Reactor Physics: The Diffusion of Neutrons

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[Based on Chapter 5 of Lamarsh since the treatment there is good]

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Summary:
Neutron movement is modelled herein as a diffusion process. Mono-energetic neutrons are used for illustration purposes. Analytical solutions of the neutron distribution are sought for some simple cases involving fixed sources.

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1 Introduction

1.1 Overview

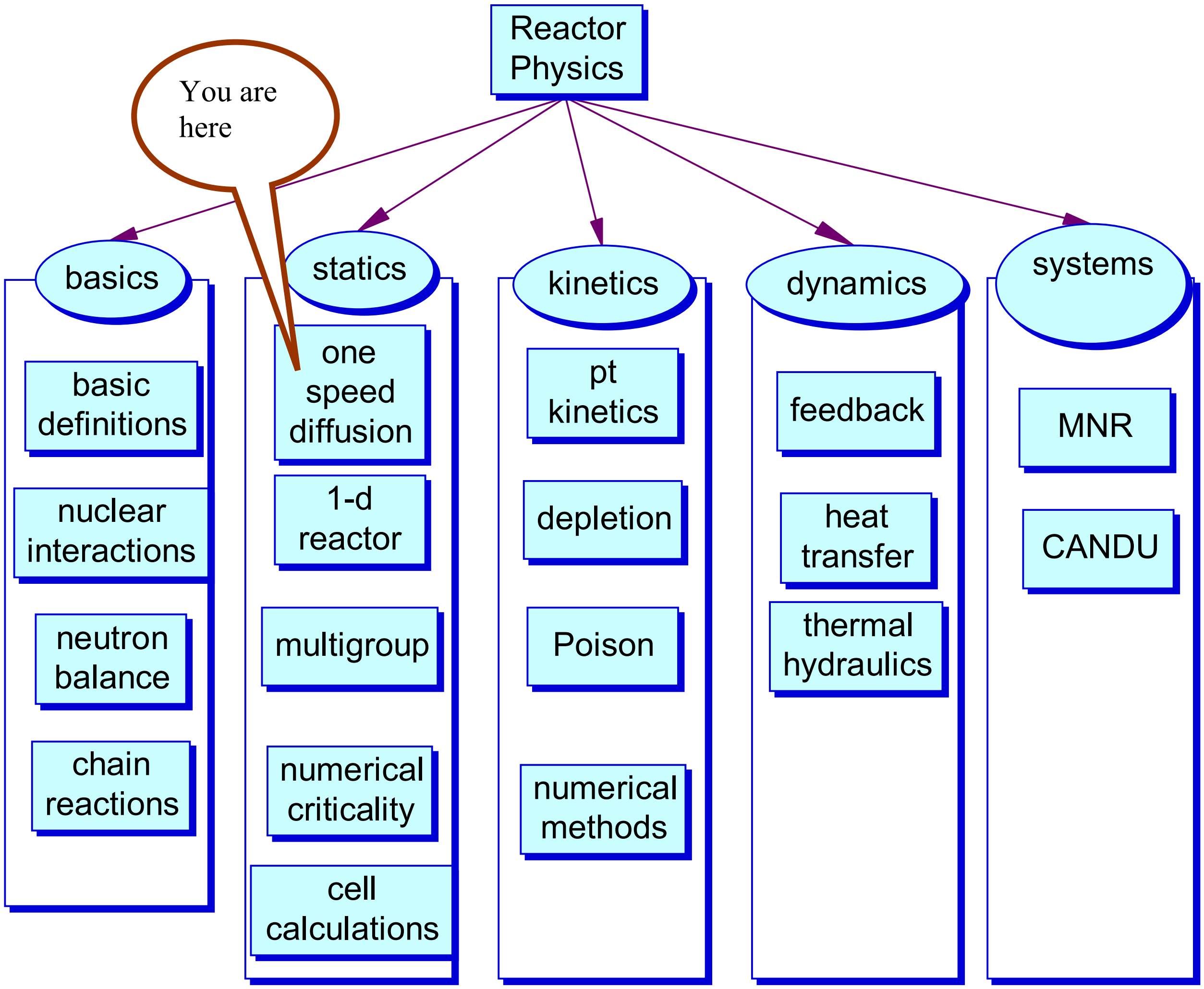


Figure 1 Course Map

- We will consider one speed diffusion
- This is a simple model that illustrates many concepts without too many complications.
- Will represent neutrons of energy range 10^{-3} to 10^{7} eV by one speed!
- Itinerary:
  - Derivation of balance equation
  - Fick's law and its limitations
  - B.C.
  - Analytical solutions for non-multiplying media

1.1 Learning Outcomes

The goal of this chapter is for the student to understand:

- physical process of diffusion of neutrons
- limitations of diffusion
- the neutron balance equation
- analytical solutions to the one speed neutron diffusion equation
- boundary condition rationale

2 Why Diffusion

Movement of neutrons is similar to movement of gas particles. Transport theory provides the general transport equation or the Boltzmann equation. This is good material for graduate courses and as a means of providing a unified approach from which the many approximations can be derived. However, much practical reactor design work is done using a simplification called diffusion theory. Once the general principles have been covered, the many ideas can be unified. It turns out that it is better to go from the particular to the general rather than from the general to the particular when learning this material; it doesn't strain the mind as much as it is much more conclusive to obtaining a feel for the subject.

One velocity (i.e. speed) neutrons are considered for the moment.

3 Interaction Rates and Neutron Flux

Interaction rate = $I\ \Sigma_t$ for beam of mono-energetic neutrons of intensity I neutrons/cm^2 – sec hitting a target of cross section, Σ_t.

Since neutrons do not interact with one another, if there is more than one beam, the interaction rate is

$(I_A + I_B + I_C \ldots)\ \Sigma_t$ interactions/cm^3 - sec

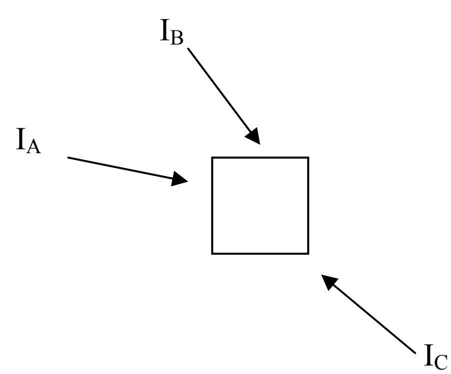


Figure 2 Superposition of interactions

Interactions in a nuclear reactor occur in a similar manner. The neutrons can move in any direction. We define the neutron density function:

$n(\mathbf{r}, \omega)$ — ω: generalized angular vector; $\mathbf{r}$: generalized space coordinate (vector)

$n(\mathbf{r}, T)$ = # of neutrons/cm^3 at $\mathbf{r}$ whose velocity vector lies within the differential solid angle, d Ω about the direction T.

and

$$n(\mathbf{r}) = \int_{4\pi} n(\mathbf{r}, \boldsymbol{\omega})\, d\Omega$$

Thus, a differential beam of intensity d I ($\mathbf{r}$, T) is:

$$dI(\mathbf{r}, \boldsymbol{\omega}) = n(\mathbf{r}, \boldsymbol{\omega}) \underbrace{v}_{\text{neutron speed}} d\Omega$$

We define the interaction rate dF ($\mathbf{r}$, T):

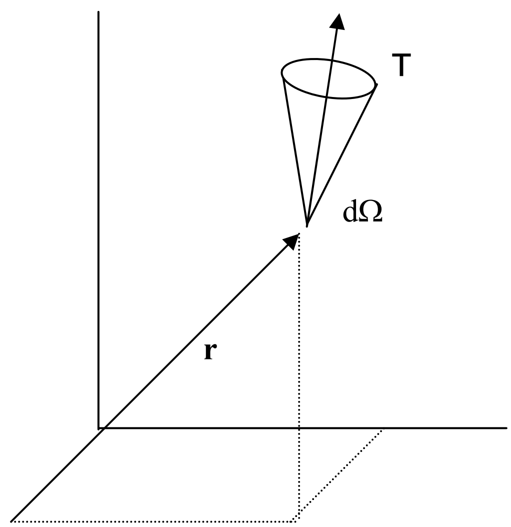


Figure 3 Differential beam

$$dF(\mathbf{r}, \boldsymbol{\omega}) = \Sigma_t\, dI(\mathbf{r}, \boldsymbol{\omega})$$

$$\therefore\ F(\mathbf{r}) = \int_{\boldsymbol{\omega}} dF(\mathbf{r}, \boldsymbol{\omega}) = \Sigma_t \int_{4\pi} v\, n(\mathbf{r}, \boldsymbol{\omega})\, d\Omega$$

$$= v\, \Sigma_t\, n(\mathbf{r})$$

$$F(\mathbf{r}) = \Sigma_t\, \phi(\mathbf{r})$$

Where the neutron flux, ϕ = nv as previously defined. This can easily be generalized to include energy dependence as follows:

$N(\mathbf{r}, E, T)\, dE\, d\Omega$ = # of neutrons/cm^3 with energy (E, E + dE) moving in a solid angle, d Ω, about T.

Hence:

$$n(\mathbf{r}, E)\, dE = \int_{4\pi} n(\mathbf{r}, E, \boldsymbol{\omega})\, d\Omega dE$$

and

$$n(\mathbf{r}) = \int_0^{\infty} \int_{4\pi} n(\mathbf{r}, E, \boldsymbol{\omega})\, d\Omega dE$$

and

$$F(\mathbf{r}, E)dE = \Sigma_t(E)n(\mathbf{r}, E)v(E)dE$$

= number of interactions occurring per cm^3 per sec at **r** in the energy interval dE.

$$= \Sigma_t(E)\phi(\mathbf{r}, E)dE$$

Finally,

$$F(\mathbf{r}) = \int_o^\infty \Sigma_t(E)\phi(\mathbf{r}, E)dE$$

Thus knowing the material properties, Σ_t, and the neutron flux, ϕ, as a function of space and energy, we can calculate the interaction rate throughout the reactor.

We can similarly arrive at interaction rates for scattering, etc.

$$F_S(r) = \int_o^\infty \Sigma_S(E)\phi(\mathbf{r}, E)dE \quad \text{, etc.}$$

Note: ϕ, neutron flux, is a scalar; whereas, most fluxes (ie. heat flux) are vectors.

ϕ is <u>not</u> the flow of neutrons. There may be no flow of neutrons, yet many interactions may occur. The neutrons move in a random fashion and hence may not flow. ϕ is more closely related to densities. Just as mass and heat flow when there is a density difference in space, neutrons will exhibit a net flow when there are spatial differences in density. Hence we <u>can</u> have a flux of neutron flux!

4 Neutron Current Density

This flux of neutron flux is called the current density.

$\mathbf{J} \equiv$ neutron current density (vector quantity)

We can redefine the intensity of neutrons in the vector sense:

$$d\,\mathbf{I}(\mathbf{r},\boldsymbol{\omega}) = n(\mathbf{r},\boldsymbol{\omega}) \underbrace{\mathbf{v}}_{\text{true velocity (vector)}} d\Omega$$

$$\mathbf{J} \equiv \int_{4\pi} n(\mathbf{r},\boldsymbol{\omega})\mathbf{v}\,d\Omega \quad \Leftarrow \quad \text{Physical significance = net motion or flux}$$

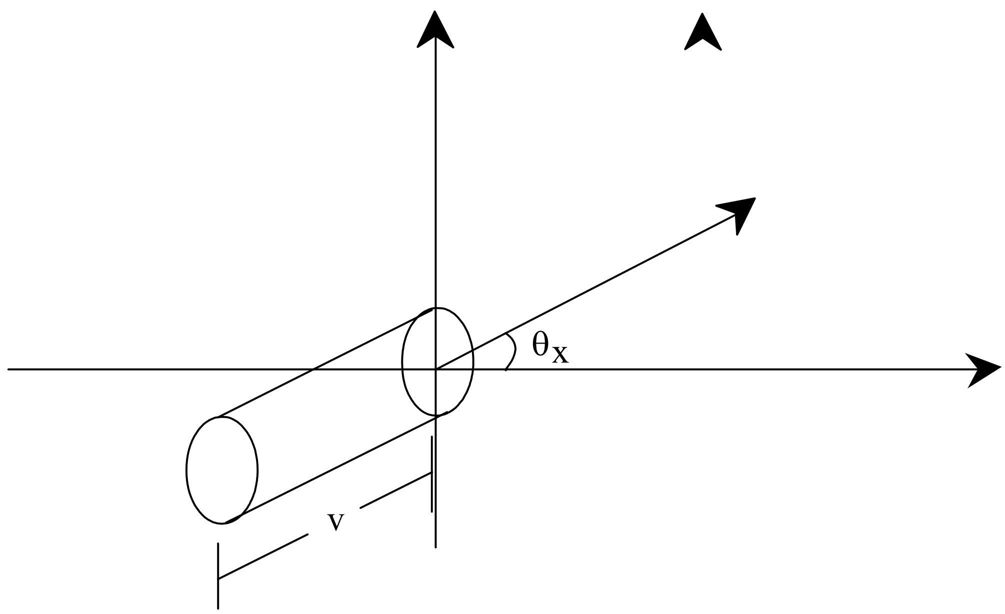


Figure 4 Neutron current

J_x = x component = $\int_{4\pi} n(\mathbf{r}, \boldsymbol{\omega}) v \cos\theta_x \, d\Omega$

= number of neutrons passing through the end of the cylinder <u>in the x direction</u>.

In general $J_n = \mathbf{J} \bullet \mathbf{n}$ = flow of neutrons in direction normal to **n**.

5 Equation of Continuity

Rate of change of neutron density = production rate – absorption rate – leakage rate

Rate of change of neutron density = $\frac{\partial}{\partial t}\int_{\forall} n(\mathbf{r}, t)d\forall$, where $\forall$ is the volume

Production rate = $\int_{\forall} S(\mathbf{r},t)\, d\forall$, where S is the source distribution function

Absorption rate = $\int_{\forall} \Sigma_a(\mathbf{r})\phi(\mathbf{r},t)d\forall$

Leakage rate = $\int_A \mathbf{J}(\mathbf{r}, t) \bullet \mathbf{n}dA$, where $\mathbf{J}$ is the current and A is the surface area bounding $\forall$

$\mathbf{n}$ = normal to area A.

Thus,

$$\frac{\partial}{\partial t}\int_{\forall} n(\mathbf{r},t)\, d\forall = \int_{\forall} S(\mathbf{r},t)\, d\forall - \int_{\forall} \Sigma_a(\mathbf{r})\phi(\mathbf{r},t)d\forall - \int_A \mathbf{J}(\mathbf{r},t)\bullet\mathbf{n}dA$$

But Gauss' Divergence theorem says:

$$\int_A \mathbf{J}(\mathbf{r}, t) \bullet \mathbf{n}dA = \int_{\forall} \nabla \bullet \mathbf{J}(\mathbf{r}, t)d\forall$$

Thus, by dropping the integral over $\forall$:

$$\boxed{\frac{1}{v}\frac{\partial}{\partial t}\phi(\mathbf{r}, t) \equiv \frac{\partial}{\partial t}n(\mathbf{r},t) = s(\mathbf{r},t) - \Sigma_a(\mathbf{r})\phi(\mathbf{r},t) - \nabla\bullet\mathbf{J}(\mathbf{r},t)} \quad (5.1)$$

This is the Equation of Continuity. It is very important to reactor theory.

We will show that $\mathbf{J} \doteq -D\,\nabla\phi$. Thus we have that $-\nabla\bullet\mathbf{J}(\mathbf{r},t) \doteq +\nabla\bullet D\nabla\phi(\mathbf{r},t)$

For steady state, $\frac{\partial}{\partial t} = 0$, $\quad \nabla\bullet \mathbf{J}(\mathbf{r}) + \Sigma_a(\mathbf{r})\,\phi\,(\mathbf{r}) - s(r) = 0$

For space independence: $\nabla\, \mathbf{J} = 0$, etc.:

$$\therefore\ \frac{dn(t)}{dt} = s(t) - \Sigma_a\phi(t)$$

Note: Delayed precursors are neglected for the time being but are easily incorporated.

6 Fick's Law

We apply Fick's Law (from diffusion in liquids, gases, etc.) to the neutrons in order to supply a relationship between **J** and known quantities.

6.1 Derivation

Fick's Law was developed under the following assumptions:

1. The medium is infinite;
2. The medium is uniform, ie. $\Sigma \neq \Sigma(\mathbf{r})$;
3. There are no neutron sources in the medium;
4. Scattering is isotropic in the laboratory coordinate system;
5. The neutron flux is a slowly varying position of position;
6. The neutron flux is not a position of time.

These assumptions can be restrictive, but we shall see that they can be relaxed to some extent. Let's consider the components of **J** in a Cartesian coordinate system:

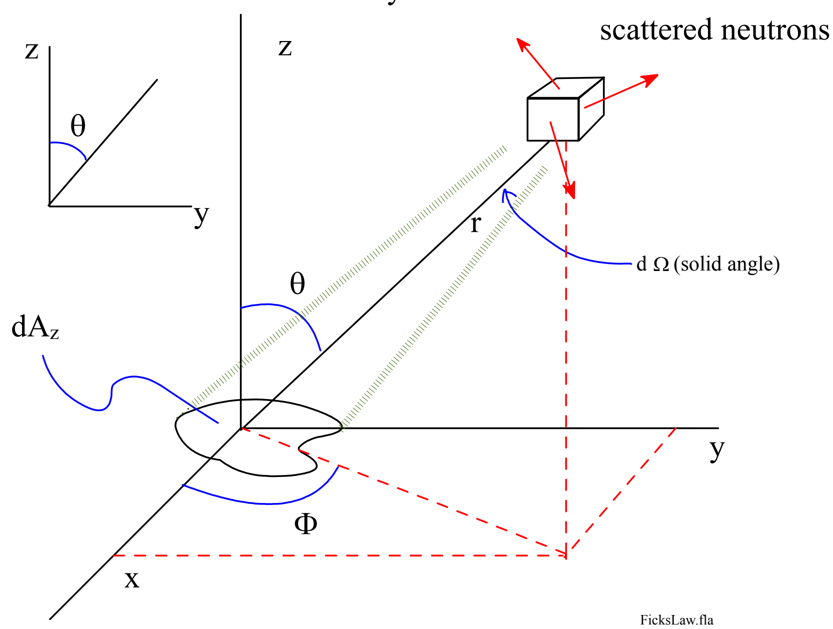


Figure 5 Scattered neutrons

We consider the number of neutrons being scatter through the origin in the x-y plane from $d\forall$ at $\forall$ at **r**. These neutrons must have been scattered since we have assumed no sources.

The number of collisions/sec at **r** is:

$$\Sigma_s(\mathbf{r})\phi(\mathbf{r})d\forall = \Sigma_s\ \phi(\mathbf{r})d\forall$$

(slowly varying: $\phi(\mathbf{r})$; assumed space independence: Σ_s)

Since the scattering is isotropic:

$$\Sigma_s\ \phi(\mathbf{r})\frac{\cos\theta\ dA_z d\forall}{4\pi r^2} \text{ neutrons/sec}$$

Head towards dA_z from $d\forall$ at **r**

But some are removed en route $(e^{-\Sigma_t r})$. (Note: As usual, we assume no buildup factor.)

Thus $\Sigma_s\ \phi(\mathbf{r})(e^{-\Sigma tr})\frac{\cos\theta\ dA_Z}{4\pi r^2}d\forall$ pass through the area dA_z.

If we write: $d\forall = r^2 \text{ Sin}\,\theta\ dr\ d\theta\ d\Phi \quad (= (r \text{ Sin}\,\theta\ d\ \Phi) \bullet rd\ \theta \bullet dr)$

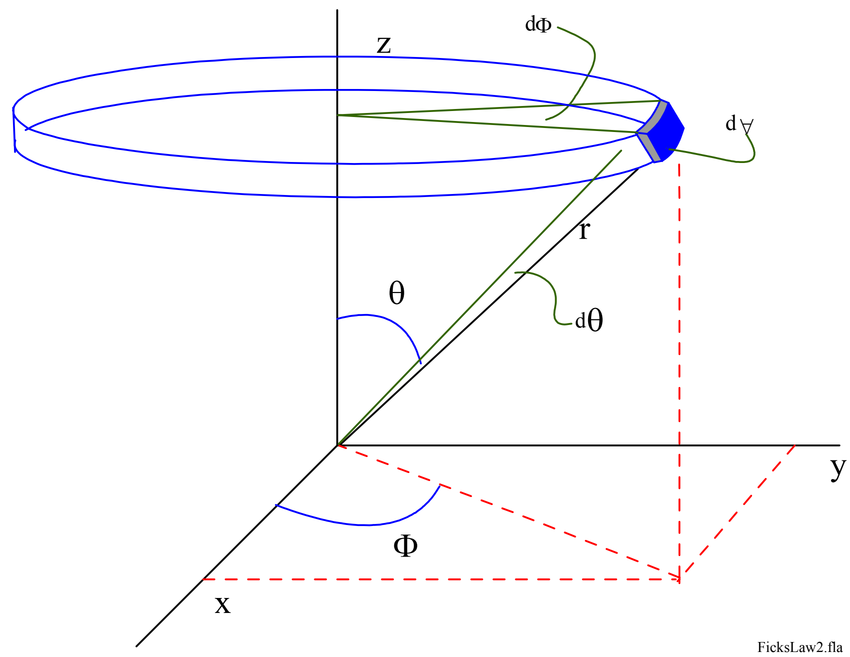


Figure 6 Fick's Law shell integration

We can integrate to get the total flow:

total flow downward through

$$\frac{\Sigma_s dA_Z}{4\pi} \int_{\Phi=0}^{2\pi} \int_{\theta=0}^{\pi/2} \int_{r=0}^{\infty} e^{-\Sigma_t r} \phi(r)\cos\theta\sin\theta dr\, d\theta\, d\Phi = J_Z^- \, dA_Z$$

Since $\phi(\mathbf{r})$ is a slowly varying function of space:

$$\phi(\mathbf{r}) = \phi_o + x\left(\frac{\partial\phi}{\partial x}\right)_o + y\left(\frac{\partial\phi}{\partial y}\right)_o + z\left(\frac{\partial\phi}{\partial z}\right)_o$$

and $x = r\,\mathrm{Sin}\theta\,\mathrm{Cos}\Phi$, $y = r\,\mathrm{Sin}\theta\,\mathrm{Sin}\Phi$, $z = r\,\mathrm{Cos}\theta$

Thus:

$$J_Z^- = \frac{\Sigma_s\,\phi_o}{4\Sigma_t} + \frac{\Sigma_s}{6\Sigma_t^2}\left(\frac{\partial\phi}{\partial z}\right)_o$$

Similarly:

$$J_Z^+ = \frac{\Sigma_s\,\phi_o}{4\Sigma_t} - \frac{\Sigma_s}{6\Sigma_t^2}\left(\frac{\partial\phi}{\partial z}\right)_o$$

Thus:

$$J_Z = \underbrace{J_Z^+ - J_Z^-}_{\text{net flow}} = -\frac{\Sigma_a}{3\Sigma_t^2}\left(\frac{\partial\phi}{\partial z}\right)_o$$

net flow

Generalizing:

$$\mathbf{J} = \hat{i}\,J_x + \hat{j}\,J_y + \hat{k}\,J_z = -\frac{\Sigma_s}{3\Sigma_t^2}\left(\frac{\partial\phi}{\partial z}\right)_o$$

In words: The current density **J** is proportional to the negative of the gradient of the neutron flux (n, v).

We define $D = \frac{\Sigma_s}{3\Sigma_{t^2}}$, thus $\mathbf{J} = -D\,\nabla\,\phi$

= diffusion coefficient

The physical interpretation is similar to fluxes of gases. The neutrons exhibit a net flow in the direction of least density. This is a natural consequence of greater collision densities at positions of greater neutron densities.

6.2 Validity of Fick's Law

We re-evaluate each assumption in turn:

1. *Infinite medium*. This assumption was necessary to allow integration over all space but flux contributions are negligible beyond a few mean free paths due to the factor, $e^{-\Sigma_t r}$.

   Thus as long as we are at least a few mean free paths from the reactor extremities, all is okay. Corrections can be made at the reactor surfaces as shown later in this chapter.

2. *Uniform medium*. A non-uniform medium $(\Sigma_s = \Sigma_s(\mathbf{r}))$ requires a re-evaluation of the derivation of Fick's Law. Now the interaction rate, $\Sigma_S\,\phi$, is a function of space due to both ϕ and Σ variations in space. Detailed calculations show, however, that the extra current (ie. scattering) contributions caused by a locally larger Σ_S are <u>exactly</u> cancelled by larger attenuations $((e^{-\Sigma_t r} = e^{-(\Sigma_s + \Sigma_a)r})$ <u>iff</u> (if and only if) $\Sigma_s >> \Sigma_a$ or Σ_s / Σ_t = constant.

   It should be noted however that $\Sigma_S(\mathbf{r})$ can lead to large values of $\frac{\partial \phi}{\partial \mathbf{r}}(\mathbf{r})$ which violates assumption (e).

3. *Sources*. As per assumption (a), we can get away with sources as long as they are more than a few mean free paths away.

4. *Isotropic scattering*. Anisotropic scattering can be corrected for by detailed considerations of transport theory in which D is re-evaluated:

$$\frac{\Sigma_s}{2}\sqrt{\frac{D}{\Sigma_a}}\ln\left[\frac{\Sigma_t + \sqrt{\Sigma_a/D}}{\Sigma_t - \sqrt{\Sigma_a/D}}\right] = \frac{1 + 3D\Sigma_s\,\bar{\mu}}{1 + 3D\Sigma_s\,\bar{\mu}}$$

   Where

$$\bar{\mu} \equiv \cos\theta \quad \text{(average of the scattering angle in the lab system)}$$

$$= \frac{2}{3A}$$

   Expanding the equation in D, above:

$$D = \frac{1}{3\Sigma_t\,(1 - \bar{\mu})(1 - 4\Sigma_a/5\Sigma_t + ...)}$$

$$= \frac{1}{3\Sigma_t\ (1 - \overline{\mu})} \quad \text{for } \Sigma_a / \Sigma_t << 1$$

$\therefore\ D = \frac{\lambda_{tr}}{3}$ as previously defined in the supplemental material at the end of the chapter on *Basic Definitions and Perspectives*

5. *Slowly varying flux*. Further expansions of φ are necessary to account for large variations in ϕ (**r**). It can be shown that 2nd order terms cancel and that third order terms are not important beyond a few mean paths. Therefore, provided $\frac{d^2\phi}{d\mathbf{r}^2}(r)$ is small over a few mean free paths, all is okay. Large variations in ϕ occur when Σ_a is large (compared to Σ_S).

6. *Time - dependent flux.* The time it takes a slow neutron to traverse 3 mean free paths (in cm.) is

$$\Delta t \sim \frac{3\lambda_s}{v} \sim \frac{3x1cm}{2x10^5 cm/s} \sim 1.5x10^{-5} s.$$

If is changed at 10%/s (a high rate), then

$$\frac{\Delta\phi}{\phi} = \frac{\Delta\phi/\phi}{\Delta t} x\Delta t \sim 0.1\Delta t = 1.5x10^{-6}.$$

This is a very small fractional change of flux amplitude in the time it takes a neutron to move a significant physical distance.

7 Question: What about the Conservation of Momentum and Energy?

Question:

This neutron balance equation:

$$\frac{\partial n(\mathbf{r}, t)}{\partial t} = s(r\,,t) - \Sigma_a(F)\phi\,(\mathbf{r}, t) + \nabla \bullet D(\mathbf{r})\phi\,(\mathbf{r}, t)$$

Is similar to the conservation of mass in fluids:

$$\frac{\partial \rho}{\partial t} = S - \nabla \bullet \rho v$$

Why do we not also consider the conservation of momentum and energy?

Solution:

Conservation of momentum and energy is used on a neutron-nucleus interaction level. Recoils, etc. lead to cross sections, which are input into the neutron balance equation. Compare this to fluid mechanics. The neutrons don't interact with each other but fluid particles do. Hence the fluid mass, energy, and momentum equations are tightly linked. The neutrons affect each other only via temperature changes, etc., brought about by fissioning, whereas, fluid particles shear off each other and with the walls. This is a fundamental difference.

In short:

1. Because of treatment of Σ as effective area of interaction
2. Too complicated.
3. Neutrons do not interact with each other!

8 The Diffusion Equation

We return now to the neutron balance equation:

$$\frac{\partial}{\partial t} n(\mathbf{r},t) = s(\mathbf{r},t) - \Sigma_a(\mathbf{r})\phi(\mathbf{r},t) - \nabla \bullet \mathbf{J}(\mathbf{r},t) \tag{8.1}$$

and substitute

$$\mathbf{J} = -D\,\nabla\phi(\mathbf{r},t)$$

to give

$$\frac{\partial}{\partial t} n(\mathbf{r},t) = s(\mathbf{r},t) - \Sigma_a(\mathbf{r})\phi(\mathbf{r},t) + \nabla \bullet D \nabla \phi(\mathbf{r},t)$$

If D = constant w.r.t. **r** then (using $n\,v = \phi$)

$$\frac{1}{v}\frac{\partial}{\partial t}\phi(\mathbf{r},t) = S(\mathbf{r},t) - \Sigma_a(\mathbf{r})\phi(\mathbf{r},t) + D\,\nabla^2\phi(\mathbf{r},t)$$ ←**Neutron diffusion equation**

For steady state $\frac{1}{v}\frac{\partial}{\partial t}\phi(\mathbf{r},t) = 0$. This steady state diffusion equations is also known as the "scalar Helmholtz equation". If s $(\mathbf{r}) = 0$ as well, it is sometimes called the "buckling equation" in analogy to the equation governing the buckling of beams in strength of materials. It is also known as the "wave equation" in analogy to vibrating strings, etc.

9 Boundary Conditions for the Steady State Diffusion Equation

For this type of equation the following applies:

On the boundaries of a region in which ϕ satisfies the differential equation, either ϕ, or the normal derivative of ϕ, or a linear combination of the two must be specified. Both ϕ and its normal derivative cannot be specified independently.

9.1 Boundary Conditions at Surfaces

One type of boundary condition occurs at surfaces, ie., interfaces between dense and sparse media recall that Fick's Law (and hence Diffusion Theory) is not valid near a surface. Therefore special consideration must be given.

We assume a boundary condition at the surface of the form

$$\frac{1}{\phi}\frac{d\phi}{dn} = -\frac{1}{d}$$

where d = extrapolated length and $\frac{d\phi}{dn}$ = normal derivative at the surface. Note that this equation satisfies the above condition regarding boundary condition forms.

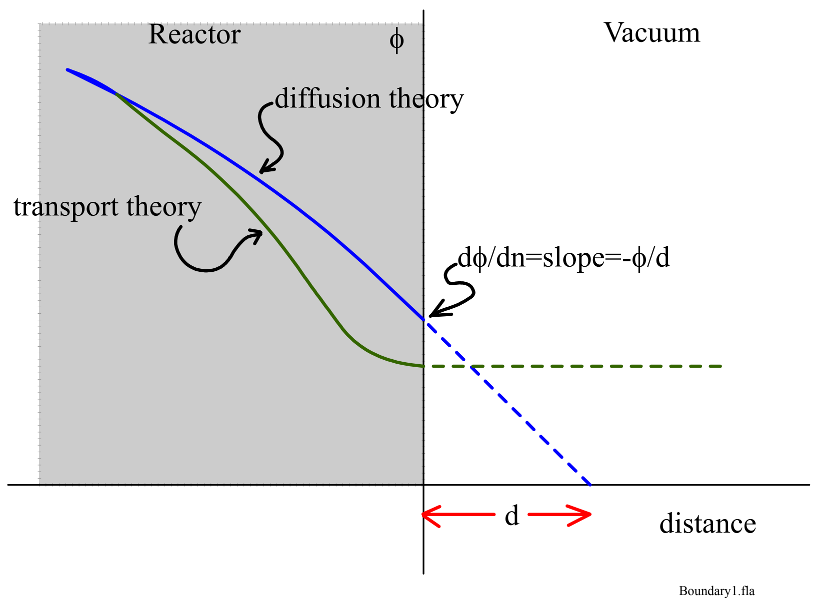


Figure 7 Extrapolated length

Imposing this B.C. on a non re-entrant surface (one in which neutrons cannot re-enter once they have left the medium) gives:

$d = 0.71\ \lambda_{tr}$ for planar surfaces

from transport theory. It can be shown that this B.C. gives an accurate value of ϕ for interior points but not near or at the surface.

In view of the fact that $\phi = 0$ at the extrapolated length and since d << system dimensions in practical situations, the above B.C. is replaced with little error by:

The solution to the diffusion equation vanishes at the extrapolation distance beyond the edge of a free surface.

The assumptions inherent in the above should be carefully noted.

Since D = diffusion coefficient = $\frac{\lambda_{tr}}{3}$ and D ~ 1 cm,

$\therefore$ d = .71 λ_{tr} ~.71 × 3 ~ 2 cm << reactor size (meters)

$\therefore$ ϕ (surface) $\approx$ 0

9.2 Boundary Conditions at an Interface

At an interface, there is no accumulation of neutrons. Therefore:

$$(J_A)_n = (J_B)_n \quad \text{where n=normal direction}$$

ie. the neutron current in region A = that of B

Also, n (**r**, t, …) and ϕ (**r**, t, …) are continuous across the interface.

9.3 Other Conditions

Physical requirements:

1. A negative or imaginary flux has no meaning. Hence: the solution to the diffusion equation must be real, non-negative and single valued in those regions where the equation applies.

2. $\phi \neq \infty$ except for singular points of source distributions. These two constraints serve to eliminate extraneous functions from the solution.

9.4 Summary of Boundary Conditions

1. $\phi = \phi$ } at an interface
2. $J = J$ } at an interface
3. ϕ defined } at a surface
4. J = defined } at a surface
5. ϕ finite
6. $\phi > 0$ and real

10 Elementary Solutions of the Steady State Diffusion Equation

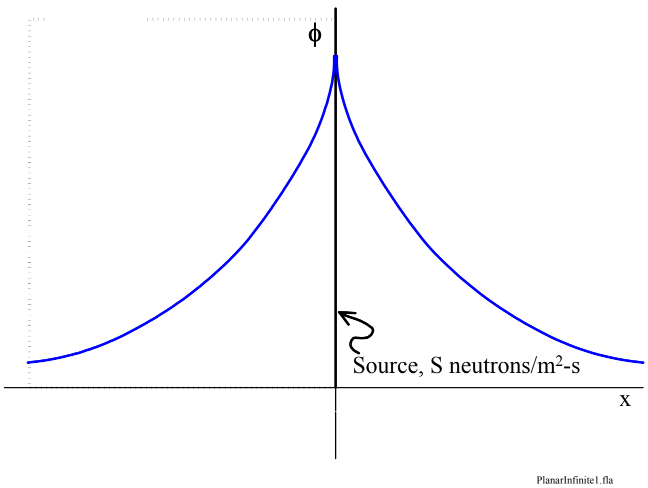
We have previously shown the Steady State Diffusion Equation to be

$$0 = S(\bar{\mathbf{r}}) - \Sigma_a(\bar{\mathbf{r}})\,\phi(\bar{\mathbf{r}}) + D\,\nabla^2\phi(\bar{\mathbf{r}})$$

Defining $L^2 = \dfrac{D}{\Sigma_a} \equiv [\mathrm{cm}^2]$; $L \equiv$ diffusion length

$$\nabla^2\phi - \frac{1}{L^2}\phi = -\frac{S}{D} \qquad (10.1)$$

10.1 Infinite Planar Source



$\delta(x) = 0,\ x \neq 0$

$\int_a^b \delta(x)\,dx = 1,\ a < 0 < b$

$= 0$ otherwise

Figure 8 Flux distribution for a planar source

Equation (10.1) reduces to:

$$\frac{d^2\phi(x)}{dx^2} - \frac{1}{L^2}\phi(x) = -\frac{S\delta(x)}{D} \qquad (10.2)$$

and for $x \neq 0$

$$\frac{d^2\phi(x)}{dx^2} - \frac{\phi(x)}{L^2} = 0 \qquad (10.3)$$

Consider the planar source (as shown in figure 9)

$$\lim_{x\to 0} \underbrace{J(x)}_{\text{current from either end}} = \frac{S}{2} \qquad (10.4)$$

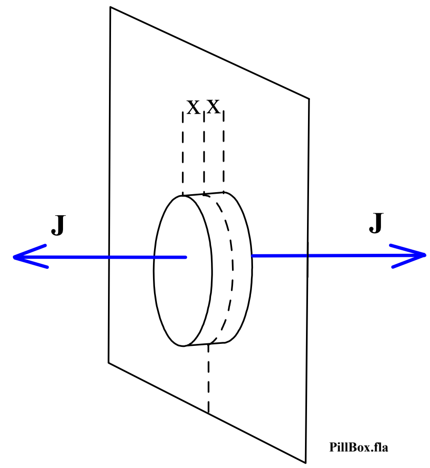


Figure 9 Current "pill box"

The solution to Equation (10.3) has the following form:

$$\phi(x) = A\, e^{-x/L} + C\, e^{x/L} \qquad (10.5)$$

For $x > 0$, $C = 0$, otherwise ϕ is non-finite as $x \to \infty$

$$\therefore\ \phi(x) = A\, e^{-x/L},\ x > 0 \qquad (10.6)$$

From Fick's Law $J\big|_0 = -D\frac{d\phi}{dx}\Big|_0 = +\frac{DA}{L}e^{-x/L}\Big|_0 = \frac{DA}{L} = \frac{S}{2}$

$$\therefore\ A = \frac{SL}{2D}$$

$$\therefore\ \phi(x) = \frac{SL}{2D}\, e^{-x/L} \qquad x > 0 \qquad (10.7)$$

Similarly for $x < 0$, giving $\boxed{\phi(x) = \frac{SL}{2D}\, e^{-|x|/L}}$. Recall that this not valid at or near $x = 0$.

This solution should make physical sense to you. The flux decays exponentially away from the source as it is absorbed by the medium. This agrees with the beam absorption laws that we have previously derived.

10.2 Point Source in an Infinite Medium

For a point source, it is appropriate to work in spherical coordinates:

$$\frac{1}{r^2}\frac{d}{dr}r^2\frac{d\phi}{dr}-\frac{1}{L^2}\phi=S\delta(r) \qquad (10.8)$$

where the source, S, is at r = 0.

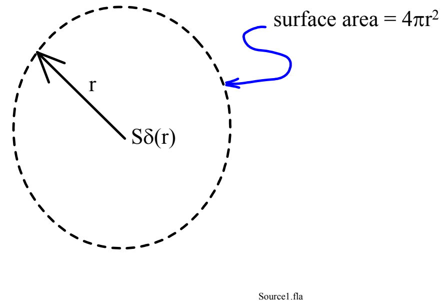


Figure 10 Point source

Now

$$4\pi r^2 J(r)=S\delta(r)=S$$

$$\therefore \lim_{\delta r\to 0} r^2 J(r)=\frac{S}{4\pi} \qquad (10.9)$$

We define a change of variable

$$\omega=r\phi \Rightarrow \frac{\partial^2\omega}{\partial r^2}-\frac{1}{L^2}\omega=0 \qquad (10.10)$$

and, therefore, as before:

$$\omega=Ae^{-r/L}+Ce^{r/L}$$

or

$$\phi=A\frac{e^{-r/L}}{r}+C\frac{e^{r/L}}{r} \quad \text{and} \quad C=0 \text{ as before.}$$

Now,

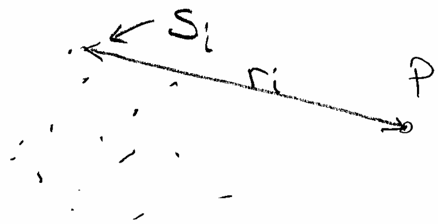
$$J=-D\frac{d\phi}{dx}=DA\left(\frac{1}{rL}+\frac{1}{r^2}\right)e^{-r/L}$$

$$\therefore r^2 J\big|_{r=0}=DA\left(\frac{r}{L}+1\right)e^{-r/L}\Big|_{r=0}=DAe^{-0/L}=DA=\frac{S}{4\pi}$$

$$\therefore A=\frac{S}{4\pi D}, \quad \therefore \boxed{\phi=\frac{S}{4\pi Dr}e^{-r/L}}$$

Note: In the above two cases, as in most reactor cases, the flux, ϕ, is proportional to the source strength, S.

ϕ from a general dist^n of S:



#/S

$$\phi(P) = \sum_i \frac{S_i}{4\pi D r_i} e^{-r_i/L}$$

if S_i is discrete point source at r_i

$$\phi(P) = \int_{-\infty}^{\infty} \frac{S(x)}{4\pi D x} e^{-x/L} \, dx$$

where $S(x) \equiv \frac{\#}{cm \cdot s}$

(i.e. $S(x)dx = \#/s$)

or in general

$$\phi(P) = \int_x \int_y \int_z \frac{S(x,y,z)}{4\pi D r} e^{-r/L} \, dx\,dy\,dz$$

where r = dist. from point P

$S(x,y,z) = \#/cm^3 \cdot s$

i.e. ($S(x) \, d\forall = \#/s$)

10.3 Systems with a Free Surface

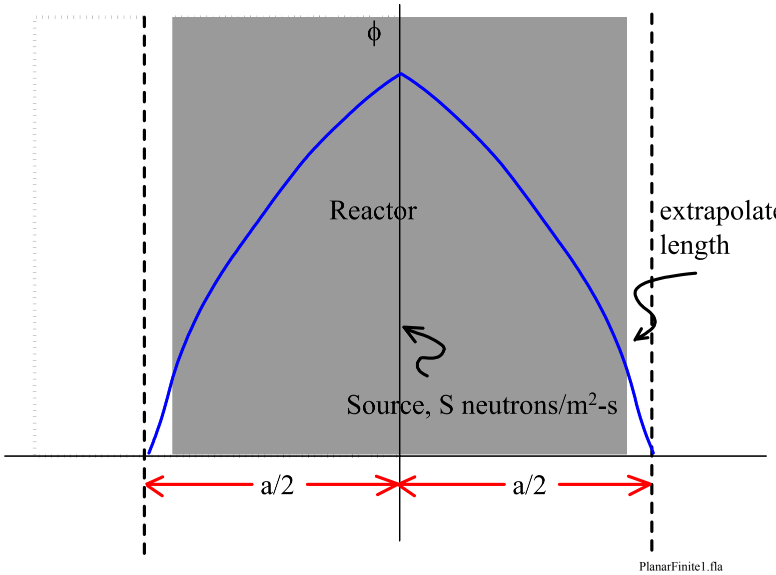


Figure 11 Planar source in a finite slab

Consider an infinite slab of thickness a-2d, where d=0.71 λ_{tr}, hence $\phi(\pm a/2) = 0$. The governing 1 speed neutron diffusion equation in steady state is

$$\frac{d^2\phi(x)}{dx^2} - \frac{1}{L^2}\phi(x) = -\frac{S\delta(x)}{D}$$

We try a solution of the form:

$$\phi(x) = A\,e^{-x/L} + C\,e^{x/L}$$

For x>0,

$$\phi\left(\tfrac{a}{2}\right) = 0 = Ae^{-a/L} + Ce^{+a/L}$$

$$\therefore C = -Ae^{-a/L}$$

$$\therefore \phi = A\left(e^{-x/L} - e^{+(x-a)/L}\right)$$

As before

$$J = -D\frac{d\phi}{dx} = \frac{S}{2} = +\frac{DA}{L}\left[e^{-x/L} + e^{(x-a/L)}\right]_{x=0}$$

$$= \frac{DA}{L}\left[1 + e^{-a/L}\right]$$

$$\therefore \phi = \frac{SL}{2D}\left[1 + e^{-a/L}\right]^{-1}\left(e^{-x/L} - e^{(x-a)/L}\right)$$

From symmetry, we conclude

$$\therefore \phi = \frac{SL}{2D}\left[1+e^{-a/L}\right]^{-1}\left(e^{-|x|/L} - e^{(|x|-a)/L}\right)$$

$$= \frac{SL}{2D}\frac{\sinh\left[\left(a-2|x|\right)/2L\right]}{\cosh\left(a/2L\right)}$$

We could have started with

$$\phi = A\cosh(x/L) + C\sinh(x/L)$$

to get the same answer.

Lamarsh suggests using hyperbolic trial solutions for finite media and exponentials for infinite media for the following reasons:

1. Sinh x has a zero – good for finite media
2. $e^{-x} \rightarrow 0$ at x = ∞ - good for infinite media
3. sinh x and cosh x $\rightarrow\infty$ at x = ∞ - bad for infinite media
4. cosh x is an even function – good for symmetry.

When working with distributed sources, solutions usually are:

1. Cosine, sine for Cartesian coordinates
2. Bessel for cylindrical geometry.

Math aside:

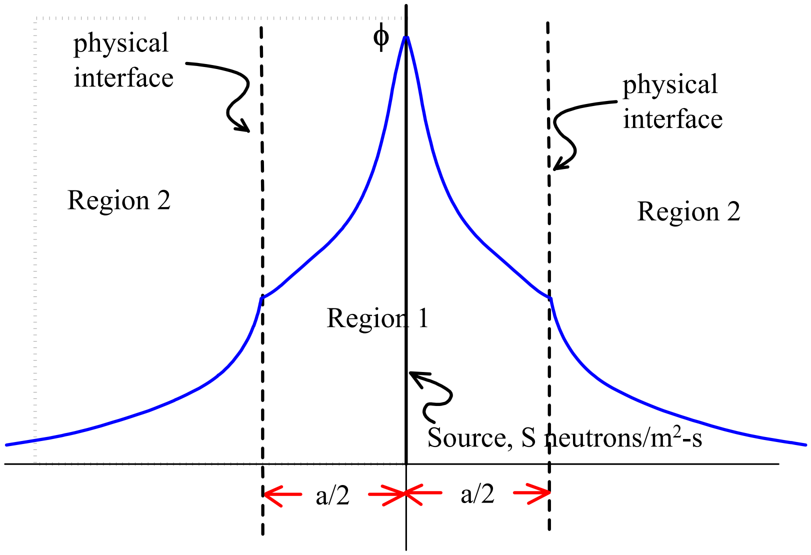
$$\frac{e^{-x/L} - e^{(x-a)L}}{1+e^{-a/L}} = \frac{e^{a/2L}(e^{-x/L} - e^{(x-a)/L})}{e^{a/2L} - e^{-a/2L}}$$

$$= \frac{e^{(a-2x)/2L} - e-^{(a-2x)/2L}}{2\cosh(a/2L)} = \frac{\sinh[(a-2x)/2L)]}{\cosh(a/2L)}$$

$$\sinh x = \frac{e^x - e^{-x}}{2}, \quad \cosh x = \frac{e^x + e^{-x}}{2}$$

10.4 Multi-region Problems

Consider the infinite slab of the previous medium surrounded by a second medium of infinite thickness.



We have:

$$\frac{d^2\phi_1}{dx^2} - \frac{1}{L_1^2}\phi_1 = 0 \quad \text{for } |x| \prec a/2,\ x \neq 0$$

$$\frac{d^2\phi_2}{dx^2} - \frac{1}{L_2^2}\phi_2 = 0 \quad \text{for } |x| \succ a/2$$

Boundary conditions

1. $\phi_2 =$ finite as $|x| \to \infty$
2. $\lim_{x\to 0} J(x) = \frac{S}{2}$ (source condition, normalization)
3. $\phi_1(\pm a/2) = \phi_2(\pm a/2)$ continuity of flux at the interface
4. $D_1 \left.\frac{d\phi_1}{dx}\right|_{x=\pm a/2} = D_2 \left.\frac{d\phi_2}{dx}\right|_{x=\pm a/2}$ continuity of current at the interface

We try:

$$\phi_1(x) = A_1 \cosh(x/L_1) + C_1 \sinh(x/L_1)$$

$$\phi_2(x) = A_2 e^{-x/L_2} + C_2 e^{x/L_2}$$

Thus we have 4 unknowns and 4 boundary conditions.

Immediately: $C_2=0$ from B.C. (1) above.

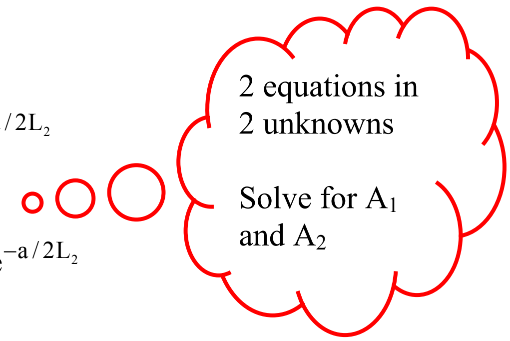
From (2):

$$\frac{S}{2} = J(0) = -D_1 \left.\frac{d\phi_1}{dx}\right|_0 = -\frac{D_1 A_1}{L_1}\left[\frac{e^{x/L_1} - e^{-x/L_1}}{2}\right]_{x=0} - \frac{D_1 C_1}{L_1}\left[\frac{e^{x/L_1} + e^{-x/L_1}}{2}\right]_{x=0}$$

$$\therefore C_1 = -\frac{SL_1}{2D_1}$$

From B.C. (3)

$$-A_1 \cosh\frac{a}{2L_1} + \frac{SL_1}{2D_1}\sinh\frac{a}{2L_1} = A_2 e^{-a/2L_2}$$



From B.C. (4)

$$-\frac{D_1 A_1}{L_1}\sinh\frac{a}{2L_1} + \frac{S}{2}\cosh\frac{a}{2L_1} = \frac{D_2 A_2}{L_2} e^{-a/2L_2}$$

Solving gives:

$$A_1 = \frac{SL_1}{2D_1}\frac{D_1 L_2 \cosh(a/2L_1) + D_2 L_1 \sinh(a/2L_1)}{D_2 L_1 \cosh(a/2L_1) + D_1 L_2 \sinh(a/2L_1)}$$

$$A_2 = \frac{SL_1 L_2}{2}\frac{e^{a/2L_1}}{D_2 L_1 \cosh(a/2L_1) + D_1 L_2 \sinh(a/2L_1)}$$

Therefore, we know ϕ_1 and ϕ_2 .

Notes:

1. ϕ_1 and ϕ_2 are proportional to S.
2. It can be shown that ϕ is continuous at $\pm a/2$ and that $d\phi/dx$ is not continuous but D $d\phi/dx$ is. This results from the boundary conditions imposed. Only if $D_1=D_2$ is continuous.

11 Diffusion Length

We have shown that, for a point source of neutrons, S at r = 0,

$$\phi(\mathbf{r}) = \frac{Se^{-r/L}}{4\pi rD}, \text{where L is the characteristic length for fall off of } \phi$$

The number of neutrons absorbed between r and r+dr is:

$$dN = \Sigma_a\, \phi(r)dV = \frac{\Sigma_a\, Se^{-r/L}}{D4\pi r} 4\pi r^2 dr$$
$$= \frac{\Sigma_a}{D} Se^{-r/L} rdr = \frac{S}{L^2} re^{-r/L} dr$$

Thus the probability of interaction (absorption) is

$$\frac{dN}{S} = p(r)dr = \frac{r}{L^2} e^{-r/L} dr$$

$$\therefore \int_0^\infty p(r)dr = \int_0^\infty \frac{r}{L^2} e^{-r/L} dr = 1 \quad (\text{recall that} \int_0^\infty x^n e^{-ax} dx = \frac{n!}{a^{n+1}})$$

ie, all the neutrons are absorbed eventually.

The second moment of is defined:

$$\overline{r^2} = \int_0^\infty \frac{r^2 re^{-r/L}}{L^2} dr = 6L^2$$

Thus

$$L^2 = \frac{1}{6}\overline{r^2}$$

That is, the diffusion length $L^2 = D/\Sigma_a = (1/6)x$ average of the square of crow-flight distance of the neutron.

L^2 is often called the diffusion area.

We define $L^2 \equiv \frac{1}{6}\overline{r^2}$. For the approximation of diffusion theory $L^2 \equiv D/\Sigma_a$ as well.

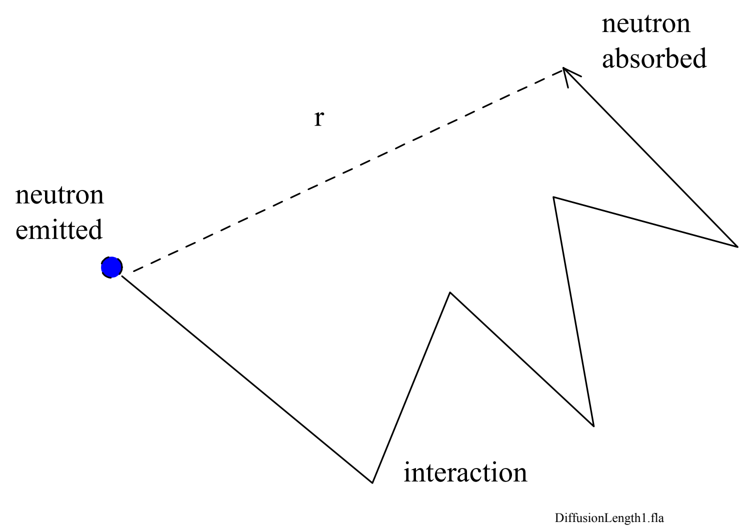


Figure 12 Diffusion Length

About this document:

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Notes: