_{g'} + \Sigma_{ag} \phi_g - \Sigma_{sg} \phi_g = s_g$$

This appears to be different from the diffusion equation we established before. We will prove that it is, in fact, identical.

Scattering cross sections satisfy:

$$\sum_{g'=1}^{G} \Sigma_{sg\to g'} = \Sigma_{sg}$$

Substituting the above into the steady-state multigroup diffusion equation, one obtains:

$g' = g$

$\Sigma_{s g' \to g} \phi_{g'}$

$\Sigma_{s g \to g'} \phi_g$

$$-D_g \nabla^2 \phi_g - \sum_{g'=1}^{G} \Sigma_{sg'\to g} \phi_{g'} + \Sigma_{ag} \phi_g + \sum_{g'=1}^{G} \Sigma_{sg\to g'} \phi_g = s_g$$

Which, considering that the intragroup scattering cancels out, can be rewritten:

$$-D_g \nabla^2 \phi_g - \sum_{g'=1, g'\neq g}^{G} \Sigma_{sg'\to g} \phi_{g'} + \Sigma_{ag} \phi_g + \sum_{g'=1, g'\neq g}^{G} \Sigma_{sg\to g'} \phi_g = s_g$$

Which is exactly what we had obtained before using directly a group-balance argument.

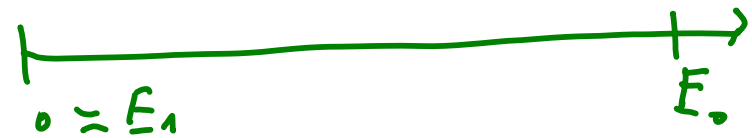
Additionally, if we assume no upscattering, we can write:

$$\Sigma_{sg \to g'} = 0 \quad \text{for} \quad g' < g$$

and thus process the multigroup diffusion equation to:

$$-D_g \nabla^2 \phi_g - \sum_{g'=1}^{g-1} \Sigma_{sg' \to g} \phi_{g'} + \Sigma_{ag} \phi_g + \sum_{g'=g+1}^{G} \Sigma_{sg \to g'} \phi_g = s_g$$

Particular Cases of the Diffusion Equation



One-Group Diffusion Equation

The entire energy range is included in just one group

The multigroup diffusion equation

$$-D_g \nabla^2 \phi_g - \sum_{g'=1}^{g-1} \Sigma_{sg' \to g} \phi_g + \Sigma_{ag} \phi_g + \sum_{g'=g+1}^{G} \Sigma_{sg \to g'} \phi_g = s_g$$

becomes:

$$-D_1 \nabla^2 \phi_1 + \Sigma_{a1} \phi_1 = s_1$$

We can drop the group index:

$$-D \nabla^2 \phi + \Sigma_a \phi = s$$

$G = 2$

Two-Group Diffusion Equation

Start with the multigroup diffusion equation

$$-D_g \nabla^2 \phi_g - \sum_{g'=1}^{g-1} \Sigma_{sg' \to g} \phi_{g'} + \Sigma_{ag} \phi_g + \sum_{g'=g+1}^{G} \Sigma_{sg \to g'} \phi_g = s_g$$

Group 1 (fast):

$$-D_1 \nabla^2 \phi_1 + \Sigma_{a1} \phi_1 + \Sigma_{s1 \to 2} \phi_1 = s_1$$

Group 2 (slow, thermal):

$$-D_2 \nabla^2 \phi_2 - \Sigma_{s1 \to 2} \phi_1 + \Sigma_{a2} \phi_2 = s_2$$

Two-group diffusion equations:

$$-D_1 \nabla^2 \phi_1 + \Sigma_{a1} \phi_1 + \Sigma_{s1 \to 2} \phi_1 = s_1$$

$$-D_2 \nabla^2 \phi_2 - \Sigma_{s1 \to 2} \phi_1 + \Sigma_{a2} \phi_2 = s_2$$

Solving the Diffusion Equation for Simple Cases

One Group, Infinite Homogeneous Medium, Uniformly Distributed Source

$$-D\nabla^2\phi(\vec{r}) + \Sigma_a\phi(\vec{r}) = s$$

Infinite, homogeneous medium

$$\phi(\vec{r}) = \phi = ct$$

$$\nabla^2\phi = 0$$

The equation becomes:

$$\Sigma_a\phi = s$$

Solving for the flux, we obtain:

$$\phi = \frac{S}{\Sigma_a}$$

Two Groups, Infinite Homogeneous Medium, Uniformly Distributed Source

$$-D_1\nabla^2\phi_1 + \Sigma_{a1}\phi_1 + \Sigma_{s1\to 2}\phi_1 = s_1$$
$$-D_2\nabla^2\phi_2 - \Sigma_{s1\to 2}\phi_1 + \Sigma_{a2}\phi_2 = s_2$$

For an infinite and homogeneous medium with uniformly-distributed source:

$$\phi_1(\vec{r}) = \phi_1 = ct$$
$$\phi_2(\vec{r}) = \phi_2 = ct$$
$$\nabla^2\phi_1 = 0$$
$$\nabla^2\phi_2 = 0$$

The two-group equations become:

$$\Sigma_{a1}\phi_1 + \Sigma_{s1\to 2}\phi_1 = s_1$$
$$-\Sigma_{s1\to 2}\phi_1 + \Sigma_{a2}\phi_2 = s_2$$

The first equation can be easily solved to yield:

$$\phi_1 = \frac{s_1}{\Sigma_{a1} + \Sigma_{s1\to 2}} = \frac{s_1}{\Sigma_r}$$

$$\Sigma_r = \text{removal cross section}$$

The second equation can be rewritten as:

$$\Sigma_{a2}\phi_2 = s_2 + \Sigma_{s1\to 2}\phi_1$$

$$\Sigma_{s1\to 2}\phi_1 = q_T = \text{slowing down density}$$

Using the expression found for the fast flux, we have:

$$\Sigma_{a2}\phi_2 = s_2 + \Sigma_{s1\to 2}\frac{s_1}{\Sigma_r}$$

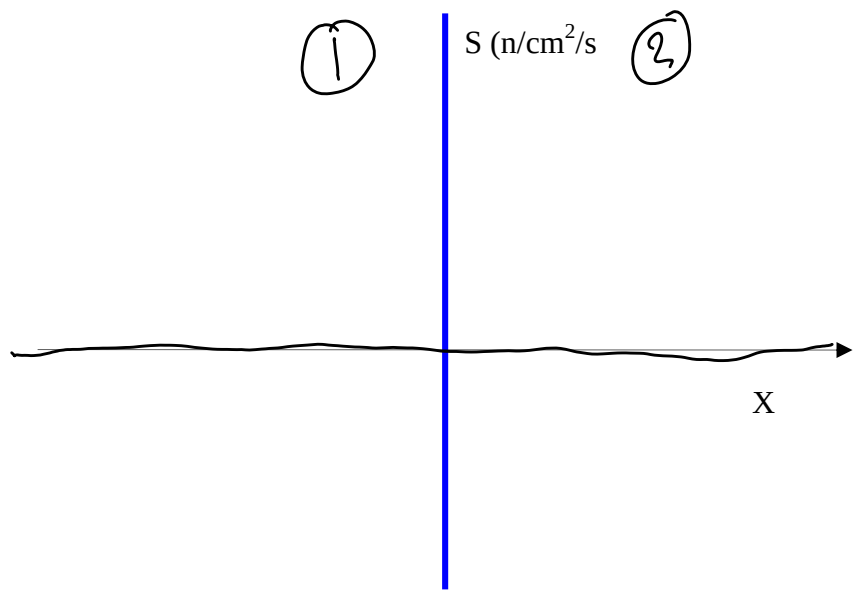
The thermal flux is hence:

$$\phi_2 = \frac{s_2 + \Sigma_{s1\to 2}\dfrac{s_1}{\Sigma_r}}{\Sigma_{a2}}$$

If there is no external thermal source $(s_2 = 0)$, then the solution simplifies to:

$$\phi_2 = \Sigma_{s1\to 2}\frac{s_1}{\Sigma_r \Sigma_{a2}}$$

One Group Diffusion for an Infinite Planar Source Situated in an Infinite Homogeneous Medium



Equivalent to two half-spaces (left and right)

$$-D\nabla^2\phi(x,y,z)+\Sigma_a\phi(x,y,z)=0 \quad \text{for } x \neq 0$$

Because of the symmetry, $\phi = \phi(x)$

The equation becomes:

$$-D\frac{\partial^2\phi(x)}{\partial\phi^2(x)} + \Sigma_a\phi(x) = 0 \qquad L^2 = \frac{D}{\Sigma_a}$$

Using the diffusion length notation:

$$\frac{d^2\phi}{dx^2} - \frac{1}{L^2}\phi = 0, \quad x \neq 0$$

e^{rx} - Particular Solution

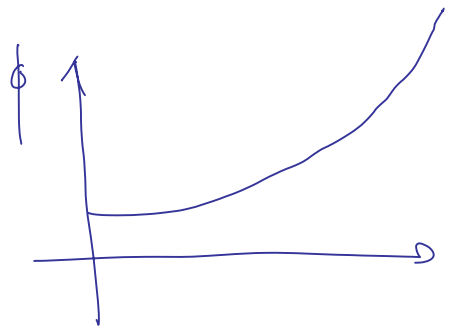
$$r^2 \cancel{e^{rx}} - \frac{1}{L^2}\cancel{e^{rx}} = 0$$

$r^2 - \frac{1}{L^2} = 0 \Rightarrow$ characteristic eq.

$r^2 = \frac{1}{L^2} \Rightarrow r = \pm \frac{1}{L}$; General solution $Ae^{-\frac{x}{L}} + Ce^{\frac{x}{L}}$

This is a homogeneous second order linear differential equation with constant coefficients. The general solution is of the type:

$$\phi(x) = Ae^{\frac{-x}{L}} + \cancel{C}e^{\frac{x}{L}}$$



Because the flux needs to be finite, we have C=0. Hence:

$$\phi(x) = Ae^{-x/L}$$

The current is:

$$J(x) = -D\frac{d}{dx}\left(Ae^{-x/L}\right) = A\frac{D}{L}e^{-x/L}$$

To find A, we use the boundary condition:

$$\lim_{x\to+0} J(x) = \frac{S}{2}$$

The initial condition yields:

$$J(x) = A\frac{D}{L}e^{0/L} = \frac{s}{2} \Rightarrow A\frac{D}{L} = \frac{s}{2} \Rightarrow A = \frac{sL}{2D}$$

The flux for x>0 is hence:

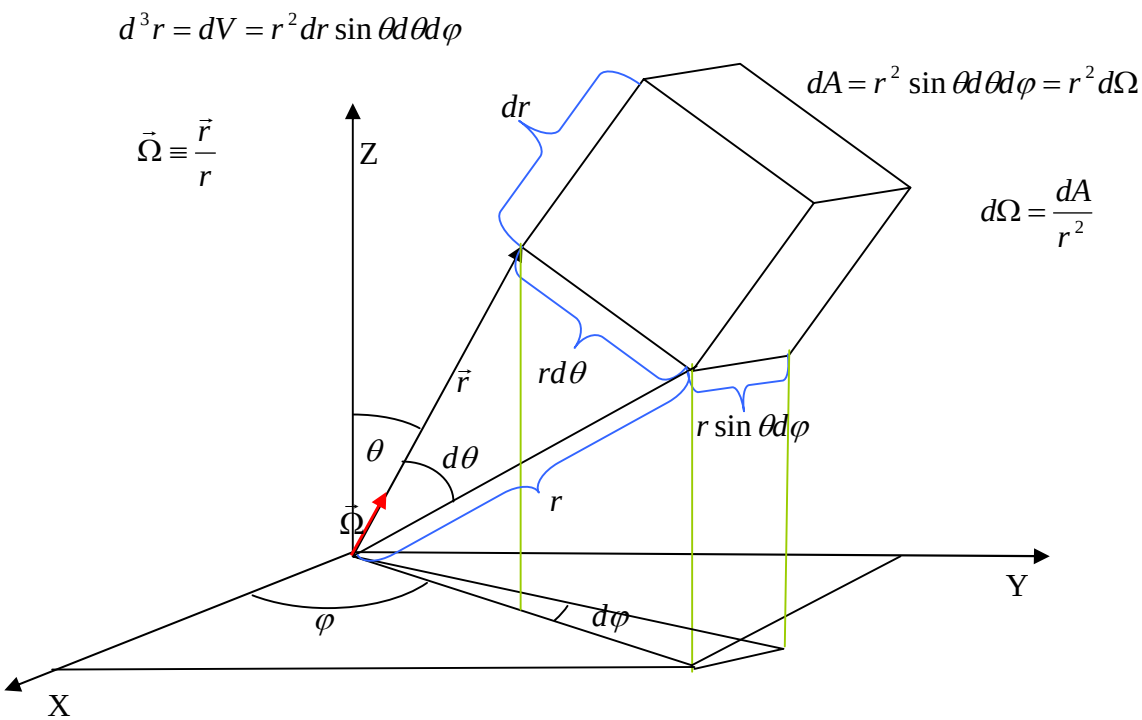
$$\phi = \frac{SL}{2D}e^{-x/L}$$

Analogously, the flux for x<0 is:

$$\phi = \frac{SL}{2D}e^{x/L}$$

One Group Diffusion for a Point Source Situated in an Infinite Homogeneous Medium

Use spherical coordinates with the source placed at the center



Because the problem is symmetrical with respect to both θ and φ (spherical symmetry) , the flux will only depend on r.

$$\phi = \phi(r)$$

Expression of Laplacian in spherical coordinates for a function with spherical symmetry, f(r).

$$\nabla^2 f(r) = \frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{df}{dr}\right)$$

The diffusion equation becomes:

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\phi}{dr}\right)-\frac{1}{L^2}\phi=0$$

This is a homogeneous second order differential equation with constant coefficients.

The boundary condition is

$$\lim_{r\to 0}\left(r^2 J(r)\right)=\frac{S}{4\pi}$$

$$\lim_{r\to 0} J(r)\cdot 4\pi r^2 = S$$

$$J\, r^2 = \frac{S}{4\pi}$$

$$\lim_{r\to\infty}\phi(r) = \text{finite}$$

To solve the equation, we make the substitution:

$$w = r\phi \Leftrightarrow \phi = \frac{w}{r}$$

The equation becomes:

$$\frac{1}{r^2}\frac{d}{dr}\left[r^2 \frac{d}{dr}\left(\frac{w}{r}\right)\right] - \frac{1}{L^2}\frac{w}{r} = 0$$

which yields:

$$\frac{d^2 w}{dr^2} - \frac{1}{L^2} w = 0$$

$$w = A e^{-\frac{r}{L}} + \cancel{C}^{\,0} e^{\frac{r}{L}}$$

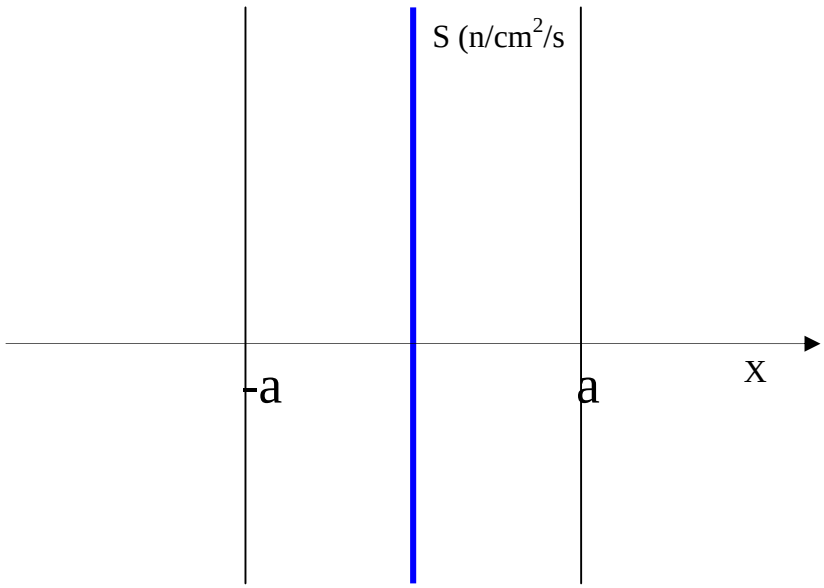
Following a similar treatment as for the plane source, we find:

$\phi = \frac{w}{r}$

$$\phi = \frac{S}{4\pi D}\frac{e^{-x/L}}{r}$$

non-multiplicative medium = no fissions
multiplicative medium = has fissions

One Group Diffusion for a Bare Slab with an Infinite Planar Source Situated in the Middle



The problem is symmetric with respect to the source and also has planar symmetry

$$\phi = \phi(x)$$

Diffusion equation:

$$\frac{d^2\phi}{dx^2} - \frac{1}{L^2}\phi = 0, \quad x \neq 0$$

Will treat the right half.

This is a homogeneous second order linear differential equation with constant coefficients. The general solution is of the type:

$$\phi(x) = Ae^{\frac{-x}{L}} + Ce^{\frac{x}{L}}$$

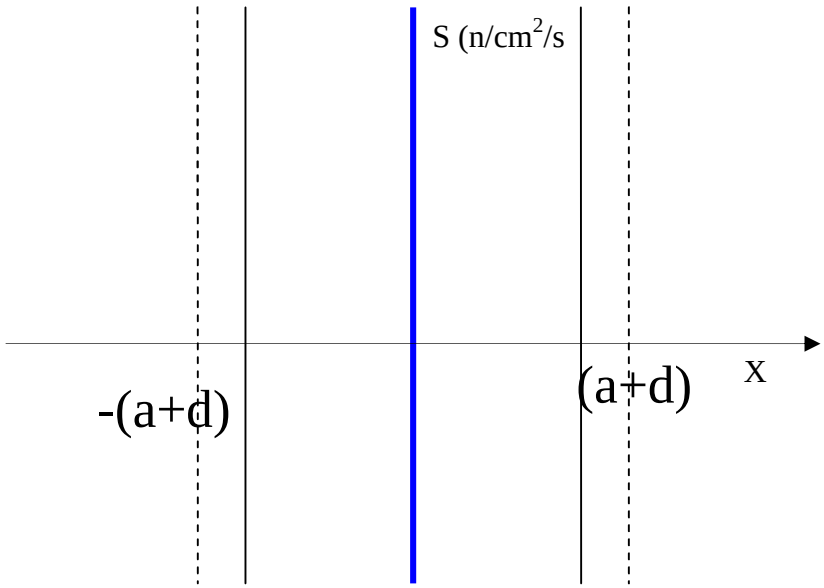
The left boundary condition is, just as before:

$$\lim_{x \to +0} J(x) = \frac{S}{2}$$

yielding:

$$J(x) = -\frac{D}{L} A e^{\frac{-x}{L}} + \frac{D}{L} C e^{\frac{x}{L}} \Bigg|_{x=0} = -\frac{D}{L} A + \frac{D}{L} C = \frac{S}{2}$$

The right boundary condition is now a vacuum boundary condition, that is the flux vanishes at the extrapolated boundary.



$$\phi(\tilde{a}) = 0$$

where

$$\tilde{a} = a + d$$

The above yields:

$$\phi(\tilde{a}) = Ae^{\frac{-\tilde{a}}{L}} + Ce^{\frac{\tilde{a}}{L}} = 0$$

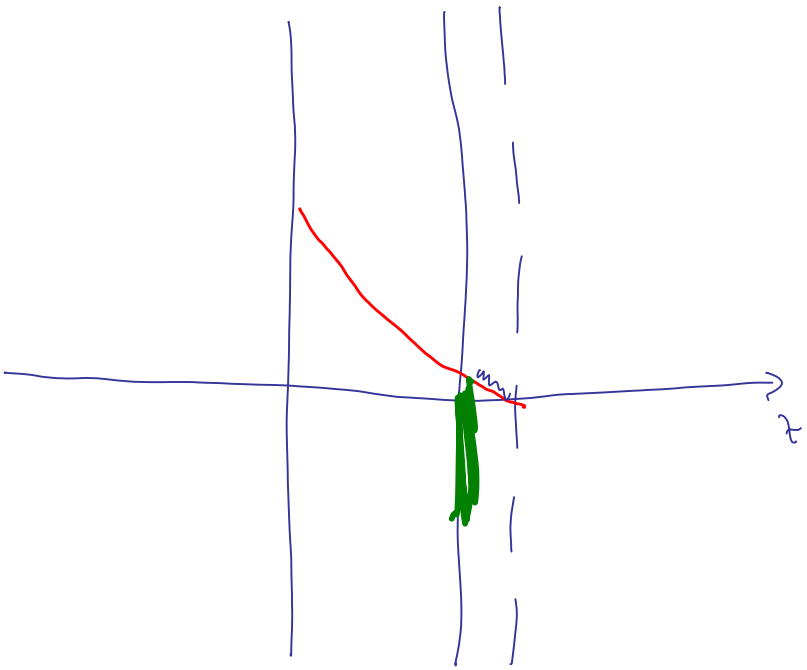
We obtain A and C by solving the system:

$$-\frac{D}{L}A + \frac{D}{L}C = \frac{S}{2}$$

$$\phi(\tilde{a}) = Ae^{\frac{-\tilde{a}}{L}} + Ce^{\frac{\tilde{a}}{L}} = 0$$

The final solution is:

$$\phi(x) = \frac{SL}{2D} \frac{e^{\frac{x}{L}} - e^{\frac{x-2\tilde{a}}{L}}}{1 + e^{-\frac{2\tilde{a}}{L}}}$$



Neutron Moderation (two group treatment)

Assume fast absorption is negligible

Two-group diffusion

$$-D_1\nabla^2\phi_1 + \Sigma_{s1\to 2}\phi_1 = 0$$

$$-D_2\nabla^2\phi_2 - \Sigma_{s1\to 2}\phi_1 + \Sigma_{a2}\phi_2 = 0$$

The two equations can be rearranged to:

$$-D_1\nabla^2\phi_1 + \Sigma_{s1\to 2}\phi_1 = S_1$$

$$-D_2\nabla^2\phi_2 + \Sigma_{a2}\phi_2 = \Sigma_{s1\to 2}\phi_1$$

We make the following notations:

$$\frac{D_1}{\Sigma_{s1\to 2}} = \tau_T \quad = \text{age}$$

Fermi

$$\frac{D_2}{\Sigma_{a2}} = L_T^2 \quad = \text{thermal diffusion area}$$

$$L_T = \text{thermal diffusion lenght}$$

With the new notations, the equations are written:

$$-\nabla^2\phi_1 + \frac{1}{\tau_T}\phi_1 = S_1$$

$$-\nabla^2\phi_2 + \frac{1}{L_T^2}\phi_2 = \frac{D_1}{D_2\tau_T}\phi_1$$

These can be solved for different configurations. For point source the solution is given in the textbook.

Nuclear Reactor Theory

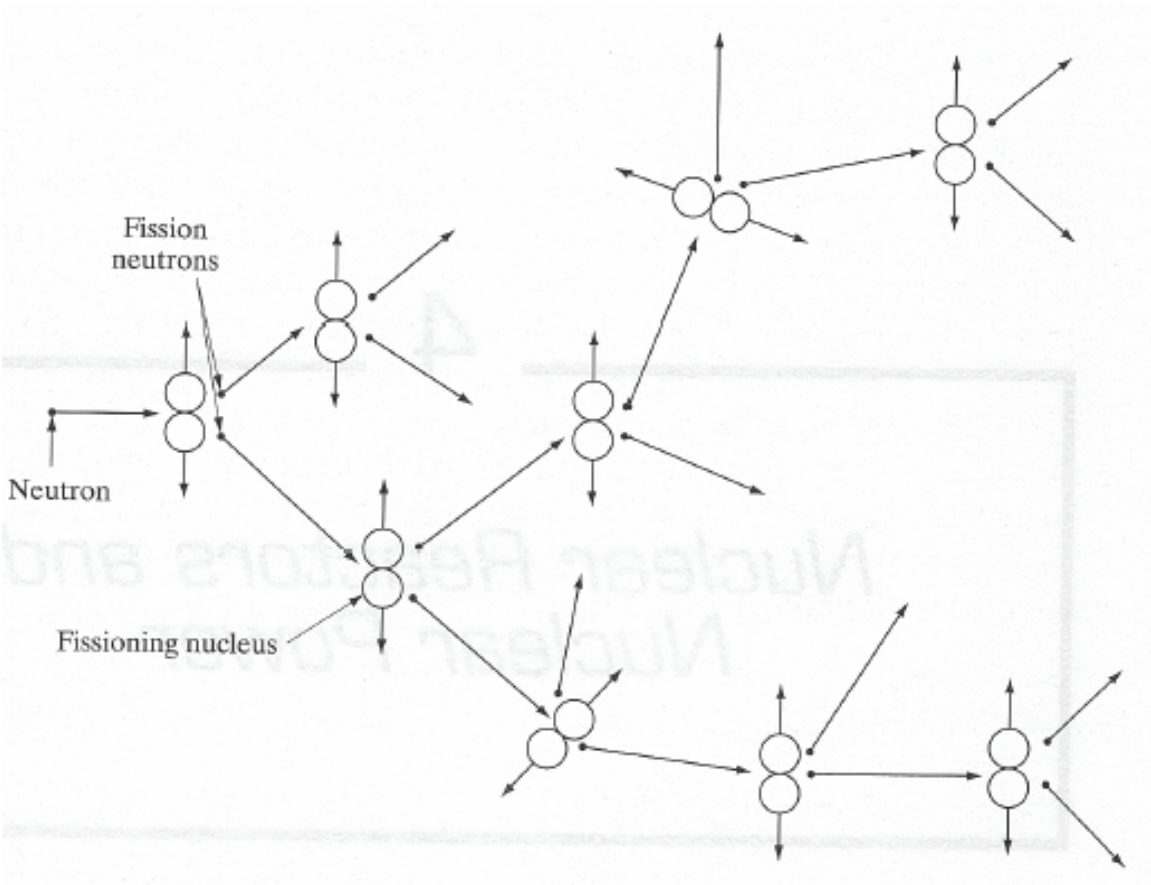
2005 E. Nichita

Preliminaries - Neutron Fluence

Neutron *fluence* is defined as the time integral of the flux

$$\psi = \int_{t_1}^{t_2} \phi(t)\,dt$$

Fission Chain Reaction



Each fission produces 2-3 more neutrons which can, in principle, induce new fissions in avalanche. This is not desirable.

However, not all neutrons resulting from fission induce new fissions. Some undergo gamma capture.

If two few neutrons (less than one per fission) induce new fissions the fission reaction dies down. Not desirable either.

The trick is to have a fission rate that is constant in time. A reactor operating at a constant fission rate is said to be *critical*.

Infinite Homogeneous Reactor
(One-Group Diffusion Approximation)

The steady-state diffusion equation is written:

$$-D\nabla^2\phi + \Sigma_a\phi = S$$

The source now consists of fission neutrons:

$$S = \nu\Sigma_f\phi$$

So the equation becomes:

$$-D\nabla^2\phi + \Sigma_a\phi = \nu\Sigma_f\phi$$

The flux is constant in space because the medium is infinite and homogeneous, so the equation becomes.

$$\Sigma_a \phi = \nu \Sigma_f \phi$$

It is obvious that the above cannot be satisfied, unless

$$\Sigma_a = \nu \Sigma_f$$

If that is not the case, then the source is artificially divided by a factor k, just to balance the equation.

$$\Sigma_a \phi = \frac{1}{k} \nu \Sigma_f \phi$$

$$\Sigma_a = \frac{1}{k} \nu \Sigma_f$$

k is called the *multiplication constant (factor)*. For an infinite medium, it is called the *infinite multiplication constant* and denoted by k_∞.

It is obvious that, for the one-group homogeneous reactor case:

$$k_\infty = \frac{\nu\Sigma_f}{\Sigma_a}$$

It is also obvious that the value of the flux cannot be determined because once the appropriate k is used, any value of the flux will satisfy the balance equation.

$$\Sigma_a\phi = \frac{1}{k_\infty}\nu\Sigma_f\phi \Rightarrow \Sigma_a\phi = \frac{1}{\frac{\nu\Sigma_f}{\Sigma_a}}\nu\Sigma_f\phi$$

Interpretation of k

Since the balance equation is written:

$$\Sigma_a \phi = \frac{1}{k_\infty} \nu \Sigma_f \phi$$

We have:

$$k_\infty = \frac{\nu \Sigma_f \phi}{\Sigma_a \phi}$$

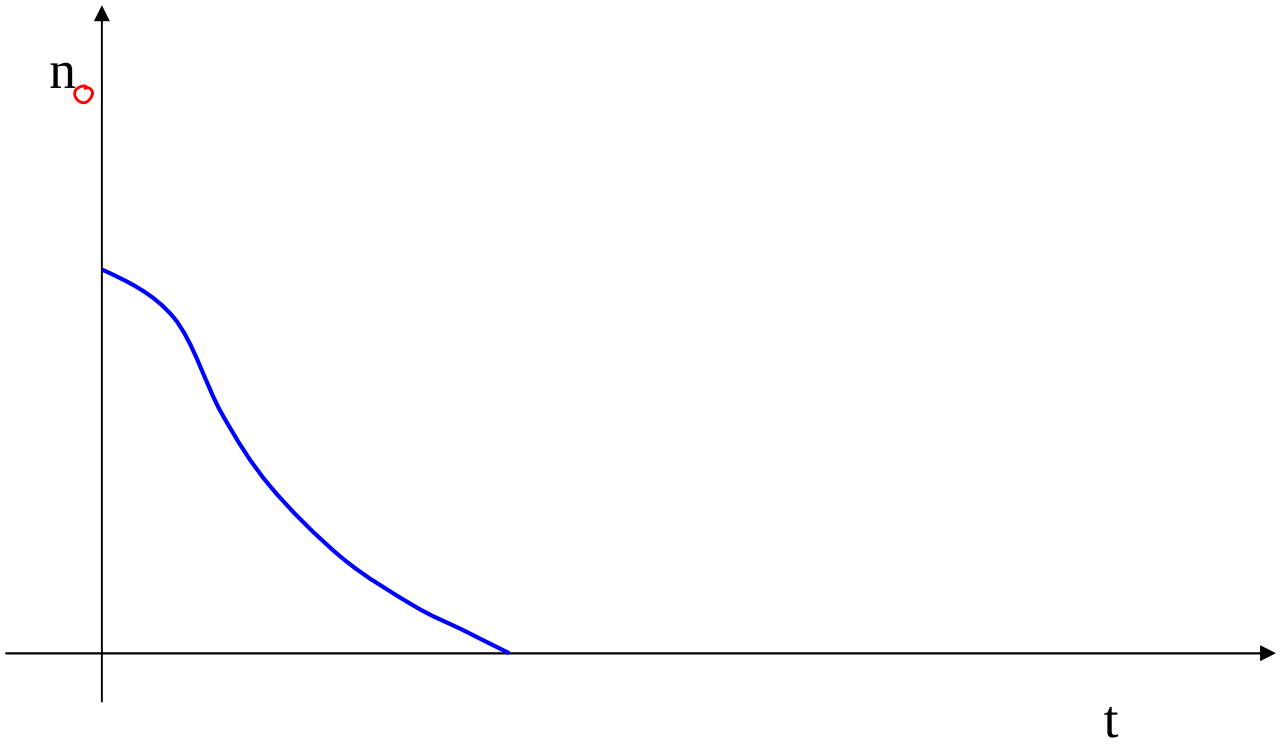
So k can be interpreted as the ratio of the neutron production rate and the neutron loss rate.

The name "multiplication factor" is used because k represents the ratio between the neutron density for one generation of neutrons, divided by the neutron density for the previous generation. This needs some explaining.

Consider a bare infinite homogeneous reactor. Initially there are no neutrons present.

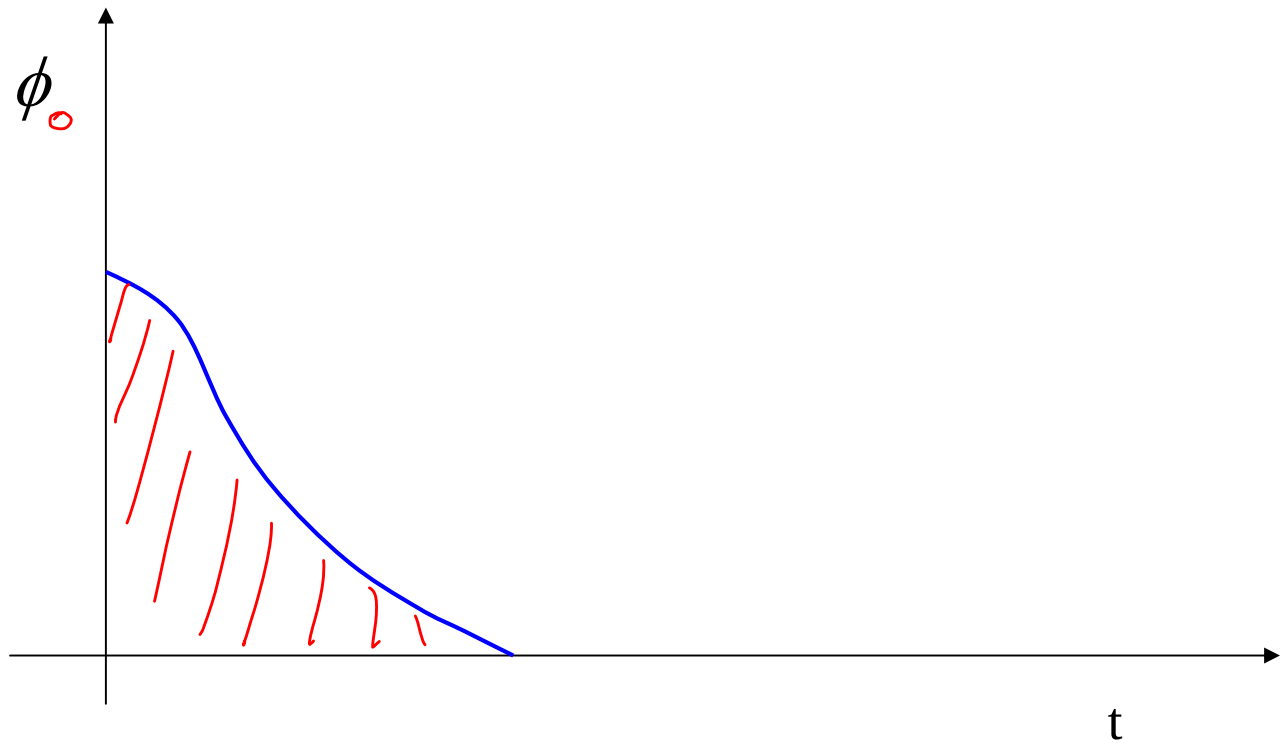
Now, assume some neutrons, with density n_0 are introduced in the reactor. Let's call these "generation 0" neutrons. These neutrons will fly around, producing a flux $\phi_0(t) = n_0(t)\mathrm{v}$ which will decrease as the neutrons are absorbed, until all neutrons are eventually absorbed.

The time dependence of the zero-generation neutrons looks something like this:



$\Sigma_a = \Sigma_f + \Sigma_\gamma$

The flux, has a similar shape



As these zero-generation neutrons are absorbed, some of them produce fissions. We consider the neutrons born out of these fissions first generation neutrons. They are produced at a rate:

$$\nu\Sigma_f \phi_0(t)$$

and are absorbed at a rate

$$\Sigma_a \phi_1(t)$$

Overall, the number of first-generation neutrons that are produced per unit volume is:

rate of production of firstgen neutrons

$$n_1 = \int_0^\infty \overbrace{\nu\Sigma_f \phi_0(t)}^{\text{rate of production of firstgen neutrons}} dt = \nu\Sigma_f \int_0^\infty \phi_0(t)dt = \nu\Sigma_f \psi_0$$

The total number of absorptions of first-generation neutrons is:

$$\int_0^\infty \underbrace{\Sigma_a \phi_1(t)}_{\text{rate of abs, of 1st gen neutrons}} dt = \Sigma_a \int_0^\infty \phi_1(t)dt = \Sigma_a \psi_1$$

Since, in the end, all first-generation neutrons get absorbed, we have:

$$\Sigma_a \psi_1 = \nu \Sigma_f \psi_0$$

which yields:

$$\psi_1 = \frac{\nu \Sigma_f}{\Sigma_a} \psi_0 = k_\infty \psi_0$$

The first-generation neutrons, in turn, produce second generation neutrons. Their number is:

$$n_2 = \nu \Sigma_f \psi_1 = \nu \Sigma_f k_\infty \psi_0 = k_\infty n_1$$

$k_\infty = \frac{\nu \Sigma_f}{\Sigma_a}$

The process continues:

$$n_3 = k_\infty n_2$$

and so on.

The number of neutrons in each generation is equal to the number in the previous generation multiplied by k_∞. Hence the name *multiplication factor*.

Infinite Homogeneous Reactor
(Two-Group Diffusion Approximation)

Diffusion equations:

$$-D_1\nabla^2\phi_1 + \Sigma_{a1}\phi_1 + \Sigma_{s1\to 2}\phi_1 = \nu\Sigma_{f1}\phi_1 + \nu\Sigma_{f2}\phi_2$$
$$-D_2\nabla^2\phi_2 - \Sigma_{s1\to 2}\phi_1 + \Sigma_{a2}\phi_2 = 0$$

Because the reactor is infinite and the flux (both fast and thermal) is constant in space, we have:

$$\Sigma_{a1}\phi_1 + \Sigma_{s1\to 2}\phi_1 = \nu\Sigma_{f1}\phi_1 + \nu\Sigma_{f2}\phi_2$$
$$-\Sigma_{s1\to 2}\phi_1 + \Sigma_{a2}\phi_2 = 0$$

Attempt to solve the system:

Group 2 equation yields:

$$\phi_2 = \frac{\Sigma_{s1\to 2}}{\Sigma_{a2}}\phi_1$$

Substituting into the group 1 equation, we obtain:

ϕ_2

$$\Sigma_{a1}\phi_1 + \Sigma_{s1\to 2}\phi_1 = \nu\Sigma_{f1}\phi_1 + \nu\Sigma_{f2}\frac{\Sigma_{s1\to 2}}{\Sigma_{a2}}\phi_1$$

Obviously, the above is only satisfied if:

$$\Sigma_{a1} + \Sigma_{s1\to 2} = \nu\Sigma_{f1} + \nu\Sigma_{f2}\frac{\Sigma_{s1\to 2}}{\Sigma_{a2}}$$

which may not always be the case. This means that unless the above is satisfied, we cannot have a steady-state solution to our diffusion equations.

To force the system of equations to have a (steady-state) solution, we resort to the same trick as before: use a "fudge factor" 1/k that multiplies fission productions.

Thus, our equations become:

$$\Sigma_{a1}\phi_1 + \Sigma_{s1\to 2}\phi_1 = \frac{1}{k_\infty}\left(\nu\Sigma_{f1}\phi_1 + \nu\Sigma_{f2}\phi_2\right)$$

$$-\Sigma_{s1\to 2}\phi_1 + \Sigma_{a2}\phi_2 = 0$$

And, by substituting $\phi_2 = \frac{\Sigma_{s1\to 2}}{\Sigma_{a2}}\phi_1$ into the fast-group equation, we obtain:

$$\Sigma_{a1}\phi_1 + \Sigma_{s1\to 2}\phi_1 = \frac{1}{k}\left(\nu\Sigma_{f1}\phi_1 + \nu\Sigma_{f2}\frac{\Sigma_{s1\to 2}}{\Sigma_{a2}}\phi_1\right)$$

$$\left(\Sigma_{a1} + \Sigma_{s12}\right)\phi_1 = \frac{1}{k}\left(\nu\Sigma_{f1} + \nu\Sigma_{f2}\frac{\Sigma_{s1\to 2}}{\Sigma_{a2}}\right)\phi_1$$

Dividing by the flux, we obtain:

$$\Sigma_{a1} + \Sigma_{s1\to 2} = \frac{1}{k_\infty}\left(\nu\Sigma_{f1} + \nu\Sigma_{f2}\frac{\Sigma_{s1\to 2}}{\Sigma_{a2}}\right)$$

We can now solve for k_∞ .

$$k_\infty = \frac{\nu\Sigma_{f1} + \nu\Sigma_{f2}\dfrac{\Sigma_{s1\to 2}}{\Sigma_{a2}}}{\Sigma_{a1} + \Sigma_{s1\to 2}}$$

K_∞

Choosing k_∞ to have the above value ensures the system admits a solution. That solution is $\phi_2 = \frac{\Sigma_{s1\to 2}}{\Sigma_{a2}}\phi_1$:

We cannot find the fast flux explicitly.

A close look at the system of equations

$$\Sigma_{a1}\phi_1 + \Sigma_{s1\to 2}\phi_1 = \frac{1}{k_\infty}\left(\nu\Sigma_{f1}\phi_1 + \nu\Sigma_{f2}\phi_2\right)$$

$$-\Sigma_{s1\to 2}\phi_1 + \Sigma_{a2}\phi_2 = 0$$

$$\begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}$$

reveals that it is a homogeneous system of linear equations which defines an eigenvalue/eigenvector problem. The eigenvalue is $1/k_\infty$ and, as expected, the eigenvector can only

be determined up to a multiplicative constant which, in our solution, is ϕ_1.

k_∞ can, in the two-group case be interpreted in three different ways:

1. the eigenvalue that allows the system of equations to have a solution
2. the ratio of productions over losses
3. the factor by which the number of neutrons gets multiplied from one generation to the next

Criticality

K<1 - Subcritical

- Number of neutrons decreases form one generation to the next
- Rate of neutron production smaller than rate of neutron loss

K=1 - Critical

- Number of neutrons stays constant form one generation to the next
- Rate of neutron production equals rate of neutron loss

K>1 - Supercritical

- Number of neutrons increases form one generation to the next
- Rate of neutron production larger than rate of neutron loss

The Four-Factor Formula

Let us look at the group 1 equation in the two-group approximation.

$$\Sigma_{a1}\phi_1 + \Sigma_{s1\to 2}\phi_1 = \frac{1}{k_\infty}\left(\nu\Sigma_{f1}\phi_1 + \nu\Sigma_{f2}\phi_2\right)$$

Solving for the multiplication factor, we obtain:

$$k_\infty = \frac{\nu\Sigma_{f1}\phi_1 + \nu\Sigma_{f2}\phi_2}{\Sigma_{a1}\phi_1 + \Sigma_{s1\to 2}\phi_1}$$

The above can be processed as follows:

$$k_\infty = \frac{\nu\Sigma_{f1}\phi_1 + \nu\Sigma_{f2}\phi_2}{\Sigma_{a1}\phi_1 + \Sigma_{s1\to 2}\phi_1} = \frac{\nu\Sigma_{f1}\phi_1 + \nu\Sigma_{f2}\phi_2}{\Sigma_{a1}\phi_1 + \Sigma_{s1\to 2}\phi_1}\,\frac{\nu\Sigma_{f2}\phi_2}{\nu\Sigma_{f2}\phi_2} =$$

$$= \frac{\nu\Sigma_{f1}\phi_1 + \nu\Sigma_{f2}\phi_2}{\nu\Sigma_{f2}\phi_2}\,\frac{\nu\Sigma_{f2}\phi_2}{\Sigma_{a1}\phi_1 + \Sigma_{s1\to 2}\phi_1}$$

By making the notation:

$$\varepsilon = \frac{\nu\Sigma_{f1}\phi_1 + \nu\Sigma_{f2}\phi_2}{\nu\Sigma_{f2}\phi_2}$$

We obtain:

$$k_\infty = \varepsilon \frac{\nu\Sigma_{f2}\phi_2}{\Sigma_{a1}\phi_1 + \Sigma_{s1\to 2}\phi_1}$$

We can continue the processing:

$$k_\infty = \varepsilon \frac{\nu\Sigma_{f2}\phi_2}{\Sigma_{a1}\phi_1 + \Sigma_{s1\to 2}\phi_1}\frac{\Sigma_{a2}\phi_2}{\Sigma_{a2}\phi_2} = \varepsilon \underbrace{\frac{\Sigma_{a2}\phi_2}{\Sigma_{a1}\phi_1 + \Sigma_{s1\to 2}\phi_1}}_{p}\frac{\nu\Sigma_{f2}\phi_2}{\Sigma_{a2}\phi_2}$$

Denoting:

$$p = \frac{\Sigma_{a2}\phi_2}{\Sigma_{a1}\phi_1 + \Sigma_{s1\to 2}\phi_1}$$

We have:

$$k_\infty = \varepsilon p \frac{\nu \Sigma_{f2} \phi_2}{\Sigma_{a2} \phi_2}$$

We can, moreover divide the thermal absorption cross section into the absorption cross section for fuel, and the one for moderator.

$$\Sigma_{a2} = \Sigma_{a2}^{\text{fuel}} + \Sigma_{a2}^{\text{moderator}}$$

With this, we can rewrite the formula for the multiplication factor as follows:

$$k_\infty = \varepsilon p \frac{\nu\Sigma_{f2}\phi_2}{\Sigma_{a2}\phi_2}\frac{\Sigma_{a2}^{\text{fuel}}\phi_2}{\Sigma_{a2}^{\text{fuel}}\phi_2} = \varepsilon p \underbrace{\frac{\Sigma_{a2}^{\text{fuel}}\phi_2}{\Sigma_{a2}\phi_2}}_{f}\underbrace{\frac{\nu\Sigma_{f2}\phi_2}{\Sigma_{a2}^{\text{fuel}}\phi_2}}_{\eta}$$

Denoting:

$$f = \frac{\Sigma_{a2}^{\text{fuel}}\phi_2}{\Sigma_{a2}\phi_2}$$

and

$$\eta = \frac{\nu\Sigma_{f2}\phi_2}{\Sigma_{a2}^{\text{fuel}}\phi_2}$$

We obtain:

$$k_\infty = \varepsilon p f \eta$$

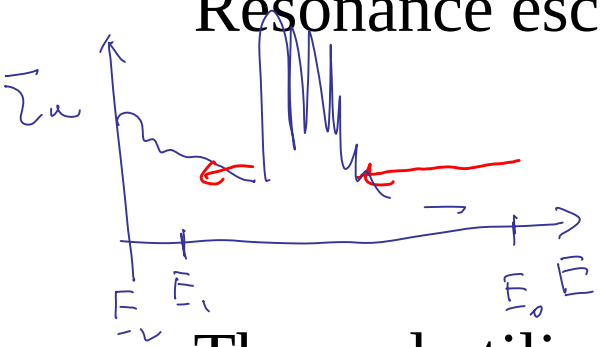
This is known as the *four factor formula*.

The names and interpretation of the factors are as follows:

Fast fission factor

$$\varepsilon = \frac{\overbrace{\nu\Sigma_{f1}\phi_1 + \nu\Sigma_{f2}\phi_2}^{\text{Total fission rate (fast + thermal)}}}{\nu\Sigma_{f2}\phi_2 \rightarrow \text{thermal fission rate}}$$

Resonance escape probability



$$p = \frac{\overbrace{\Sigma_{a2}\phi_2}^{\text{Thermal absorption rate}}}{\underbrace{\Sigma_{a1}\phi_1 + \Sigma_{s1\rightarrow 2}\phi_1}_{\text{Fast removal rate}}}$$

Thermal utilization factor

$$f = \frac{\overbrace{\Sigma_{a2}^{\text{fuel}}\phi_2}^{\text{Absorption rate in fuel}}}{\underbrace{\Sigma_{a2}\phi_2}_{\text{Total absorption rate}}}$$

η (number of neutrons produced per neutron absorbed in fuel)

$$\eta = \frac{\nu\Sigma_{f2}\phi_2}{\Sigma_{a2}^{\text{fuel}}\phi_2}$$

} - rate of neutron production

↳ rate of absorption in fuel.

One-Group Treatment of Finite Reactors

Diffusion Equation

$$D\nabla^2\phi - \Sigma_a\phi + \frac{1}{k}\nu\Sigma_{\mathrm{f}}\phi = 0$$

$$\nabla^2\phi + \frac{1}{D}\left(-\Sigma_a + \frac{1}{k}\nu\Sigma_{\mathrm{f}}\right)\phi = 0$$

Notation:

$$B^2 = \frac{1}{D}\left(-\Sigma_a + \frac{1}{k}\nu\Sigma_{\mathrm{f}}\right)$$

Equation can be rewritten:

$$\nabla^2 \phi + B^2 \phi = 0$$

B depends on k. It turns out that B cannot take just any value. It has to be equal to the value imposed by the geometry, called the geometrical buckling.

$$B^2 = B_g^2$$

Then:

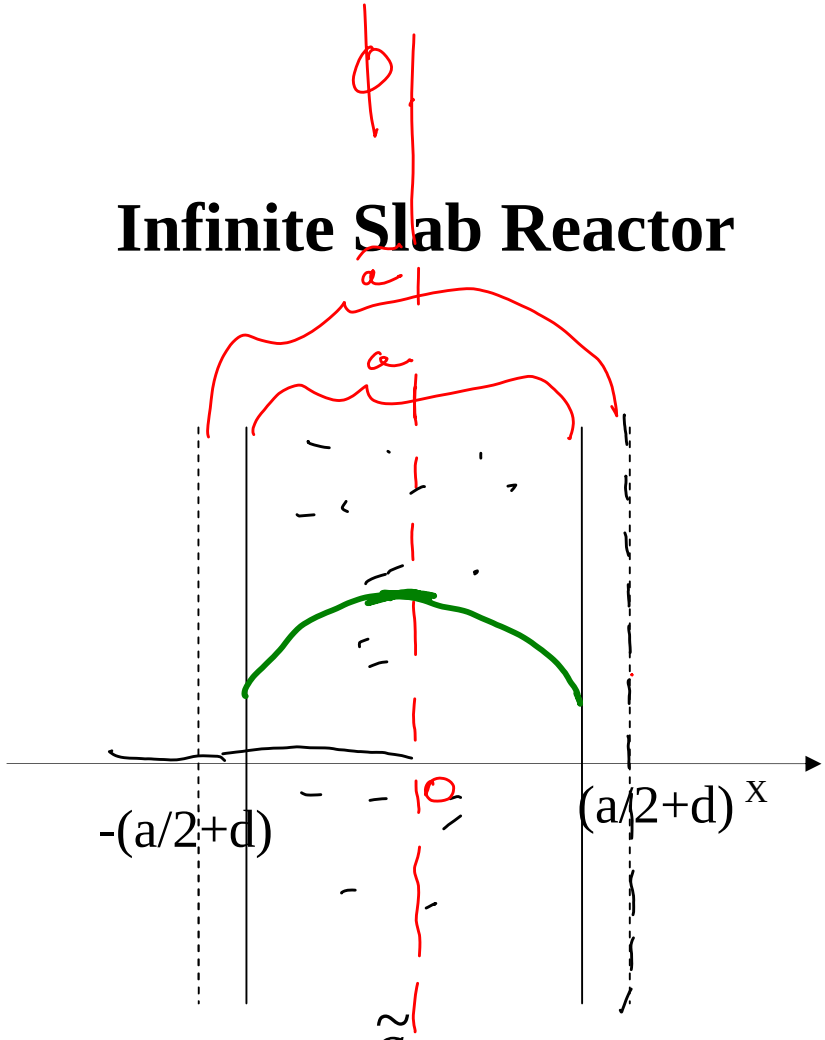
$$\frac{1}{D}\left(-\Sigma_a + \frac{1}{k}\nu\Sigma_{\mathrm{f}}\right) = B_g^2$$

offers an equation for k.

$$k = \frac{\nu\Sigma_{\mathrm{f}}}{B_g^2 D + \Sigma_a}$$

Things will become clearer by showing an example.

Infinite Slab Reactor



$$\phi\left(\frac{\tilde{a}}{2}\right) = 0$$

where

$$\tilde{a} = a + 2d$$

We then have:

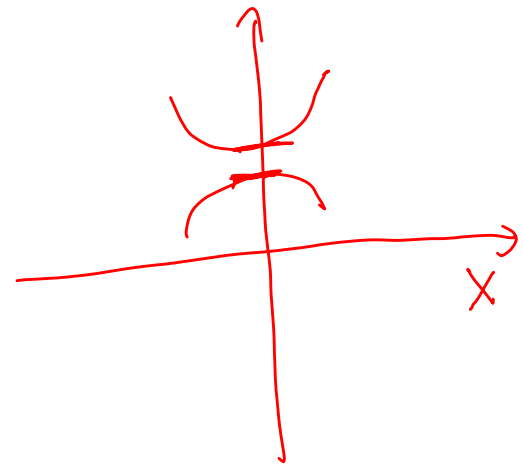
$$\frac{d^2\phi}{dx^2} + B^2\phi = 0$$

e^{rx}

$r^2 e^{rx} + B^2 e^{rx} = 0$

$r^2 + B^2 = 0 \Rightarrow r^2 = -B^2$

$r = \pm Bi$



Boundary conditions:

$$\phi\left(\frac{\tilde{a}}{2}\right) = \phi\left(\frac{-\tilde{a}}{2}\right) = 0$$

Symmetry of problem implies:

$$\left.\frac{d\phi}{dx}\right|_{x=0} = 0$$

General Solution:

$$\phi(x) = A\cos Bx + C\sin Bx$$

$$\left.\frac{d\phi(x)}{dx}\right|_{x=0} = \left[-\cancel{AB\sin Bx} + CB\underbrace{\cos Bx}_{1}\right]\Bigg|_{x=0} =$$

$$= CB = 0 \Rightarrow C = 0$$

Hence:

$$\phi(x) = A\cos Bx$$

Vacuum B.C.

$$\phi\left(\frac{\tilde{a}}{2}\right) = A\cos\left(\frac{B\tilde{a}}{2}\right) = 0$$

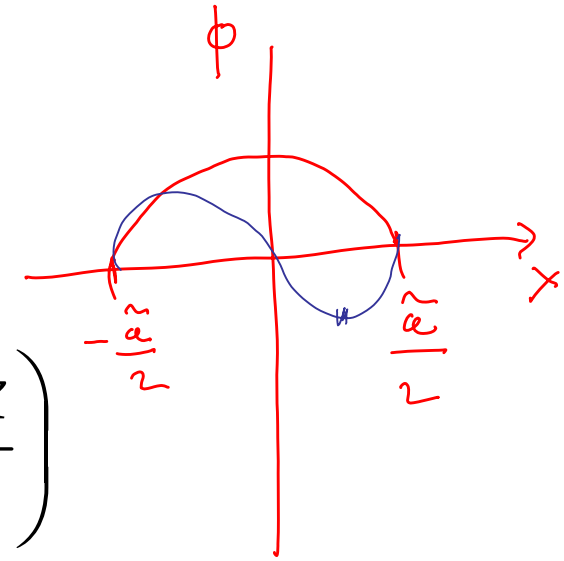
Implies:

$$\cos\left(\frac{B\tilde{a}}{2}\right) = 0$$

Yields:

$$B_n = \frac{n\pi}{\tilde{a}}$$

Fundamental solution



$$\phi(x) = A\cos B_1 x = A\cos\left(\frac{\pi x}{\tilde{a}}\right)$$

B_1 is the geometrical buckling $\quad B_1 = B_g = \frac{\pi}{\tilde{a}}$

A cannot be determined from the diffusion equation. It can be determined from the condition on the reactor power.

$$P = E_R \Sigma_f \int_{-a/2}^{a/2} \phi(x)dx$$

$$P = \frac{2\tilde{a}E_R\Sigma_f A\sin\left(\frac{\pi a}{2\tilde{a}}\right)}{\pi}$$

$$A = \frac{P\pi}{2\tilde{a}E_R\Sigma_f \sin\frac{\pi a}{2\tilde{a}}}$$

$$\phi(x) = \frac{\pi P}{2aE_R\Sigma_f}\cos\left(\frac{\pi x}{a}\right)$$

$$k = \frac{\nu\Sigma_{\mathrm{f}}}{B_g^2 D + \Sigma_a} = \frac{\nu\Sigma_{\mathrm{f}}}{\left(\frac{\pi}{\tilde{a}}\right)^2 D + \Sigma_a}$$

Spherical Reactor

We have, in sequence:

$\Delta\phi$

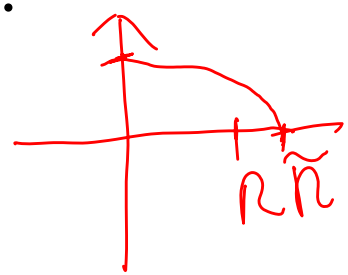
$$\frac{1}{r^2}\frac{d}{dr}r^2\frac{d\phi}{dr}+B^2\phi=0$$

$\phi=\frac{u}{r}$

$$\phi=A\frac{\sin Br}{r}+C\frac{\cos Br}{r}$$

Because the flux has to be finite at r=0, we have:

$$\phi=A\frac{\sin Br}{r}$$



$$B_1^2 = \left(\frac{\pi}{\tilde{R}} \right)^2$$

$$\phi = A \frac{\sin \pi r / \tilde{R}}{r}$$

$$P = E_R \Sigma_f \int \phi(r) dV$$

$$\phi = \frac{P}{4 E_R \Sigma_f R^2} \frac{\sin\left(\pi r / \tilde{R} \right)}{r}$$

Infinite Cylinder

We have in sequence:

$\nabla^2\phi$ in cylindrical coordinates

$$\overbrace{\frac{1}{r}\frac{d}{dr}r\frac{d\phi}{dr}}+B^2\phi=0$$

$$\frac{d^2\phi}{dr^2}+\frac{1}{r}\frac{d\phi}{dr}+B^2\phi=0$$

$m=0$

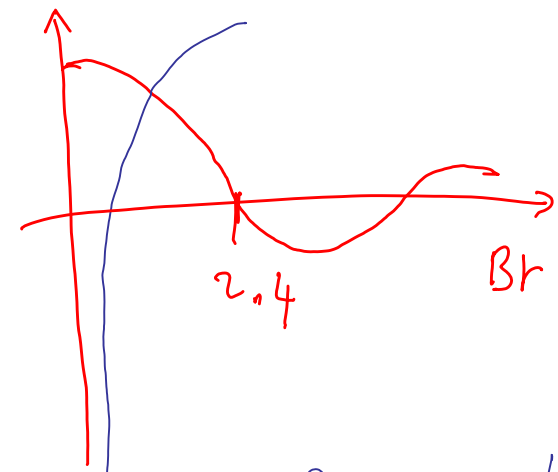
$$\frac{d^2\phi}{dr^2}+\frac{1}{r}\frac{d\phi}{dr}+\left(B^2-\frac{m^2}{r^2}\right)\phi=0$$

J_0

Solution: Bessel functions of first and second kind:

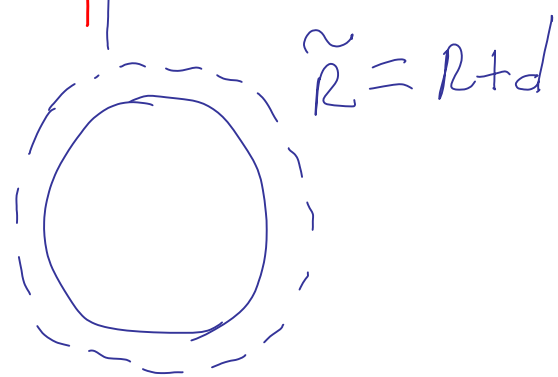
$$\phi = AJ_0(Br) + \cancel{CY_0(Br)}$$

$C = 0$



Y_0 infinite at origin (fig. 6.3)

$$\phi = AJ_0(Br)$$



B.C. Vacuum

$$\phi(\tilde{R}) = AJ_0(Br) = 0$$

$B\tilde{R} = j_1 = 2.4$

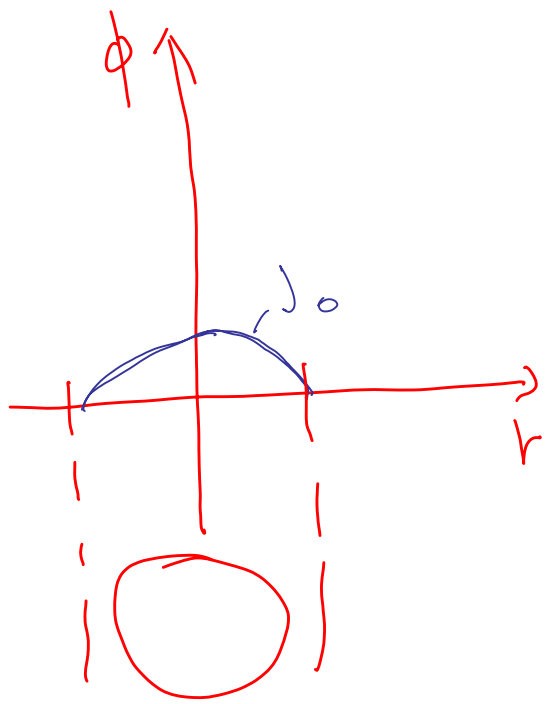
$B = \frac{j_1}{\tilde{R}} = \frac{2.4}{\tilde{R}}$

$x_1 \equiv j_1$

$$B_1^2 = \left(\frac{x_1}{\tilde{R}}\right)^2 = \left(\frac{2.405}{\tilde{R}}\right)^2$$

$$\phi = AJ_0\left(\frac{2.405r}{\tilde{R}}\right)$$

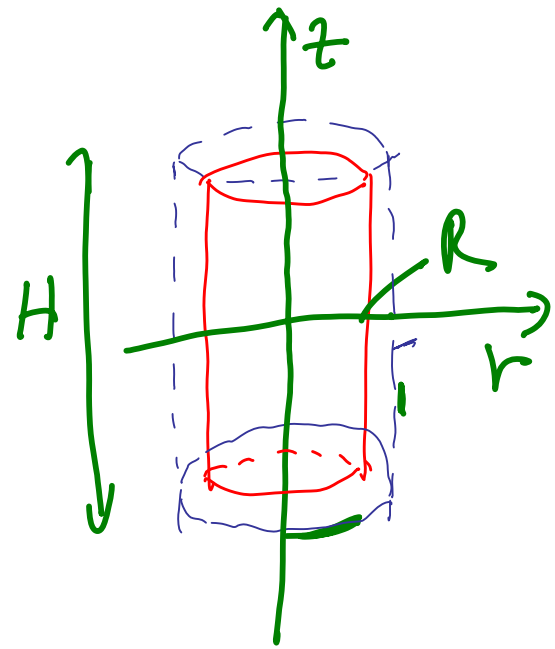
B



Finite Cylinder

$\phi = \phi(r,z)$

$$\underbrace{\frac{d^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{\partial^2 \phi}{\partial z^2}}_{\Delta \phi(r,z)} + B^2\phi = 0$$



B.C.

$$\phi(\tilde{R}, z) = 0$$

$$\phi(r, \frac{\tilde{H}}{2}) = 0$$

Separation of Variables

$$\cancel{Z}\frac{1}{r}\frac{d}{dr}r\frac{dR}{dr}+\cancel{R}\frac{d^2Z}{dz^2}+B^2RZ=0$$

Divide by ZR

$$\phi(r,z)=R(r)Z(z)$$

$$\underbrace{\frac{1}{R}\frac{1}{r}\frac{d}{\partial r}r\frac{\partial R}{\partial r}}_{f(r)}+\underbrace{\frac{1}{Z}\frac{\partial^2 Z}{\partial z^2}}_{f(z)}=-B^2$$

$$\frac{1}{R}\frac{1}{r}\frac{d}{\partial r}r\frac{\partial R}{\partial r}=-B_r^2 \Rightarrow \underbrace{\frac{1}{r}\frac{d}{dr}r\frac{dR}{dr}+B_r^2R=0}_{\text{infinite cylin}}$$

$$\frac{1}{Z}\frac{\partial^2 Z}{\partial z^2}=-B_z^2 \Rightarrow \underbrace{\frac{d^2Z}{dz^2}+B_z^2Z=0}_{\text{infinite slab}}$$

Solution:

$$\underbrace{\phi(r,z) = A\overbrace{J_0\left(\frac{2.405r}{\tilde{R}}\right)}^{\text{infinite cylinder}}\overbrace{\cos\frac{\pi z}{\tilde{H}}}^{\text{infinite slab}}}_{\text{finite cylinder.}}$$

Reactor Kinetics

Time-Dependent Phenomena

- Short Time Phenomena (ms, s)
  - accidents
  - experiments
  - startup
- Medium Time Phenomena (hrs, days)
  - fission product poisoning
    - Xe
    - Sm
- Long Time Phenomena (months, years)
  - fuel burnup with consequent change in composition

Time-Dependent Phenomena

- No feedback (approximation)
  - Changes in flux level do not induce changes in the absorption or production properties of the reactor.
- Feedback
  - Changes in flux level do induce changes in the absorption or production properties of the reactor.

Point Kinetics Equations

part 1

all neutrons emitted in a fission
are assumed prompt

One-energy-group diffusion equation

- time-dependent diffusion (results from neutron balance)

$$\frac{\partial n(\vec{r},t)}{\partial t} = \nu\Sigma_f \Phi(\vec{r},t) + D\nabla^2\Phi(\vec{r},t) - \Sigma_a \Phi(\vec{r},t)$$

- If sources are exactly equal to sinks, then the static equation results (no time dependence)

$$0 = \nu\Sigma_f \Phi(\vec{r}) + D\nabla^2\Phi(\vec{r}) - \Sigma_a \Phi(\vec{r}) \Leftrightarrow -D\nabla^2\Phi(\vec{r}) + \Sigma_a \Phi(\vec{r}) = \nu\Sigma_f \Phi(\vec{r})$$

One-energy-group diffusion equation

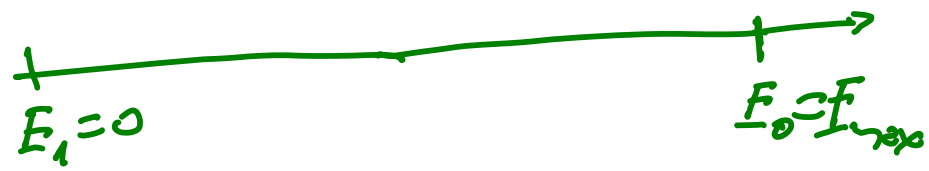
- To keep the static form of the diffusion equation even when the sources do not exactly equal the sinks, we introduced K, (multiplication factor) to artificially adjust the sources.

$$0 = \frac{1}{k}\nu\Sigma_f\Phi(\vec{r}) + D\nabla^2\Phi(\vec{r}) - \Sigma_a\Phi(\vec{r}) \Leftrightarrow -D\nabla^2\Phi(\vec{r}) + \Sigma_a\Phi(\vec{r}) = \frac{1}{k}\nu\Sigma_f\Phi(\vec{r})$$

- Now, we will not use k any more, but rather concentrate on the time-dependent equation

Time-dependent one-energy-group diffusion equation

$$\frac{\partial n(\vec{r},t)}{\partial t} = \nu\Sigma_f \Phi(\vec{r},t) + D\nabla^2\Phi(\vec{r},t) - \Sigma_a \Phi(\vec{r},t)$$



- But let's remember:

$$\Phi = n\bar{v} \Leftrightarrow n = \frac{\Phi}{\bar{v}}$$

- so:

$$\frac{\partial n}{\partial t} = \frac{1}{\bar{v}}\frac{\partial \Phi}{\partial t}$$

Time-dependence of the neutron flux and neutron density

- We can now write the time-dependent diffusion: in two separate ways
  - concentrate on the flux

$$\frac{1}{\bar{v}}\frac{\partial \Phi(\vec{r},t)}{\partial t} = \nu\Sigma_f \Phi(\vec{r},t) + D\nabla^2\Phi(\vec{r},t) - \Sigma_a \Phi(\vec{r},t)$$

  - concentrate on the neutron density

$$\frac{\partial n(\vec{r},t)}{\partial t} = \nu\Sigma_f \bar{v} n(\vec{r},t) + D\nabla^2 \bar{v} n(\vec{r},t) - \Sigma_a \bar{v} n(\vec{r},t)$$

Some assumptions

- Static one-energy-group diffusion equation for a critical reactor
  - $-D\nabla^2\Phi(\vec{r}) + \Sigma_a\Phi(\vec{r}) = \nu\Sigma_f\Phi(\vec{r})$
- It can be rewritten as:

$$\nabla^2\Phi(\vec{r}) + B^2\Phi(\vec{r}) = 0$$

- Where

$$\frac{\nu\Sigma_f - \Sigma_a}{D} = B^2$$

Some assumptions

- Assume that the equation satisfied by the time-independent flux in a critical reactor is also satisfied, at any time t, by the time-dependent flux in a non-critical reactor.

$$\nabla^2\Phi(\vec{r}) + B^2\Phi(\vec{r}) = 0$$

$$\nabla^2\Phi(\vec{r},t) + B^2\Phi(\vec{r},t) = 0 \Leftrightarrow \nabla^2\Phi(\vec{r},t) = -B^2\Phi(\vec{r},t)$$

- This is equivalent to assuming that the spatial shape of the flux does not change with time

$$\Phi(\vec{r},t) = \varphi_0(r) \cdot T(t)$$

Back to the time-dependence of the neutron flux and neutron density

- We can now write the time-dependent diffusion
  - for the flux

$B^2\phi = -\nabla^2\phi$

$$\frac{1}{\bar{v}}\frac{\partial\Phi(\vec{r},t)}{\partial t} = \nu\Sigma_f\Phi(\vec{r},t) - DB^2\Phi(\vec{r},t) - \Sigma_a\Phi(\vec{r},t)$$

  - for the neutron density

$$\frac{\partial n(\vec{r},t)}{\partial t} = \nu\Sigma_f\bar{v}n(\vec{r},t) - DB^2\bar{v}n(\vec{r},t) - \Sigma_a\bar{v}n(\vec{r},t)$$

Time-dependence of the neutron density

$$\frac{\partial n(\vec{r},t)}{\partial t} = \left(\nu\Sigma_f \bar{\mathrm{v}} - DB^2\bar{\mathrm{v}} - \Sigma_a \bar{\mathrm{v}}\right) n(\vec{r},t) \Leftrightarrow \frac{\partial n(\vec{r},t)}{\partial t} = \alpha n(\vec{r},t)$$

- where

$$\alpha = \left(\nu\Sigma_f - DB^2 - \Sigma_a\right)\bar{\mathrm{v}}$$

Time-dependence of the neutron density

$d^3\vec{r} \equiv dV$

$$\frac{\partial n(\vec{r},t)}{\partial t} = \alpha n(\vec{r},t)$$

- Integrating over the entire reactor we obtain:

$$\int_V \frac{\partial n(\vec{r},t)}{\partial t} d^3\vec{r} = \int_V \alpha n(\vec{r},t) d^3\vec{r}$$

$$\frac{d}{dt}\int_V n(\vec{r},t) d^3\vec{r} = \alpha \int_V n(\vec{r},t) d^3\vec{r}$$

Total neutron population in the reactor

Time-dependence of the total neutron population

- Total neutron population

$$n(t) = \int_V n(\vec{r},t)d^3\vec{r} = \int_V n(\vec{r},t)\,dV$$

- Equation governing the time behaviour of the total neutron population

$$\frac{d}{dt}n(t) = \alpha n(t) \Leftrightarrow \dot{n}(t) = \alpha n(t)$$

$$\dot{f} = \frac{df}{dt}$$

- solution

$$n(t) = n_0 e^{\alpha t}$$

Time dependence of the neutron flux

$$\frac{1}{\bar{v}}\frac{\partial\Phi(\vec{r},t)}{\partial t} = \nu\Sigma_f\Phi(\vec{r},t) - DB^2\Phi(\vec{r},t) - \Sigma_a\Phi(\vec{r},t)$$

$$\frac{1}{\bar{v}}\frac{\partial\Phi(\vec{r},t)}{\partial t} = \left(\nu\Sigma_f - DB^2 - \Sigma_a\right)\Phi(\vec{r},t)$$

$$\frac{\partial\Phi(\vec{r},t)}{\partial t} = \left(\nu\Sigma_f - DB^2 - \Sigma_a\right)\bar{v}\Phi(\vec{r},t) \Leftrightarrow \boxed{\frac{\partial\Phi(\vec{r},t)}{\partial t} = \alpha\Phi(\vec{r},t)}$$

- The results are analogous to those for the neutron density.

Time dependence of the neutron flux

- Integrating over the volume of the reactor:

$$\int_V \frac{\partial \Phi(\vec{r},t)}{\partial t} d^3\vec{r} = \int_V \alpha \Phi(\vec{r},t) d^3\vec{r}$$

$$\frac{d}{dt}\int_V \Phi(\vec{r},t) d^3\vec{r} = \alpha \int_V \Phi(\vec{r},t) d^3\vec{r}$$

$$\frac{d}{dt}\hat{\phi}(t) = \alpha\hat{\phi}(t) \Leftrightarrow \dot{\hat{\phi}}(t) = \alpha\hat{\phi}(t)$$

- where:

$$\hat{\phi}(t) = \int_V \Phi(\vec{r},t) d^3\vec{r}$$

total neutron population

$$\hat{\phi}(t) = n(t) \cdot \overline{V}$$

Observations

- The total neutron population and the volume integrated flux obey the same equation.
- The relation between the volume integrated flux and the total neutron population is the same as that between the flux and neutron density.

$$\Phi(\vec{r},t) = n(\vec{r},t)\overline{\mathrm{v}}$$

$$\int_V \Phi(\vec{r},t)d^3\vec{r} = \overline{\mathrm{v}}\int_V n(\vec{r},t)d^3\vec{r} \Leftrightarrow \hat{\phi}(t) = n(t)\overline{\mathrm{v}}$$

Point Kinetics Equation without Delayed Neutrons

- Just a special way of arranging the coefficients.
- Usually written for the neutron population, but similar equation can be written for the volume-integrated flux.

k_∞ – infinite medium
$k < k_{eff}$ – finite medium

Point Kinetics Equation without Delayed Neutrons

- Multiplication constant $k_{eff} = \dfrac{\nu\Sigma_f}{\Sigma_a + DB^2}$

- Reactivity $\rho = \dfrac{k-1}{k} = 1 - \dfrac{1}{k}$

$\rho > 0 \Rightarrow k > 1 \Rightarrow$ Supercritical
$\rho = 0 \Rightarrow k = 1 \Rightarrow$ Critical
$\rho < 1 \Rightarrow k < 1 \Rightarrow$ Subcritical

- we can write:

$$\alpha = \left(\nu\Sigma_f - DB^2 - \Sigma_a\right)\bar{v} = \nu\Sigma_f \bar{v} \frac{\nu\Sigma_f - DB^2 - \Sigma_a}{\nu\Sigma_f} =$$

$$\nu\Sigma_f \bar{v}\left(\frac{\nu\Sigma_f}{\nu\Sigma_f} - \frac{DB^2 + \Sigma_a}{\nu\Sigma_f}\right) = \nu\Sigma_f \bar{v}\underbrace{\left(1 - \frac{1}{k_{eff}}\right)}_{\rho} = \nu\Sigma_f \bar{v}\rho$$

Point Kinetics Equation without Delayed Neutrons

- Notation: $\Lambda = \dfrac{1}{v\Sigma_f \bar{v}}$

- It follows that:

$$\alpha = \frac{\rho}{\Lambda}$$

- The equation for the neutron population can then be written

$$\frac{dn(t)}{dt} = \frac{\rho}{\Lambda} n(t)$$ = Point kinetics eq. w/o dn

Solution: $n_0 e^{\frac{\rho}{\Lambda} t}$

Point Kinetics Equation without Delayed Neutrons

- A similar equation can be written for the volume-integrated flux.

$$\frac{d\hat{\phi}(t)}{dt} = \frac{\rho}{\Lambda}\hat{\phi}(t)$$

- Alternative processing:

$$\alpha = \left(\nu\Sigma_f - DB^2 - \Sigma_a\right)\bar{v} = \left(DB^2 + \Sigma_a\right)\bar{v}\,\frac{\nu\Sigma_f - DB^2 - \Sigma_a}{DB^2 + \Sigma_a} =$$

$$\left(DB^2 + \Sigma_a\right)\bar{v}\left(\frac{\nu\Sigma_f}{DB^2 + \Sigma_a} - \frac{DB^2 + \Sigma_a}{DB^2 + \Sigma_a}\right) = \left(DB^2 + \Sigma_a\right)\bar{v}\left(k - 1\right)$$

Point Kinetics Equation without Delayed Neutrons

- New notation

$$\ell = \frac{1}{\left(DB^2 + \Sigma_a\right)\bar{v}}$$

$$\alpha = \frac{k-1}{\ell}$$

- With the new notation the point kinetics eq. can be written (a less common form):

$$\frac{dn(t)}{dt} = \frac{k-1}{\ell} n(t)$$

- and, for the flux:

$$\frac{d\hat{\phi}(t)}{dt} = \frac{k-1}{\ell} \hat{\phi}(t)$$

Point kinetics equation(s)

- Nomenclature – called point-kinetics because the reactor is reduced to a point – no accounting for spatial or energy dependence.
- Can be derived starting from a more general, space and energy dependent, flux.

Names and interpretations of symbols

- Neutron generation time

$$\Lambda = \frac{1}{\bar{\nu}\nu\Sigma_f} = \frac{h}{\underbrace{\bar{\nu}n}_{\phi}\nu\Sigma_f} = \frac{h}{\underbrace{\phi\nu\Sigma_f}_{\text{total generation rate}}}$$

$h \rightarrow$ total neutron population

- Interpretations
  - Average time between two neutron births in successive generations
  - Time it would take to generate the current number of neutrons at the current generation rate.
  - Average "age" of neutrons in the reactor. (Note that this is a time, and not the Fermi age).

Names and interpretations of symbols

total neutron pop.

- Neutron life time

$$\ell = \frac{1}{\bar{v}} \frac{1}{\Sigma_a + DB^2} = \frac{n}{n\bar{v}(\Sigma_a + DB^2)} = \frac{n}{\hat{\phi}(\Sigma_a + DB^2)}$$

- For an infinite reactor: $\ell_\infty = \frac{1}{\bar{v}} \frac{1}{\Sigma_a}$ $\quad B^2 = 0$
- Interpretations
  - average time between the birth and death of a neutron
  - Time necessary to lose all the neutrons in the reactor at the current loss rate.
  - Average life expectancy for neutrons in the reactor.

$$p = \bar{v}\left(\Sigma_a + DB^2\right)$$

$$dn = -p\,n \qquad \text{similar to} \qquad dN = -\lambda N$$

$$T_{2v} = \frac{1}{\lambda}$$

$$T_{avg}\,n = \frac{1}{p} = \frac{1}{\bar{v}\left(\Sigma_a + DB^2\right)} \equiv l$$

Point Kinetics Equations

part 2

Accounting for Delayed Neutrons

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Point Kinetics with Only One Delayed Neutron Group

- We make the same assumptions about the buckling staying constant as in the case with no delayed neutrons.
- We write directly the equation for the entire reactor (volume-integrated quantities)
- Some neutrons are emitted directly from fission
- Some neutrons come from the decay of precursors.

Neutron Balance Equation for the Entire Reactor

$\nu = \nu_p + \nu_d$

$\frac{\nu_d}{\nu} = \beta$

- Sources
  - Prompt neutrons from fission

$$\int_V \nu_p \Sigma_f \Phi(\vec{r}) d^3\vec{r} = \nu_p \Sigma_f \int_V \Phi(\vec{r}) d^3\vec{r} = \nu_p \Sigma_f \hat{\phi} = (\nu - \nu_d)\Sigma_f \hat{\phi} = \nu(1-\beta)\Sigma_f \hat{\phi}$$

  - Delayed neutrons from the decay of precursors

$$\lambda \hat{C} \qquad (\hat{C} = \text{total number of precursors in the reactor})$$

Neutron Balance Equation for the Entire Reactor

- Sinks
  - Absorption

$$\int_V \Sigma_a \Phi(\vec{r}) d^3\vec{r} = \Sigma_a \int_V \Phi(\vec{r}) d^3\vec{r} = \Sigma_a \hat{\phi}$$

  - Leakage

$$\int_V DB^2 \Phi(\vec{r}) d^3\vec{r} = DB^2 \int_V \Phi(\vec{r}) d^3\vec{r} = DB^2 \hat{\phi}$$

Precursor Balance Equation for the Entire Reactor

- Source

$$\int_V \nu_d \Sigma_f \Phi(\vec{r}) d^3\vec{r} = \nu_d \Sigma_f \int_V \Phi(\vec{r}) d^3\vec{r} = \nu_d \Sigma_f \hat{\phi} = \nu \beta \Sigma_f \hat{\phi}$$

- Sink

$$\lambda \hat{C}$$

Neutron and Precursor balance Equations

- Neutron Balance

$$\frac{dn(t)}{dt} = \nu_p \Sigma_f \hat{\phi} - \Sigma_a \hat{\phi} - DB^2 \hat{\phi} + \lambda \ \hat{C}$$

- Precursor Balance

$$\frac{d\hat{C}(t)}{dt} = \nu_d \Sigma_f \hat{\phi} - \lambda \hat{C}(t)$$

- We now have a system of two (coupled) differential equations.

Point Kinetics Equations with One Group of Delayed Neutrons

- Rearrange the first equation in a few steps

$$\frac{dn(t)}{dt} = \left[\nu(1-\beta)\Sigma_f - \Sigma_a - DB^2\right]\hat{\phi} + \lambda\,\hat{C}$$

$$\frac{dn(t)}{dt} = \nu\Sigma_f \frac{\left[\nu(1-\beta)\Sigma_f - \Sigma_a - DB^2\right]}{\nu\Sigma_f}\hat{\phi} + \lambda\,\hat{C}$$

$$\frac{dn(t)}{dt} = \bar{\nu}\,\nu\Sigma_f \left[\frac{\nu\Sigma_f - \Sigma_a - DB^2}{\nu\Sigma_f} - \frac{\beta\cancel{\nu\Sigma_f}}{\cancel{\nu\Sigma_f}}\right]\frac{\hat{\phi}}{\bar{\nu}} + \lambda\,\hat{C}$$

$$\frac{dn(t)}{dt} = \bar{\nu}\,\nu\Sigma_f \left[1 - \frac{\Sigma_a + DB^2}{\nu\Sigma_f} - \beta\right]\frac{\hat{\phi}}{\bar{\nu}} + \lambda\,\hat{C}$$

$\frac{1}{k}$

$1 - \frac{1}{k} = \bar{\rho}$

Point Kinetics Equations with One Group of Delayed Neutrons

- Rearrange the second equation

$$\frac{d\hat{C}(t)}{dt} = \underbrace{\nu_d \Sigma_f \hat{\phi}} - \lambda \hat{C}(t)$$

$\nu_d = \nu\beta$

$$\frac{d\hat{C}(t)}{dt} = \nu\beta\Sigma_f \hat{\phi} - \lambda \hat{C}(t)$$

$$\frac{d\hat{C}(t)}{dt} = \beta\bar{v}\nu\Sigma_f \frac{\hat{\phi}}{\bar{v}} - \lambda \hat{C}(t)$$

Point Kinetics Equations with One Group of Delayed Neutrons

- Make the same notations and observations as for the case with no delayed neutrons

$$\Lambda = \frac{1}{\nu\Sigma_f \bar{v}}$$

$$\hat{\phi}(t) = n(t)\bar{\mathrm{v}} \Leftrightarrow n(t) = \frac{\hat{\phi}(t)}{\bar{\mathrm{v}}}$$

$$k = \frac{\nu\Sigma_f}{\Sigma_a + DB^2}$$

$$\rho = 1 - \frac{1}{k} = 1 - \frac{\Sigma_a + DB^2}{\nu\Sigma_f}$$

Point Kinetics Equations with One Group of Delayed Neutrons

- Neutron Balance Equation

$$\frac{dn(t)}{dt} = \underbrace{\bar{v}\nu\Sigma_f}_{\frac{1}{\Lambda}} \left[\underbrace{1 - \frac{\Sigma_a + DB^2}{\nu\Sigma_f}}_{\rho} - \beta \right] \underbrace{\frac{\hat{\phi}}{\bar{v}}}_{n} + \lambda \hat{C}$$

$$\frac{dn(t)}{dt} = \frac{\rho - \beta}{\Lambda} n(t) + \lambda \hat{C}$$

- Precursor Balance Equation

$$\frac{d\hat{C}(t)}{dt} = \beta \underbrace{\bar{v}\nu\Sigma_f}_{\frac{1}{\Lambda}} \underbrace{\frac{\hat{\phi}}{\bar{v}}}_{=n} - \lambda \hat{C}(t)$$

$$\frac{d\hat{C}(t)}{dt} = \frac{\beta}{\Lambda} n(t) - \lambda \hat{C}(t)$$

Point Kinetics Equations with One Group of Delayed Neutrons

- Final form of kinetics equations using the neutron population

$$\frac{dn(t)}{dt} = \frac{\rho - \beta}{\Lambda} n(t) + \lambda \hat{C}$$ => Neutron balance

$$\frac{d\hat{C}(t)}{dt} = \frac{\beta}{\Lambda} n(t) - \lambda \hat{C}(t)$$ => precursor balance.

Point Kinetics Equations with One Group of Delayed Neutrons

- Final form of the point kinetics equations using the volume-integrated flux

$$\hat{\phi}(t) = n(t)\bar{\mathrm{v}} \Leftrightarrow n(t) = \frac{\hat{\phi}(t)}{\bar{\mathrm{v}}}$$

$$\frac{d\hat{\phi}(t)}{dt} = \frac{\rho - \beta}{\Lambda}\hat{\phi}(t) + \bar{\mathrm{v}}\lambda\,\hat{C} \Leftrightarrow \frac{d\hat{\phi}(t)}{dt} = \frac{\rho - \beta}{\Lambda}\hat{\phi}(t) + \frac{1}{\Lambda\nu\Sigma_f}\lambda\,\hat{C}$$

$$\frac{d\hat{C}(t)}{dt} = \frac{\beta}{\Lambda}\frac{1}{\bar{\mathrm{v}}}\hat{\phi}(t) - \lambda\hat{C}(t) \Leftrightarrow \frac{d\hat{C}(t)}{dt} = \beta\nu\Sigma_f\hat{\phi}(t) - \lambda\hat{C}(t)$$

Point Kinetics Equations with Six Groups of Delayed Neutrons

- Equations using the neutron population (7 coupled differential equations)

$$\frac{dn(t)}{dt} = \frac{\rho - \beta}{\Lambda} n(t) + \sum_{k=1}^{6} \lambda_k \hat{C}_k$$

$$\beta_k = \frac{\nu_{dk}}{\nu}$$

$$\frac{d\hat{C}_k(t)}{dt} = \frac{\beta_k}{\Lambda} n(t) - \lambda_k \hat{C}_k(t), \quad k = 1 \ldots 6$$

Point Kinetics Equations with Six Groups of Delayed Neutrons

- Equations using the volume-integrated flux (7 coupled differential equations)

$$\frac{d\hat{\phi}(t)}{dt} = \frac{\rho-\beta}{\Lambda}\hat{\phi}(t) + \bar{\nu}\sum_{k=1}^{6}\lambda_k \hat{C}_k \Leftrightarrow \frac{d\hat{\phi}(t)}{dt} = \frac{\rho-\beta}{\Lambda}\hat{\phi}(t) + \frac{1}{\Lambda \nu\Sigma_f}\sum_{k=1}^{6}\lambda_k \hat{C}_k$$

$$\frac{d\hat{C}_k(t)}{dt} = \frac{\beta_k}{\Lambda}\frac{1}{\bar{\nu}}\hat{\phi}(t) - \lambda_k \hat{C}_k(t) \Leftrightarrow \frac{d\hat{C}_k(t)}{dt} = \beta_k \nu\Sigma_f \hat{\phi}(t) - \lambda_k \hat{C}_k(t)$$

Inhour Equation

Start with the point kinetics equations

$$\frac{dn(t)}{dt} = \frac{\rho - \beta}{\Lambda} n(t) + \sum_{k=1}^{6} \lambda_k \hat{C}_k$$

$$\frac{d\hat{C}_k(t)}{dt} = \frac{\beta_k}{\Lambda} n(t) - \lambda_k \hat{C}_k(t), \quad k = 1...6$$

This is a system of seven coupled differential equations with constant coefficients.

Solutions of the form

$$Y = \begin{bmatrix} \tilde{n} e^{\omega t} \\ \tilde{c}_1 e^{\omega t} \\ \vdots \\ \tilde{c}_6 e^{\omega t} \end{bmatrix}$$

$$\frac{d}{dt}\left(h e^{\omega t}\right) = \omega h e^{\omega t}$$

General solution

$$\sum_{m=1}^{7} C_m Y_m$$

Substituting the above form we obtain:

$$\left.\begin{aligned} \omega\tilde{n} &= \frac{\rho-\beta}{\Lambda}\tilde{n} + \sum_{k=1}^{6}\lambda_k\tilde{c}_k \\ \omega c_k &= \frac{\beta_k}{\Lambda}\tilde{n} - \lambda_k\tilde{c}_k, \quad k = 1\ldots 6 \end{aligned}\right\}$$

Characteristic system

Solving for c_k in the precursor equations, we obtain:

$$\tilde{c}_k = \frac{\beta_k}{\Lambda(\omega+\lambda_k)}\tilde{n}, \quad k = 1\ldots 6$$

Substituting into the neutron population equation we obtain:

$$\omega\tilde{n} = \frac{\rho - \beta}{\Lambda}\tilde{n} + \sum_{k=1}^{6} \lambda_k \frac{\beta_k}{\Lambda(\omega + \lambda_k)}\tilde{n}$$

which, after division by $\tilde{n}$ and multiplication by Λ, becomes:

$$\Lambda\omega = \rho - \beta + \sum_{k=1}^{6} \lambda_k \frac{\beta_k}{(\omega + \lambda_k)}$$

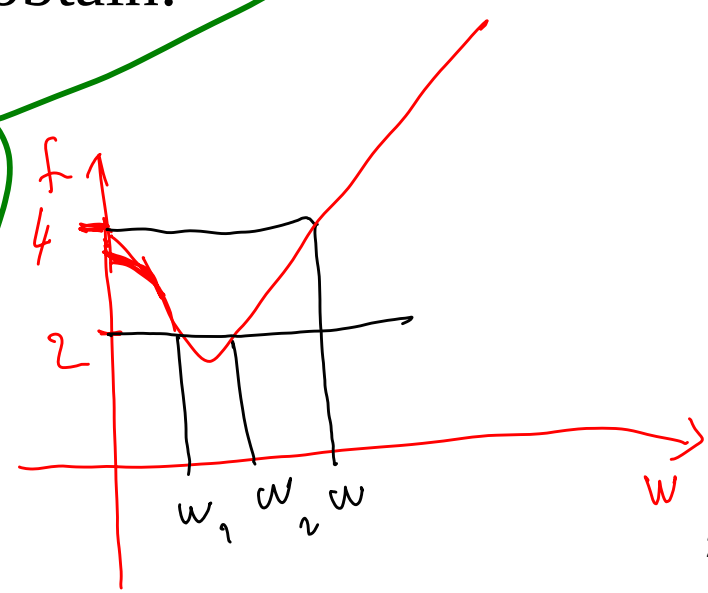
Inhour Equatio

Solving formally for ρ, we obtain:

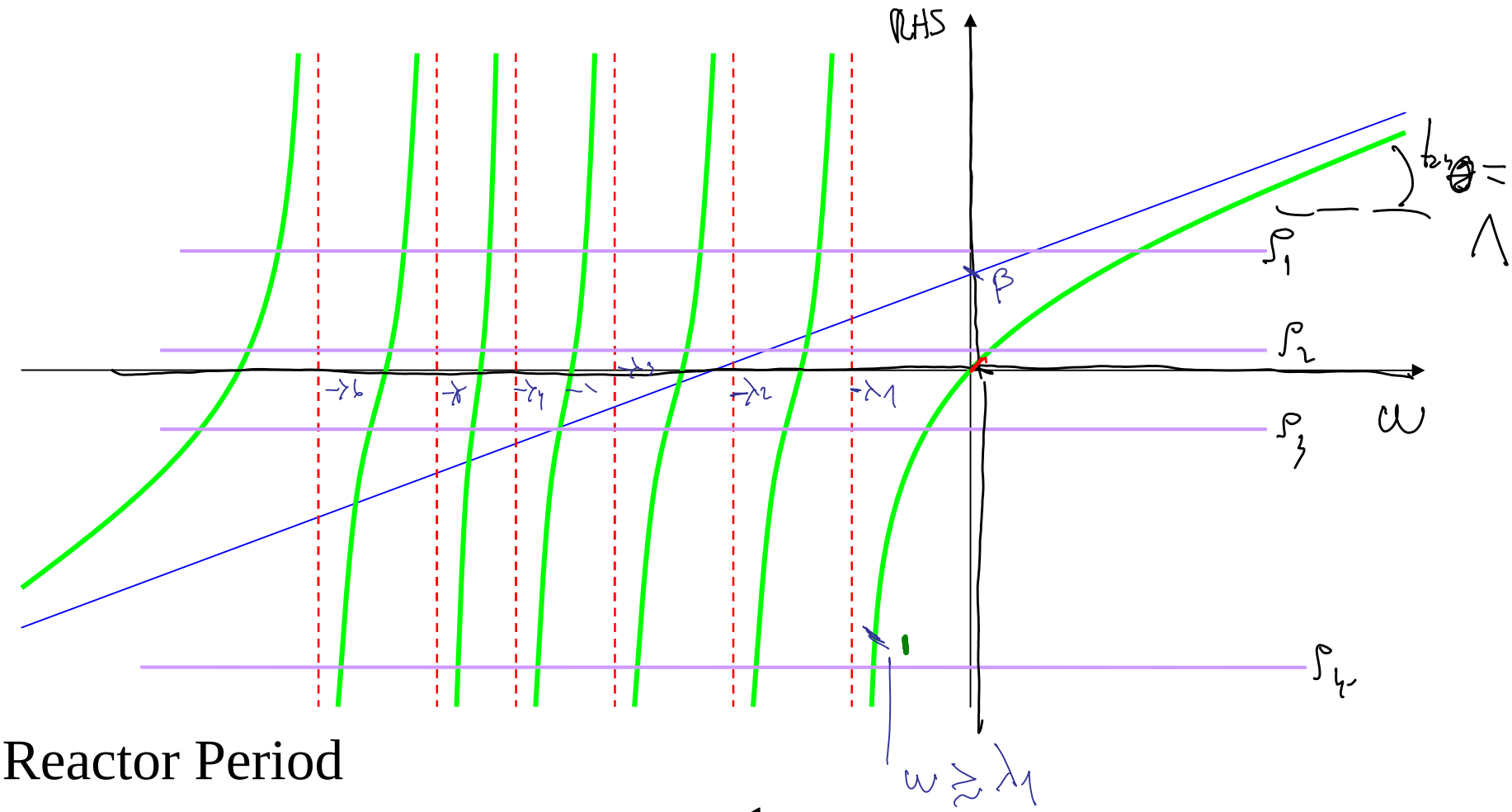
$$\rho = \Lambda\omega + \beta - \sum_{k=1}^{6} \lambda_k \frac{\beta_k}{(\omega + \lambda_k)}$$

$\rho = f(\omega)$

$RHS(\omega = 0) = \beta - \sum_k \beta_k = 0$



We can solve graphically for ω by plotting the RHS and intersecting it with a horizontal line at y=ρ.



Reactor Period

$$T = \frac{1}{\omega_{\max}}$$

ω_{max}

$h = c_1 \tilde{h}_1 e^{\omega_1 t} + c_2 \tilde{h}_2 e^{\omega_2 t} \ldots c_7 \tilde{h}_7 e^{\omega_7 t} \longrightarrow c_1 \tilde{h}_1 e^{\omega_1 t}$

Perturbation Theory

Perturbation Theory

- Consider a reactor with the following parameters (This is called the “reference”, or “unperturbed” reactor):

$$D_0(\vec{r})$$

$$\Sigma_{a0}(\vec{r})$$

$$\nu\Sigma_{f0}(\vec{r})$$

- K can be calculated by solving:

$$-D_0(\vec{r})\nabla^2\Phi_0(\vec{r}) + \Sigma_{a0}(\vec{r})\Phi_0(\vec{r}) = \frac{1}{k_0}\nu\Sigma_f(\vec{r})\Phi_0(\vec{r})$$

Perturbation Theory

- Consider now a different reactor, with the same shape, but different absorption and fission cross sections (This is called the "perturbed" reactor).

$$\Sigma_a(\vec{r}) = \Sigma_{a0}(\vec{r}) + \delta\Sigma_a(\vec{r})$$

$$\Sigma_f(\vec{r}) = \Sigma_{f0}(\vec{r}) + \delta\Sigma_f(\vec{r})$$

- K can be calculated by solving:

$$-D_0\nabla^2\Phi(\vec{r}) + \Sigma_a\Phi(\vec{r}) = \frac{1}{k}\nu\Sigma_f\Phi(\vec{r})$$

Perturbation Theory

- Assuming that

$$\left|\delta\Sigma_a(\vec{r})\right| \text{ is small}$$

$$\left|\delta\Sigma_f(\vec{r})\right| \text{ is small}$$

- we can find k by a simple formula, without having to solve the diffusion equation again
- NOTE:
  - $\delta\Sigma_a$ is called the perturbation in the absorption cross section
  - $\delta\Sigma_f$ is called the perturbation in the fission cross section

Perturbation Theory

- Perturbation Formula for finding k (no proof):

$$\frac{1}{k} - \frac{1}{k_0} \cong \frac{\int_V \delta\Sigma_a(\vec{r})\Phi_0^2(\vec{r})dV - \frac{1}{k_0}\int_V \nu\delta\Sigma_f(\vec{r})\Phi_0^2(\vec{r})dV}{\int_V \nu\Sigma_{f0}(\vec{r})\Phi_0^2(\vec{r})dV}$$

- The formula is good even if the reactor is not homogeneous.

Perturbation Theory

- Q: What does “small” mean for a perturbation?
- A:

$$\int_V \left|\delta\Sigma_a(\vec{r})\right| dV << \int_V \Sigma_{a0}(\vec{r}) dV$$

$$\int_V \left|\delta\Sigma_f(\vec{r})\right| dV << \int_V \Sigma_{f0}(\vec{r}) dV$$

- A perturbation can be small in two ways:
  - a) $\left|\delta\Sigma_a(\vec{r})\right| << \Sigma_{a0}(\vec{r})$

    $\left|\delta\Sigma_f(\vec{r})\right| << \Sigma_{f0}(\vec{r})$
  - b) The perturbation only affects a small part of the reactor.

Reactivity Induced by a Perturbation in a Initially Critical Reactor

- Reactor initially critical $k_0 = 1$
- Introduce a perturbation
- The new k can be found using the perturbation formula:

$$\frac{1}{k} - \frac{1}{k_0} \cong \frac{\overbrace{\int_V \delta\Sigma_a(\vec{r})\Phi_0^2(\vec{r})dV}^{\text{absorption change}} - \frac{1}{k_0}\overbrace{\int_V \nu\delta\Sigma_f(\vec{r})\Phi_0^2(\vec{r})dV}^{\text{production change}}}{\int_V \nu\Sigma_{f0}(\vec{r})\Phi_0^2(\vec{r})dV}$$

$k_0 = 1$

Reactivity Induced by a Perturbation in a Critical Reactor

- Since $k_0 = 1$
- we can write:

$$\frac{1}{k} - 1 \cong \frac{\int_V \delta\Sigma_a(\vec{r})\Phi_0^2(\vec{r})dV - \int_V \nu\delta\Sigma_f(\vec{r})\Phi_0^2(\vec{r})dV}{\int_V \nu\Sigma_{f0}(\vec{r})\Phi_0^2(\vec{r})dV}$$

$$\rho = 1 - \frac{1}{k} \cong \frac{\int_V \nu\delta\Sigma_f(\vec{r})\Phi_0^2(\vec{r})dV - \int_V \delta\Sigma_a(\vec{r})\Phi_0^2(\vec{r})dV}{\int_V \nu\Sigma_{f0}(\vec{r})\Phi_0^2(\vec{r})dV}$$

Reactivity Induced by a Perturbation in a Critical Reactor

- Remember the definition of the static reactivity:

$$\rho = 1 - \frac{1}{k}$$

- Hence, we can write the perturbation formula for the reactivity:

$$\rho \cong \frac{\int_V \nu \delta \Sigma_f(\vec{r}) \Phi_0^2(\vec{r}) dV - \int_V \delta \Sigma_a(\vec{r}) \Phi_0^2(\vec{r}) dV}{\int_V \nu \Sigma_{f0}(\vec{r}) \Phi_0^2(\vec{r}) dV}$$

Generalized Definition of Reactivity

- Diffusion equation

$$-D_0\nabla^2\Phi(\vec{r})+\Sigma_a\Phi(\vec{r})=\frac{1}{k}\nu\Sigma_f\Phi(\vec{r})$$

- It represents the neutron balance

$$\text{Losses}=\frac{1}{k}\text{Productions}$$

- So k can be interpreted as the ratio Productions/Losses

$$k=\frac{\text{Productions}}{\text{Losses}}$$

Generalized Definition of Reactivity

- Reactivity

$$\rho = 1 - \frac{1}{k} = 1 - \frac{1}{\dfrac{\text{Productions}}{Losses}} = \frac{\text{Productions} - Losses}{\text{Productions}}$$

- The generalized definition can be also applied to non-static (time-dependent) situations

Steps for Setting up the (Quasi-)Exact Point Kinetics Equations (Case of One Energy Group)

1. Find out the neutronic parameters (diffusion coefficient and macroscopic cross sections) of the critical (unperturbed) reactor.
2. Determine the flux Φ_0 for the critical (unperturbed) reactor (Usually by solving the diffusion equation or referring to the results from a previous calculation).

3. Calculate the point kinetics parameters (except reactivity) using their formulas of definition (shown below).

$$\Lambda = \frac{1}{\bar{v}\,\nu\Sigma_f}$$

$$\beta_k = \frac{\nu_{dk}\Sigma_f}{\nu\Sigma_f}$$

$$\ell = \frac{1}{\bar{v}\left(\Sigma_a + DB^2\right)}$$

4. Calculate the reactivity using the perturbation formula:

$$\rho \cong \frac{\int_V \delta\nu\Sigma_f(\vec{r})\Phi_0^2(\vec{r})dV - \int_V \delta\Sigma_a(\vec{r})\Phi_0^2(\vec{r})dV}{\int_V \nu\Sigma_{f0}(\vec{r})\Phi_0^2(\vec{r})dV}$$

5. Write the point-kinetics equations:

$$\frac{dn(t)}{dt} = \frac{\rho - \beta}{\Lambda} n(t) + \sum_{k=1}^{6} \lambda_k \hat{C}_k(t)$$

$$\frac{d\hat{C}_k(t)}{dt} = \frac{\beta_k}{\Lambda} n(t) - \sum_{k=1}^{6} \lambda_k \hat{C}_k(t)$$

6. Solve the point-kinetics equations.

Measuring the Reactivity Worth of a Control Rod by the Source Multiplication Method

- The method is applicable for a subcritical reactor with an external neutron source.
- We need to have a second rod whose reactivity worth we already know.

Steps

1. Measure the flux at any reactor position without any of the rods inserted and record the value.
2. Insert the rod of known reactivity worth $\Delta\rho_1$.
3. Measure the flux at the same reactor position as in step 1 and record the value.
4. Remove the rod of known reactivity.
5. Insert the rod of unknown reactivity $\Delta\rho_2$.
6. Measure the flux at the same reactor position as in step 1 and record the value.
7. Calculate the unknown reactivity worth $\Delta\rho_2$.

Calculations $S = \gamma \cdot S_{true}$

For any point in a subcritical reactor with an external neutron source we have:

$$\Phi_{\det ector} = -\frac{\Lambda}{\rho} S$$

Note: we have shown that this is true for the flux at the center of the reactor and we will accept without proof that it is true for the flux at any position in the reactor.

For the reactor without any control rod we can write:

$$\Phi^{0}_{\det ector} = -\frac{\Lambda}{\rho_0} S \qquad (1)$$

or

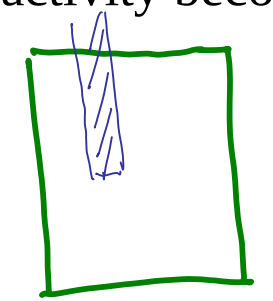
$$\frac{1}{\Phi^0_{\det ector}} = -\frac{S}{\Lambda}\rho_0 \qquad (2)$$

After we insert the rod of known reactivity, the reactivity becomes

known,

$$\rho_1 = \rho_0 + \Delta\rho_1 \qquad (3)$$

not known



The flux equation becomes

$$\frac{1}{\Phi^1_{\det ector}} = -\frac{S}{\Lambda}\left(\rho_0 + \Delta\rho_1\right) \qquad (4)$$

Dividing equation (4) by equation (2) we obtain:

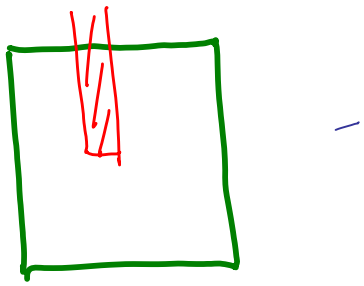
$$\frac{\Phi^{0}_{\det ector}}{\Phi^{1}_{\det ector}} = \frac{\rho_0 + \Delta\rho_1}{\rho_0} \tag{5}$$

We can now solve for ρ_0

$$\rho_0 = \frac{\Delta\rho_1}{\frac{\Phi^{0}_{\det ector}}{\Phi^{1}_{\det ector}} - 1} \tag{6}$$

After we remove the rod of known worth and insert the rod of unknown worth, we have:

$$\frac{1}{\Phi^{2}_{\det ector}} = -\frac{S}{\Lambda}\left(\rho_0 + \Delta\rho_2\right) \tag{7}$$



Dividing now equation (7) by equation (2) we obtain

$$\frac{\Phi^0_{\det ector}}{\Phi^2_{\det ector}} = \frac{\rho_0 + \Delta\rho_2}{\rho_0} \tag{8}$$

We can now solve for $\Delta\rho_2$ in (8)

$$\Delta\rho_2 = \rho_0 \left(\frac{\Phi^0_{\det ector}}{\Phi^2_{\det ector}} - 1 \right) \tag{9}$$

And substituting the expression for ρ_0 that we found in equation (6) we obtain:

$$\Delta\rho_2 = \rho_0 \left(\frac{\Phi^0_{\det ector}}{\Phi^2_{\det ector}} - 1 \right) = \underbrace{\Delta\rho_1}_{\text{Known}} \frac{\dfrac{\Phi^0_{\det ector}}{\Phi^2_{\det ector}} - 1}{\dfrac{\Phi^0_{\det ector}}{\Phi^1_{\det ector}} - 1} \tag{10}$$

So the formula for finding the reactivity worth for our rod is:

$$\Delta\rho_2 = \Delta\rho_1 \frac{\dfrac{\Phi^0_{\det ector}}{\Phi^2_{\det ector}} - 1}{\dfrac{\Phi^0_{\det ector}}{\Phi^1_{\det ector}} - 1} \qquad (11)$$

Example

A subcritical cylindrical reactor has radius 3m and height 6m. A flux detector is placed at r=1m in the midplane of the reactor, and a neutron source of unknown strength is placed in a position diametrically opposed to the detector. The detector reads initially 1000 units. A control rod worth 1 mk is then inserted into the reactor and the detector reading drops to 500 units. The rod is then removed and another rod of unknown worth is inserted. The detector now reads 250 units. What is the reactivity worth of the new rod?

Answer

Applying eq. (11) we have:

$$\Delta\rho_2 = \Delta\rho_1 \frac{\dfrac{\Phi^0_{\det ector}}{\Phi^2_{\det ector}} - 1}{\dfrac{\Phi^0_{\det ector}}{\Phi^1_{\det ector}} - 1} = 0.001 \times \frac{\dfrac{1000}{250} - 1}{\dfrac{1000}{500} - 1} = 0.001 \times \frac{4-1}{2-1} = 0.001 \times \frac{3}{1} = 0.003$$

Measuring the Reactivity Worth of a Control Rod by the Null Reactivity Method

Method 1 Steps

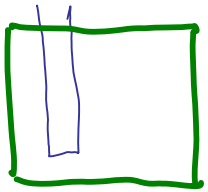
1. Insert a calibrated rod fully (up to the maximum depth, d_{max})into the reactor. (A calibrated rod is a rod for which we know what the reactivity worth is as a function of the depth of insertion $\Delta\rho_c(d)$)
2. Make the reactor critical by modifying other parameters, possibly extracting some poison.
3. Insert the rod to be measured
4. Make the reactor critical again by partially extracting the calibrated rod, up to depth d.
5. Calculate the reactivity worth of the second rod, $\Delta\rho_x$.

Calculations

Consider the reactor with the calibrated control rod extracted, but with all other parameters having the same value as when the calibrated rod was inserted.

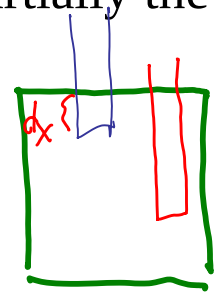
Its reactivity would be ρ_0 (unknown)

After the insertion of the calibrated rod, we know that the reactivity is zero.



$$\rho_c = \rho_0 + \Delta\rho_c(d_{\max}) = 0$$

After we insert the second rod and withdraw partially the calibrated rod, the reactor is still critical.



$$\rho_x = \rho_0 + \Delta\rho_c(d_x) + \Delta\rho_x = 0$$

Subtracting these two equations we obtain:

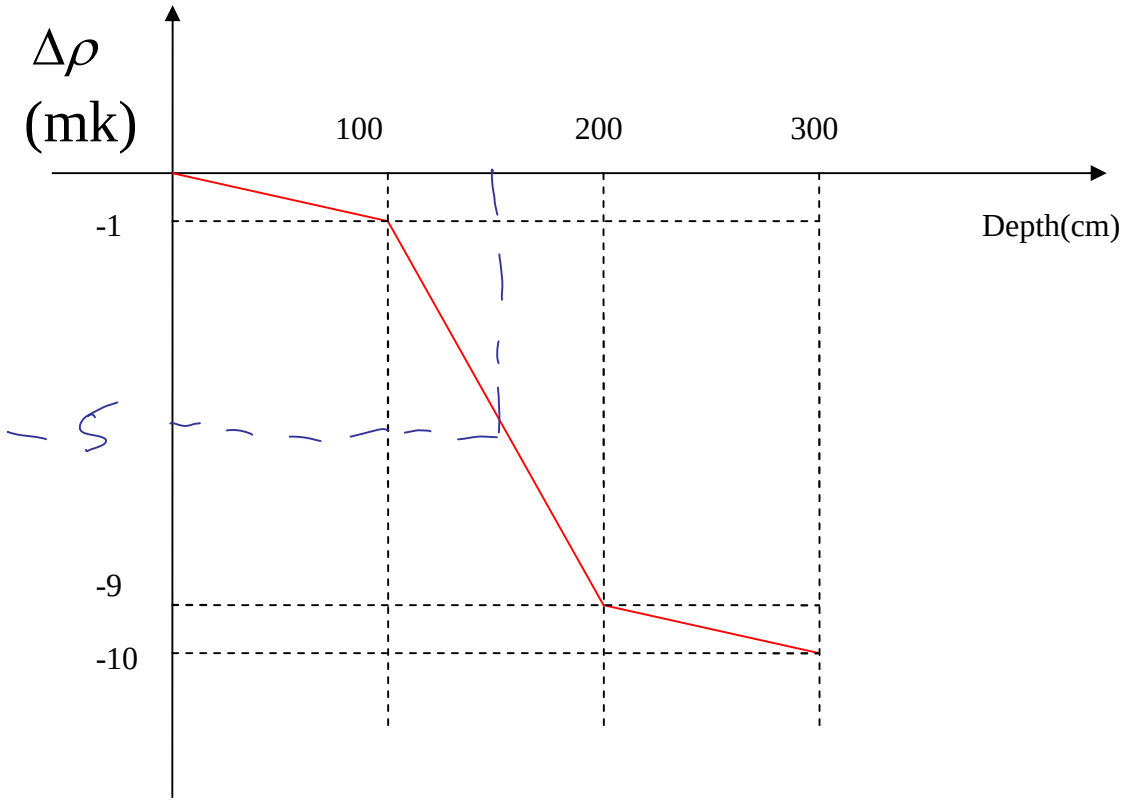
$$\Delta\rho_c(d_x) - \Delta\rho_c(d_{\max}) + \Delta\rho_x = 0$$

We can now solve for $\Delta\rho_x$

$$\Delta\rho_x = \Delta\rho_c(d_{\max}) - \Delta\rho_c(d_x)$$

Example

The reactivity worth of a calibrated control rod as a function of its depth of insertion is given in the graph below. The reactor is made critical with the calibrated rod fully inserted. A second control rod is inserted, and the reactor is again made critical by withdrawing the calibrated control rod up to a depth of 1.5 m. What is the reactivity worth $\Delta\rho_x$ of the second control rod?


Δρ
(mk)
100
200
300
Depth(cm)
-1
-5
-9
-10

Solution

Consider the reactor with the calibrated control rod extracted, but with all other parameters having the same value as when the calibrated rod was inserted.

Its reactivity would be ρ_0 (unknown)

After the insertion of the calibrated rod, we know that the reactivity is zero.

$$\rho_c = \rho_0 + \Delta\rho_c(300cm) = 0$$

After we insert the second rod and withdraw partially the calibrated rod, the reactor is still critical.

$$\rho_x = \rho_0 + \Delta\rho_c(150cm) + \Delta\rho_x = 0$$

Subtracting these two equations we obtain:

$$\Delta\rho_c(150cm) - \Delta\rho_c(300cm) + \Delta\rho_x = 0$$

We can now solve for $\Delta\rho_x$

$$\Delta\rho_x = \Delta\rho_c(300cm) - \Delta\rho_c(150cm)$$

Substituting the numerical values, we have:

$$\Delta\rho_x = -10 - (-5) = -5mk$$

Method 2 Steps

1. Obtain the reactivity calibration of a liquid poison (e.g. B) as a function of its concentration $\Delta\rho_{poison}(X)$
2. Add some poison while maintaining the reactor critical (possibly by removing some other reactivity devices).
3. Mark the poison concentration Xc.
4. Insert the rod to be measured.
5. Make the reactor critical again by removing some poison.
6. mark the new poison concentration.
7. Calculate the reactivity worth of the second rod, $\Delta\rho_x$.

Calculations

The poison reactivity worth is proportional to the poison concentration:

$$\Delta\rho_{poison}(X) = \alpha X$$

Knowing the reactivity calibration of poison means knowing α , which is usually measured in mk/ppm.

Consider the reactor in step 2 but without any poison. Its reactivity would be ρ_0 (unknown)

Now consider the (critical) reactor at step 2.

$$\rho_c = \rho_0 + \Delta\rho_{poison}(X_c) = 0$$

After we insert the rod and remove part of the poison the reactor is still critical.

$$\rho_x = \rho_0 + \Delta\rho_{poison}(X) + \Delta\rho_x = 0$$

Subtracting these two equations we obtain:

$$\Delta\rho_{poison}(X) - \Delta\rho_{poison}(X_c) + \Delta\rho_x = 0$$

We can now solve for the reactivity worth of the rod.

$$\Delta\rho_x = \Delta\rho_{poison}(X_c) - \Delta\rho_{poison}(X) = \alpha\left(X_c - X\right)$$

Example

The reactivity worth of Boron in a CANDU reactor is 7mk/ppm. The reactor is made critical by the addition of Boron. A control rod is then inserted and the reactor is maintained critical by removing 1.5 ppm of Boron. What is the reactivity worth of the rod?

Solution

By applying the formula we derived, we have:

$$\Delta\rho_x = \alpha\left(X_c - X\right) = 7 \times \left(-1.5\right) = -10.5(mk)$$

Appendix: Elements of Relativity

Introduction

Consider two Cartesian frames of reference, S and S', and assume S' is moving with respect to S with a velocity u, directed along the X axis, and oriented in the positive direction of the axis. Also assume that each frame of reference is provided with its own clock. The two clocks are of identical make, and they are set so that the origin of time (t=0) in both frames of reference coincides with the moment when the origin of S and that of S' coincide.

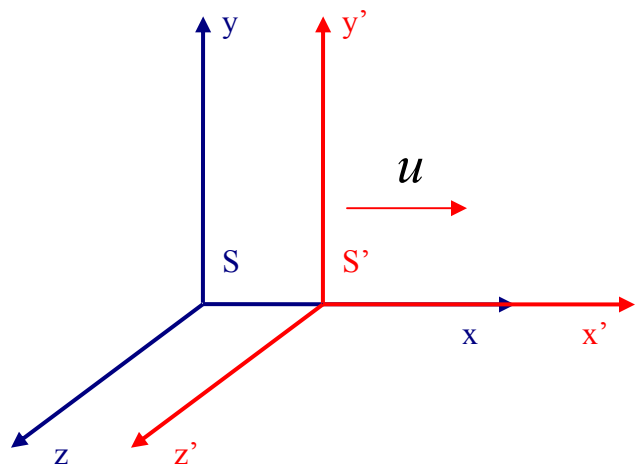


Figure 1

Consider now an event (say an electric bulb being turned on) that happens in S at position (x,y,z) and time t. We want to determine the coordinates (x', y', z') of that event in frame S', as well as the time t' indicated by the clock in S' when the said event happens.

So we are concerned with the following transformation:

$$(x, y, z, t) \rightarrow (x', y', z', t')$$

Galileo's transformation formulas

Our initial "common sense" tendency would be to say that t=t'. That is to say that, if the clocks are of identical make, and since both were set to start when the origins of S and S' coincided, they will continue to show the same time forever. If we assume this to be true, then we have the situation depicted in Figure 2.

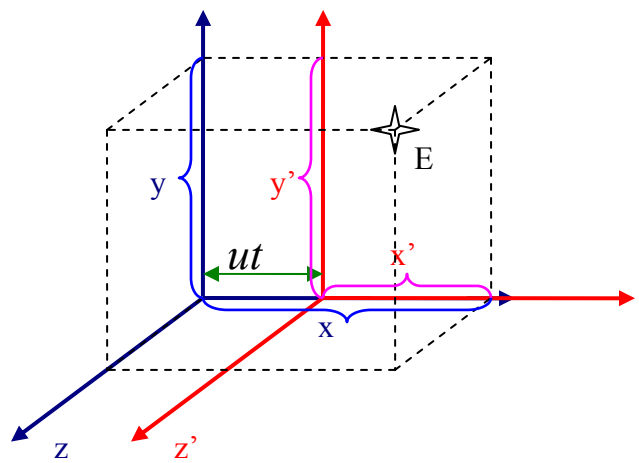


Figure 2

We can then easily infer:

$$t' = t$$
$$y' = y \text{ (because u is along x)}$$
$$z' = z \text{ (because u is along x)}$$
$$x' = x - ut$$

These are known as Galileo's transformation formulas. They include a very profound assumption: that time transforms independently of space. In other words, if we have two events happening in S at the same time t and different positions (x_1, y_1, z_1) and (x_2, y_2, z_2), both of them will happen at time $t' = t$ in S', at different coordinates, (x_1', y_1', z_1') and (x_2', y_2', z_2'), given by:

$$y_1' = y_1$$
$$z_1' = z_1$$
$$x_1' = x_1 - ut$$

and

$$y_2' = y_2$$
$$z_2' = z_2$$
$$x_2' = x_2 - ut$$

If we accept Galileo's formulas, we can easily write a formula for transforming velocities. All we have to keep in mind is that:

$$\vec{v} \equiv \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} = \begin{bmatrix} \dfrac{dx}{dt} \\ \dfrac{dy}{dt} \\ \dfrac{dz}{dt} \end{bmatrix}$$

And that:

$$\vec{v}' \equiv \begin{bmatrix} v'_x \\ v'_y \\ v'_z \end{bmatrix} = \begin{bmatrix} \dfrac{dx'}{dt'} \\ \dfrac{dy'}{dt'} \\ \dfrac{dz'}{dt'} \end{bmatrix}$$

With that in mind, we can begin to write the components of the velocity in S'.

For the x direction, we obtain:

$$v'_x = \frac{dx'}{dt'} = \frac{dx'}{dt}\frac{dt}{dt'} = \frac{d}{dt}(x - ut)\frac{dt}{dt'} = \left(\frac{dx}{dt} - u\right)\frac{dt}{dt'} = (v_x - u)\times 1 = v_x - u$$

For the y direction, we follow the same steps, but the result is even simpler:

$$v'_y = \frac{dy'}{dt'} = \frac{dy'}{dt}\frac{dt}{dt'} = \frac{dy}{dt}\frac{dt}{dt'} = v_y \frac{dt}{dt'} = v_y \times 1 = v_y$$

And similarly for the z component.

$$v'_z = \frac{dz'}{dt'} = \frac{dz'}{dt}\frac{dt}{dt'} = \frac{dz}{dt}\frac{dt}{dt'} = v_z \frac{dt}{dt'} = v_z \times 1 = v_z$$

We can now write the formula for the velocity in S' as:

$$\vec{v}' \equiv \begin{bmatrix} v'_x \\ v'_y \\ v'_z \end{bmatrix} = \begin{bmatrix} v_x - u \\ v_y \\ v_z \end{bmatrix}$$

This is the expected, "common sense", result.

The trouble with the above velocity-addition formula is that it has been shown by experiment to fail for velocities close to the speed of light in vacuum. Michelson and Morley performed a famous experiment that showed that if v=c (c = speed of light in vacuum), then the classical formula for adding velocities fails.

In Michelson and Morley's experiment, the moving frame of reference was taken to be the Earth, which was known to move around the Sun at a speed of approximately 30 Km/s. The "fixed" frame of reference was taken to be the "ether", an all-encompassing "fluid" that was presumed to fill all space and constitute the medium through which light propagates. If the classical law of velocity addition was true, then the speed of light measured in the direction of the Earth's motion (chosen as the x axis) would have had to be c-u, while the speed of light in a direction perpendicular to the Earth's velocity would have had to remain equal to c. Michelson and Morley's

experiment showed that, in fact, the speed of light in vacuum was the same in both directions (parallel, as well as perpendicular to the Earth's motion). There were two immediate implications of this experiment, and they resonated in the scientific world:

1. There exists no "ether" as the medium through which light propagates.
2. The classical law of velocity addition does not apply for velocities close to the speed of light.

Albert Einstein then devised the Theory of Relativity by postulating what was now obvious from Michelson and Morley's experiment: The speed of light in vacuum is the same, and equal to $c = 3 \times 10^8 m/s$, in any inertial frame of reference. Additionally, Einstein postulated that the laws of physics are the same in any inertial frame of reference. The two hypotheses are known as the Postulates of the theory of relativity:

1. The laws of physics are the same in all inertial reference frames.
2. The speed of light in a vacuum has the same value, $c=3x10^8$ m/s in all inertial reference frames (independent of the observer's motion).

If the above two principles are applied consistently, then alternative transformation relations are obtained for the coordinates of an event E, when moving from inertial frame S to a second inertial frame S'. (Remember that two inertial frames of reference have to move uniformly (i.e. constant velocity) relative to each other.) These are known as the Lorentz transformations, because they were first formulated by Lorenz (before Einstein came up with the theory of relativity) in the context of the electromagnetic theory of light.

The Lorenz Transformations

In trying to come up with the transformations that will satisfy the two postulates of relativity, we will look for as simple transformations as we can. The ultimate test of their validity will have to be experiment. And so far experiments have proven the Lorenz transformation to be right.

We will hence make two simplifying assumptions:

1. The transformation of y and z coordinates (the ones perpendicular to the motion) transform the same way as in the classical approach, that is y'=y, and z'=z.
2. The transformation of x and t are linear, that is of the form:

$$x' = a_{11}x + a_{12}t$$
$$t' = a_{21}x + a_{22}t$$

We can already note an important difference between the proposed transformations and those of Galileo: The value of t' is not independent of position any more. That is, if two events happen at the same time t in S, but at different positions x_1 and x_2 , they will happen at different times t'_1 and t'_2 in S'. So two events that are simultaneous in one frame of reference, may or may not be simultaneous in the other frame of reference. This is a major difference from our classical understanding of how the world works. But let us try to determine the coefficients of the transformation. To do that, we will imagine a "thought experiment.

Let us first consider a light detector placed at coordinates $(0,d,0)$ in the "stationary" frame S. At time $t = t' = 0$, the exact time when the origins of the two frames of reference coincide, a bulb is lit at the common origin, for a fraction of a second. The light then propagates to the detector. Let us consider the two events:

1. The emission of the light pulse
2. The arrival of the light pulse at the detector.

We will now try to write the coordinates of each event, including time, in both frames of reference, using Figure 3 as a guide for emission, and Figure 4 as a guide for reception.

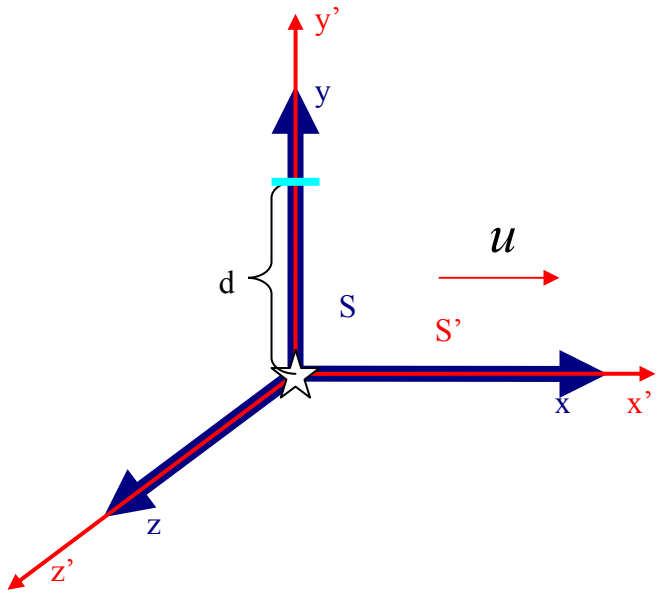


Figure 3

For the emission, according to Figure 3:

In S

$$x = 0$$
$$y = 0$$
$$z = 0$$
$$t = 0$$

In S':

$$x' = 0$$
$$y' = 0$$
$$z' = 0$$
$$t' = 0$$

For the reception we use Figure 4.:

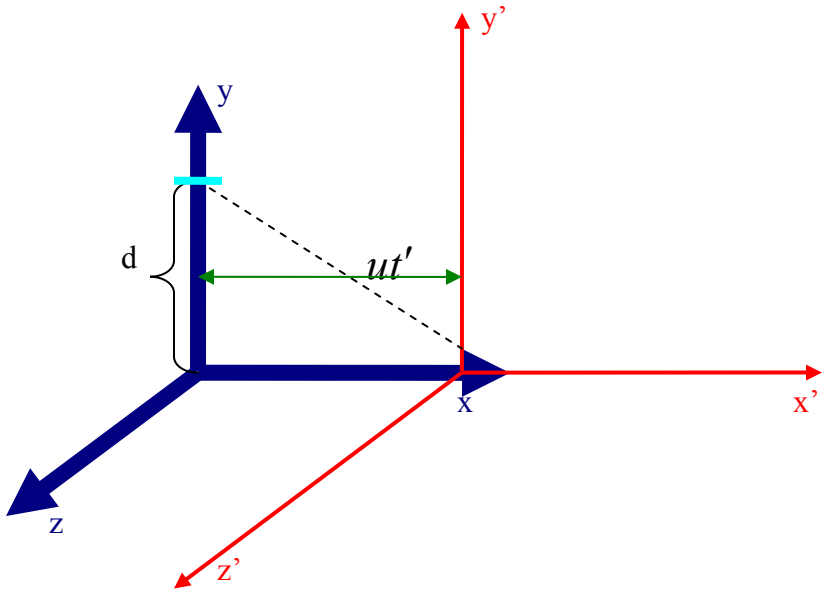


Figure 4

In S

$$x = 0$$
$$y = d$$
$$z = 0$$
$$t = \frac{d}{c}$$

In S':

$$x' = -ut'$$
$$y' = y = d$$
$$z' = 0$$
$$t' = ?$$

To determine t', we need to consider the fact that the speed of light (in a vacuum) needs to be perceived to be the same in both frames of reference. In S, the distance traveled by the light is d. In S', the distance traveled by the light is:

$$\sqrt{d^2 + (ut')^2}$$

Since the speed of light is the same in both frames of reference, it means that:

In S:

$$ct = d$$

In S’

$$ct' = \sqrt{d^2 + (ut')^2}$$

Substituting the first of the two relations in the second, we obtain:

$$ct' = \sqrt{(ct)^2 + (ut')^2}$$

which yields subsequently:

$$(ct')^2 = (ct)^2 + (ut')^2$$

then:

$$(ct')^2 - (ut')^2 = (ct)^2$$

$$c^2 t'^2 - u^2 t'^2 = c^2 t^2$$

$$(c^2 - u^2) t'^2 = c^2 t^2$$

$$t'^2 = \frac{c^2}{(c^2 - u^2)} t^2 = \frac{1}{\left(1 - \frac{u^2}{c^2}\right)} t^2$$

$$t' = \frac{1}{\sqrt{\left(1 - \frac{u^2}{c^2}\right)}} t$$

Now, considering our transformation equations for the experiment we have just described, we have:

$$x' = a_{11} \times 0 + a_{12}t = a_{12}t$$
$$t' = a_{21} \times 0 + a_{22}t = a_{22}t$$

It then follows from the second equation that:

$$a_{22} = \frac{1}{\sqrt{\left(1 - \frac{u^2}{c^2}\right)}}$$

We now consider the first transformation equation, and the fact that $x' = -ut'$, which follows directly from the fact that just as S' moves with u with respect to S, so S moves with –u with respect to S'. We also keep in mind the relation we just established between t ant t'. We then substitute the expression for t' in the first transformation equation and equal it to $x' = -ut'$:

$$x' = a_{12}t = -ut' = -u\frac{1}{\sqrt{\left(1 - \frac{u^2}{c^2}\right)}}t$$

It follows immediately that:

$$a_{12} = -u\frac{1}{\sqrt{\left(1 - \frac{u^2}{c^2}\right)}}$$

We have thus found two of the four coefficients we were after.

Now, consider the movement of the origin of S', as viewed from S. Since it moves with velocity u, its position at time t is

$$x = ut$$

At the same time, its coordinate in its own frame of reference S', is always zero $x' = 0$.

Applying the transformation formula for x, we obtain for the origin of S':

$$0 = x' = a_{11}x + a_{12}t$$
$$0 = a_{11}ut + a_{12}t$$
$$a_{11} = -\frac{a_{12}}{u}$$

Given that we have already determined a_{12} to be $a_{12} = -u\frac{1}{\sqrt{\left(1-\frac{u^2}{c^2}\right)}}$, we can now find a_{11} as:

$$a_{11} = -\frac{a_{12}}{u} - \frac{-u\frac{1}{\sqrt{\left(1-\frac{u^2}{c^2}\right)}}}{u} = \frac{1}{\sqrt{\left(1-\frac{u^2}{c^2}\right)}}$$

So:

$$a_{11} = \frac{1}{\sqrt{\left(1 - \frac{u^2}{c^2}\right)}}$$

There is now only one more coefficient left to determine: a_{21}. To find it, we consider a pulse of light that is emitted from the common origin at t=t'=0, along the x axis, in the positive direction, and consider its position in both frames of reference at time t.

In S:

$$x = ct$$

In S':

$$x' = ct'$$

We now substitute the known coefficients of the transformation from S to S', that is:

$$x' = \frac{1}{\sqrt{\left(1 - \frac{u^2}{c^2}\right)}} x - \frac{u}{\sqrt{\left(1 - \frac{u^2}{c^2}\right)}} t$$

$$t' = a_{21} x + \frac{1}{\sqrt{\left(1 - \frac{u^2}{c^2}\right)}} t$$

So now we can write:

$$ct' = \frac{1}{\sqrt{\left(1-\frac{u^2}{c^2}\right)}}ct - \frac{u}{\sqrt{\left(1-\frac{u^2}{c^2}\right)}}t$$

$$c\left(a_{21}ct + \frac{1}{\sqrt{\left(1-\frac{u^2}{c^2}\right)}}t\right) = \frac{1}{\sqrt{\left(1-\frac{u^2}{c^2}\right)}}ct - \frac{u}{\sqrt{\left(1-\frac{u^2}{c^2}\right)}}t$$

We can now solve for a_{21}.

$$a_{21}c^2 + \frac{c}{\sqrt{\left(1-\frac{u^2}{c^2}\right)}} = \frac{1}{\sqrt{\left(1-\frac{u^2}{c^2}\right)}}c - \frac{u}{\sqrt{\left(1-\frac{u^2}{c^2}\right)}}$$

$$a_{21}c^2 = -\frac{u}{\sqrt{\left(1-\frac{u^2}{c^2}\right)}}$$

$$a_{21} = -\frac{u}{c^2}\frac{1}{\sqrt{\left(1-\frac{u^2}{c^2}\right)}}$$

We now write the complete Lorentz transformations:

$$t' = -\frac{u}{c^2}\frac{1}{\sqrt{\left(1-\frac{u^2}{c^2}\right)}}x + \frac{1}{\sqrt{\left(1-\frac{u^2}{c^2}\right)}}t$$

$$x' = \frac{1}{\sqrt{\left(1-\frac{u^2}{c^2}\right)}}x - \frac{1}{\sqrt{\left(1-\frac{u^2}{c^2}\right)}}ut$$

$$y' = y$$

$$z' = z$$

Oftentimes, the Lorentz transformations are rearranged by factoring out $\frac{1}{\sqrt{\left(1-\frac{u^2}{c^2}\right)}}$ and rearranging the time equation, to:

$$t' = \frac{1}{\sqrt{\left(1-\frac{u^2}{c^2}\right)}}\left(t - \frac{u}{c^2}x\right)$$

$$x' = \frac{1}{\sqrt{\left(1-\frac{u^2}{c^2}\right)}}(x - ut)$$

$$y' = y$$

$$z' = z$$

There are a couple of notations that are usually employed:

$$\beta \equiv \frac{u}{u}$$

$$\gamma \equiv \frac{1}{\sqrt{1-\frac{u^2}{c^2}}}$$

With these notations, the Lorentz transformations can be rewritten as:

$$t' = \gamma\left(t - \frac{1}{c}\beta x\right)$$

$$x' = \gamma(x - ut)$$

$$y' = y$$

$$z' = z$$

Relativistic effects on time and distance

Time dilation

Consider two events , E_1 and E_2, that occur at times t_1 and t_2 respectively, in S, in the same place (x,y,z). (Such a time interval measured between events occurring at the same place is called proper time). Then, their corresponding times in S' will be:

$$t_1' = \frac{1}{\sqrt{\left(1-\frac{u^2}{c^2}\right)}}\left(t_1 - \frac{u}{c^2}x\right)$$

$$t_2' = \frac{1}{\sqrt{\left(1-\frac{u^2}{c^2}\right)}}\left(t_2 - \frac{u}{c^2}x\right)$$

The time elapsed in S is $\Delta t = t_2 - t_1$, while the time elapsed, as measured in S' is:

$$\Delta t' = t_2' - t_1' = \frac{1}{\sqrt{\left(1-\frac{u^2}{c^2}\right)}}\left(t_2 - \frac{u}{c^2}x\right) - \frac{1}{\sqrt{\left(1-\frac{u^2}{c^2}\right)}}\left(t_1 - \frac{u}{c^2}x\right) = \frac{1}{\sqrt{\left(1-\frac{u^2}{c^2}\right)}}\left(t_2 - t_1\right) = \frac{1}{\sqrt{\left(1-\frac{u^2}{c^2}\right)}}\Delta t$$

So, in S' the time interval appears longer. This means that the proper time (measured in a frame of reference where events occur in the same place) is always the shortest.

Length contraction

The length of an object measured in a frame of reference where the object is at rest is called proper length.

Consider a spaceship moving with speed u from planet A to planet B, separated by distance L. Consider the two events: the departure, and the arrival. In the ship's frame of reference, the two occur in the same place, so the time of travel in the ship's frame of reference is the proper time of travel, and equals Δt. On the other hand, the proper length L is measured in a frame of reference where the planets A and B are at rest. Such a frame of reference moves with –u with respect to the ship. The time of travel measured in this frame of reference will be $\Delta t' = \gamma \Delta t$. So, for the planets' frame of reference we can write:

$$L = u\gamma\Delta t$$

And for the ship's frame of reference we can write:

$$L_{ship} = u\Delta t$$

It follows by dividing the two equations, that:

$$L_{ship} = \frac{1}{\gamma} L$$

As $\gamma > 1$, this means that the length in the ship's frame of reference is contracted compared to the proper length.

Relativistic momentum and mass

The classical definition of the momentum of a particle is:

$$p = m_0 \vec{v}$$

where m_0 is the mass of the particle.

If the principles of relativity are applied rigorously, it turns out that $m_0 v$ is not conserved. It turns out (we will not give the proof) that the quantity that is conserved is

$$\vec{p} = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} m_0 \vec{v}$$

This is the relativistic expression for the momentum of a particle.

We can interpret the above as still being the product between "a" mass m and velocity $\vec{v}$ provided we define the mass of a moving particle as:

$$m = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} m_0$$

Where m is called the relativistic mass, and m_0 is called the rest mass, and corresponds to the mass of the particle as measured in a frame of reference where the particle is at rest.

The relativistic relation between momentum and force is the same as the classical one:

$$\vec{F} = \frac{d\vec{p}}{dt}$$

However, $\vec{F} \neq m\frac{dv}{dt}$ because the relativistic mass is not a constant.

Relativistic energy

Consider a particle moving on a straight line from x=a to x=b, under the influence of a force F, and assume the particle is initially at rest. Just like in classical mechanics, the kinetic energy imparted to a particle starting from rest equals the work of the force acting on it:

$$KE = \int_a^b F dx = \int_a^b \frac{dp}{dt} dx = \int_a^b \frac{dp}{dt}\frac{dx}{dt} dt = \int_a^b \frac{dp}{dt} v dt = \int_a^b \frac{dp}{dv}\frac{dv}{dt} v dt = \int_a^b \frac{dp}{dv} v dv$$

Substituting the expression for the relativistic momentum, we obtain:

$$KE = \int_a^b \frac{dp}{dv} v dv = \int_a^b \frac{d}{dv}\left(\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} m_0 v \right) v dv$$

The above needs some processing, which we do below:

$$KE = \int_a^b \left[\frac{d}{dv}\left(m_0 v\right)\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} + \frac{d}{dv}\left(\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\right) m_0 v \right] v dv =$$

$$\int_a^b \left[m_0 \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} + \left(\frac{1}{2}\frac{1}{\left(1-\frac{v^2}{c^2}\right)^{\frac{3}{3}}}\frac{2v}{c^2} \right) m_0 v \right] v dv =$$

$$\int_a^b \left[m_0 \frac{1}{\left(1-\frac{v^2}{c^2}\right)^{\frac{3}{3}}} \left[\left(1-\frac{v^2}{c^2}\right) + \left(\frac{1}{2}\frac{2v}{c^2}\right) v \right] \right] v dv$$

Further processing yields:

$$KE = \int_a^b \left[m_0 \frac{1}{\left(1-\frac{v^2}{c^2}\right)^{\frac{3}{3}}} \left[\left(1-\frac{v^2}{c^2}\right) + \left(\frac{v^2}{c^2}\right) \right] \right] v dv =$$

$$m_0 \int_a^b \frac{1}{\left(1-\frac{v^2}{c^2}\right)^{\frac{3}{3}}} v dv = m_0 \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} c^2 - m_0 c^2$$

The above can be rewritten more elegantly as:

$$KE = mc^2 - m_0 c^2$$

Where m is the relativistic mass, and m_0 is the rest mass. The term $m_0 c^2$ is called the rest energy.

Rearranging the terms, we can write:

$$KE + m_0 c^2 = mc^2$$

The term mc^2 is therefore the sum between the rest energy and the kinetic energy and is called total energy and denoted by E. We can therefore write:

$$E = mc^2$$

which is the famous relativistic energy formula that shows the equivalence between mass and energy.

The interpretation of the equivalence between mass and energy is that we should, on one hand, be able to create energy from matter, and, on the other hand, be able to create matter from energy. Both situations have been found to occur in reality. In particular, the total energy can be liberated in reactions of annihilation, where a particle and an antiparticle collide, the most common such reaction being that between an electron and a positron. A positron is a particle with mass equal to that of an electron, but with positive charge. The reverse reaction is called pair production. When a photon (a particle of light) with sufficient energy is decelerated very fast (usually by collision with a heavy particle), an electron-positron pair can be created.