APPENDIX 2

THE REYNOLDS TRANSPORT THEOREM

(SEE " FUNDAMENTAL MECHANICS OF FLUIDS , I. G. CURRIE 1974 , pg 9→12 .)

Before deriving the theorem, let's get the nominclature straight:

From the Eulerian point of view (fixed in space) a property, α, of a material (in this case, neutron density) is a function of time, t, and space, x, y, z. But from a convective or Lagrangian point of view α is a function of t and the coordinates at t_0, ie x_0, y_0, z_0. This means ~~that~~ given the starting point and the ~~starting~~ time, you can ~~say~~ where the particle is at some later time, t.

Now, following along the streamline or particle path with the same velocity as the particle, we notice a change, $\delta\alpha$. This change is due to a change in time only in the Lagrangian system and due to a change in ~~space~~ space and time from the Eulerian point of view. So:

$$\delta\alpha = \frac{\partial\alpha}{\partial t}\,\delta t + \frac{\partial\alpha}{\partial x}\,\delta x + \frac{\partial\alpha}{\partial y}\,\delta y + \frac{\partial\alpha}{\partial z}\,\delta z$$

where the δ refers to changes as seen in the Lagrangian system and the ∂ refers to the more traditional change of a function of a number of variables as seen in traditional calculus and as we are used to in Eulerian formalisms.

Dividing by δt:

$$\frac{\delta\alpha}{\delta t} = \frac{\partial\alpha}{\partial t} + \frac{\partial\alpha}{\partial x}\frac{\delta x}{\delta t} + \frac{\partial\alpha}{\partial y}\frac{\delta y}{\delta t} + \frac{\partial\alpha}{\partial z}\frac{\delta z}{\delta t}\,.$$

As $\delta t \to 0$, $\frac{\delta\alpha}{\delta t} \to \frac{D}{Dt}$, $\frac{\delta x}{\delta t} \to$ Velocity in x directi $= V_x$

Thus $\underbrace{\frac{D\alpha}{Dt}}_{\text{material or substantial derivative}} = \frac{\partial\alpha}{\partial t} + \underline{V}\cdot\nabla\alpha\,.$

Of course, you've seen this result before. This is just a reminder to reinforce the roots of the expression.

Now we turn to the Transport Theorem. The motivation for the theorem is to be able to handle the time derivative of a volume integral over a time varying volume:

$$\frac{D}{Dt}\int_{\forall(t)} \alpha(t)\, d\forall \, .$$

Note α = function of t only (in Lagrangian system)

This arises in fluid mechanics, neutron kinetics and any situation involving moving particles. Needless to say, it is of fundamental importance.

We use the definition of a differential limit to write:

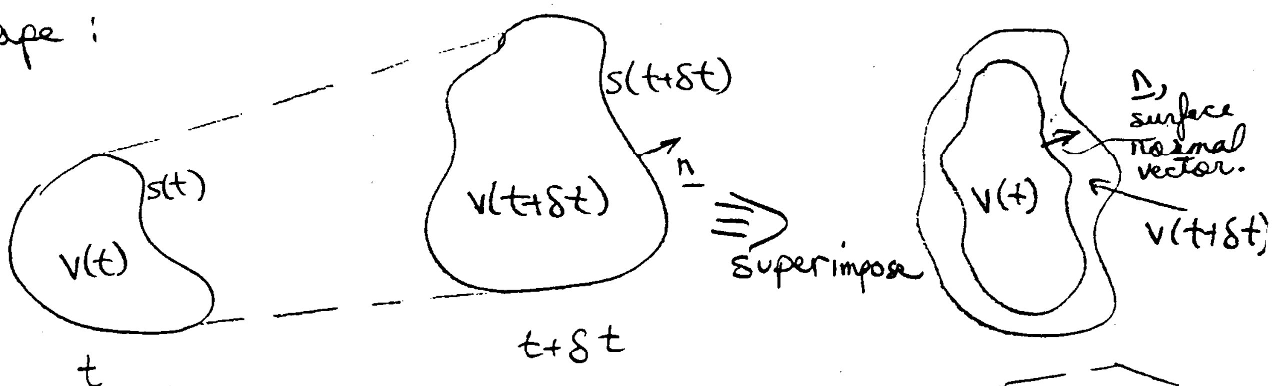
$$\frac{D}{Dt}\int_{V(t)} \alpha(t)\, d\forall = \lim_{\delta t\to 0}\left\{\frac{1}{\delta t}\left[\int_{\forall(t+\delta t)} \alpha(t+\delta t)\, d\forall - \int_{\forall(t)} \alpha(t)\, d\forall\right]\right\}$$

$$= \lim_{\delta t\to 0}\left\{\frac{1}{\delta t}\left[\int_{\forall(t+\delta t)} \alpha(t+\delta t)\, d\forall - \int_{V(t)} \alpha(t+\delta t)\, d\forall\right] + \frac{1}{\delta t}\left[\int_{V(t)} \alpha(t+\delta t)\, d\forall - \int_{V(t)} \alpha(t)\, d\forall\right]\right.$$

$$\frac{D}{Dt}\int_{V(t)} \alpha(t)\,d\forall = \lim_{\delta t \to 0}\left\{\frac{1}{\delta t}\left[\int_{V(t+\delta t)-V(t)} \alpha(t+\delta t)\,d\forall\right]\right\} + \int_{V(t)} \frac{\partial \alpha}{\partial t}\,dt \,.$$

The remaining limit corresponds to the volume changing while α remains constant.

Consider the volume as it moves and changes shape :



The change in volume is just :

$$\delta\forall = \underline{v}\cdot\underline{n}\,\delta t\,\delta s$$

Thus : $$\lim_{\delta t \to 0}\left\{\frac{1}{\delta t}\left[\int_{\forall(t+\delta t)-V(t)} \alpha(t+\delta t)\,d\forall\right]\right\} = \int_{S(t)} \alpha(t)\,\underline{v}\cdot\underline{n}\,dS \,,$$

So ~~that~~ :

$$\frac{D}{Dt}\int_{V(t)} \alpha(t)\,d\forall = \int_{V(t)} \frac{\partial \alpha(t)}{\partial t}\,d\forall + \int_{S(t)} \alpha(t)\,\underline{v}\cdot\underline{n}\,ds \,.$$

Of course, we can replace the surface integral with a volume integral using Gauss' Theorem to get:

$$\frac{D}{Dt}\int_{\forall} \alpha \, d\forall = \int_{\forall}\left[\frac{\partial \alpha}{\partial t} + \nabla \cdot (\alpha \underline{V})\right] d\forall \, .$$

These last two expressions of Reynold's Transport Theorem give us a neat way to take derivatives involving varying integrands.

I like this derivation by Currie because it helps to generate some intuition on the physical processes involved.

As you might have expected, this is not the only way to generate this result. For instance, we could listen to Truesdell :

"The classical field theories, Encyclopedia of Physics, Flugge, S, Ed., III/1, Principles of Classical Mechanics and Field Theory, Springer Verlag, 226-858

<u>Leibniz Rule</u> (Truesdell and Toupin, 1960)

$$\frac{d}{dt}\int_{V(t)} f(\underline{r},t)\,dV = \int_{V(t)} \frac{\partial f}{\partial t}(\underline{r},t)\,dV + \int_{S(t)} f\,\underline{v}_s \cdot d\underline{S}$$

$\underline{v}_s$ = velocity of displacement of surface.

If $V(t)$ is a material volume, this reduces to the Reynold's Transport Theorem.

— ref: Two-Phase Flow and Heat Transfer in the Power and Process Industries by Bergles et al, Hemisphere Publishing Corp 1981.

This is a nice crisp result (if you happen to recall the derivation of the Leibniz Rule). It also has the appeal of a dried cracker.

Aris also tries to "illuminate" the subject by an elegant approach involving the Jacobian to do coordinate transformations. Perhaps one day, I'll understand it (R. Aris, "Vectors, Tensors and the Basic Equations of Fluid Mechanics, 1962, P84-85, Prentice-Hall.)