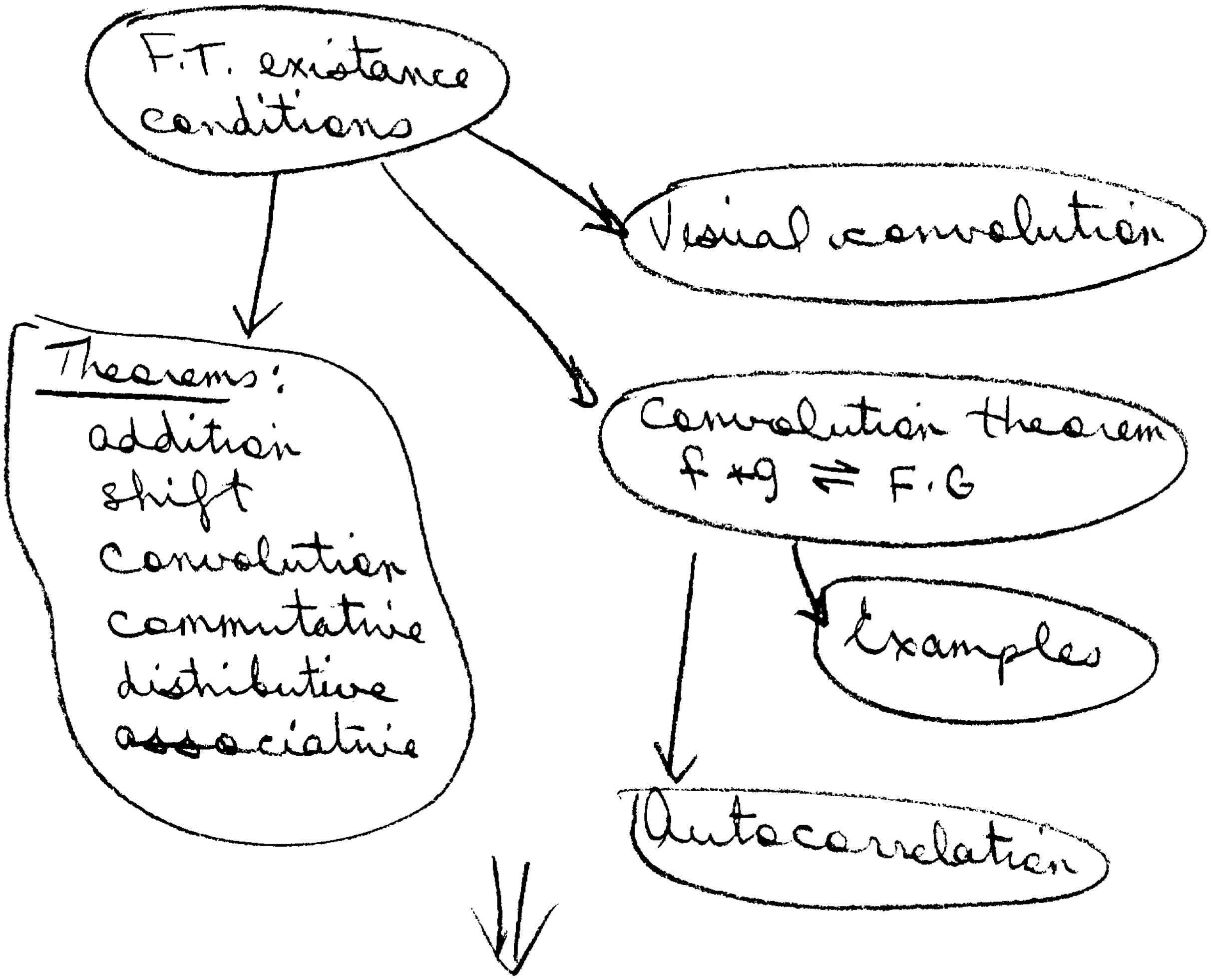
Chapter 6 - Useful Properties and Theorems

6.0 Overview



Algebra of convolution:

- Derivative Thr $\longrightarrow$ PDE sol'n
- Derivative of convolution
- Power Thr.
- Sampling Thr. $\longrightarrow$ Aliasing $\longrightarrow$ Nyquist criteria
- Similarity Thr.

6 Useful Properties and Theorems

(following James chapter 2)

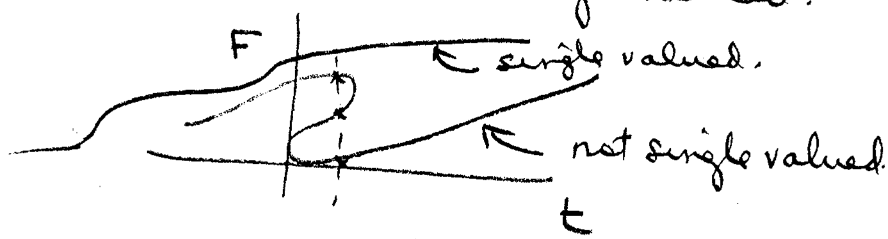
6.1 The Dirichlet conditions

Not all functions can be Fourier-transformed. They are if they fulfill the Dirichlet conditions, which are:

(1) $\int_{-\infty}^{\infty} |F(t)|^2 dt < \infty$

This implies $F(t) \to 0$ as $t \to \infty$.

(2) $F(t)$ & $\Phi(\nu)$ are single valued.



(3) $F(t)$ & $\Phi(\nu)$ are 'piece-wise continuous'

(4) $F(t)$ & $\Phi(\nu)$ should be bounded but this is not a practical engineering issue.

All natural systems obey the Dirichlet conditions.

6.2 Theorems

A bit of work here will give us tools to make the work of Fourier Transforms easier.

We assume $F_1(t) \rightleftharpoons \Phi_1(\nu)$ & $F_2(t) \rightleftharpoons \Phi_2(\nu)$

The Addition Theorem

$$F_1(t) + F_2(t) \rightleftharpoons \Phi_1(\nu) + \Phi_2(\nu)$$

(the proof is trivial - $\int (a+b)\,dx = \int a\,dx + \int b\,dx$)

The Shift Theorems:

$$F_1(t+a) \rightleftharpoons \Phi_1(\nu)\, e^{2\pi i \nu a}$$

Proof:
$$\int_{-\infty}^{\infty} F_1(t+a)\, e^{-2\pi i \nu t}\,dt = \int_{-\infty}^{\infty} F_1(z)\, e^{-2\pi i \nu (z-a)}\,dz$$

where $z = t+a$

$$= \Phi_1(\nu)\, e^{2\pi i \nu a} \qquad \text{Q.E.D.}$$

Similarly, $F_1(t-a) \rightleftharpoons \Phi_1(\nu)\, e^{-2\pi i \nu a}$

Thus
$$F_1(t+a) + F_1(t-a) = \Phi_1(\nu)\left(e^{2\pi i \nu a} - e^{-2\pi i \nu a}\right)$$
$$= 2\Phi_1(\nu) \cos 2\pi \nu a$$

Note: If F_1 is the δ-function, we have (since $\delta(t) \rightleftharpoons 1$)

$$\delta(t+a) \rightleftharpoons e^{2\pi i \nu a}$$

$$\delta(t-a) \rightleftharpoons e^{-2\pi i \nu a}$$ (consistent with what we showed before in 5.4.5)

& $\delta(t+a) + \delta(t-a) \rightleftharpoons 2\cos 2\pi \nu a$

6.3 Convolutions and the Convolution Theorem

We talked about linear systems at the beginning of the course and saw how system output, $y(t)$ is related to input, $x(t)$ by:

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, L[\delta(t-\tau)]\, d\tau$$
$$= \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

where $L[\delta(t-\tau)] = h(t-\tau)$ is the system response to an input at time τ.

One specific example of this would be when the 'system' is a measuring instrument and, instead of measuring a time signal, we measure a spectrometer signal, ie:

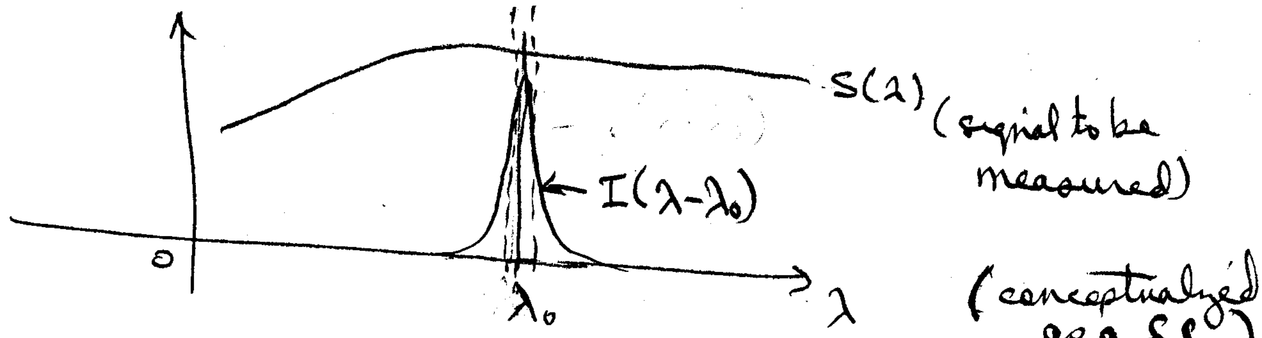
$$O(\lambda) = k\int_{-\infty}^{\infty} S(\lambda_1)\, I(\lambda - \lambda_1)\, d\lambda_1$$

where $O(\lambda)$ is the measured output

$I(\lambda - \lambda_1)$ is the instrument response at λ to an impulse at λ_1

$S(\lambda_1)$ is the spectrum source

K = geometry factor for the instrument (which could be included with I but here we normalize I via $\int I(\lambda)\, d\lambda = 1$)



So the instrument will respond to that bit of signal at λ_0 with an output that is spread out a bit. So a sharp line looks like a less distinct shape.

In general the convolution of 2 functions is $C(t) = \int_{-\infty}^{\infty} F_1(\tau)\, F_2(t-\tau)\, d\tau$

or $C(t) = F_1(t) * F_2(t)$ ← short hand notation

It is handy to know the rules of manipulation of this short hand notation.

<u>Commutative rule</u>:

$$C(t) = F_1(t) * F_2(t) = F_2(t) * F_1(t)$$

ie
$$
\begin{aligned}
C(t) &= \int_{-\infty}^{\infty} F_1(\tau)\, F_2(t-\tau)\, d\tau \\
&= \int_{\infty}^{-\infty} F_1(t-z)\, F_2(z)\, d(-z) \qquad (\text{let } \tau = t-z) \\
&= \int_{-\infty}^{\infty} F_2(z)\, F_1(t-z)\, dz \\
&= F_2(t) * F_1(t) \qquad \text{QED}
\end{aligned}
$$

Distributive rule:

$$F_1(t) * [F_2(t) + F_3(t)] = F_1(t) * F_2(t) + F_1(t) * F_3(t)$$

(follows directly from $\int a(b+c)\,dt = \int ab\,dt + \int ac\,dt$)

Associative rule:

$$F_1(t) * [F_2(t) * F_3(t)] = [F_1(t) * F_2(t)] * F_3(t)$$

Proof:

Fourier Transform

$$F_1(t) * [F_2(t) * F_3(t) \rightarrow \phi_1(\nu)[\mathcal{F}(F_2(t) * F_3(t)]$$

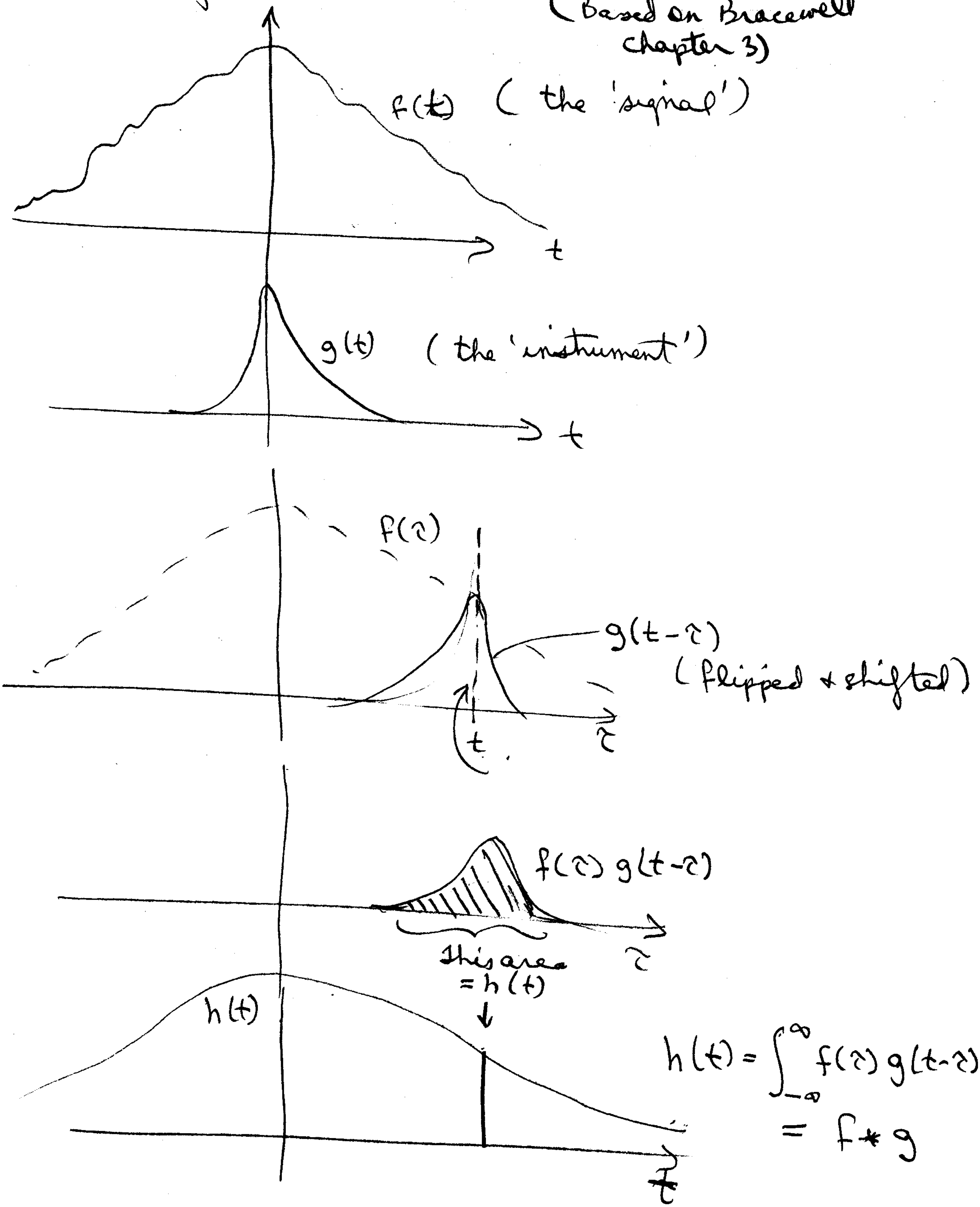
$$= \phi_1(\nu)[\phi_2(\nu) \cdot \phi_3(\nu)]$$

$$= [\phi_1(\nu) \cdot \phi_2(\nu)] \cdot \phi_3(\nu)$$

$$[F_1(t) * F_2(t)] * F_3(t) \leftarrow = \mathcal{F}[F_1(t) * F_2(t)] \cdot \phi_3(\nu)$$

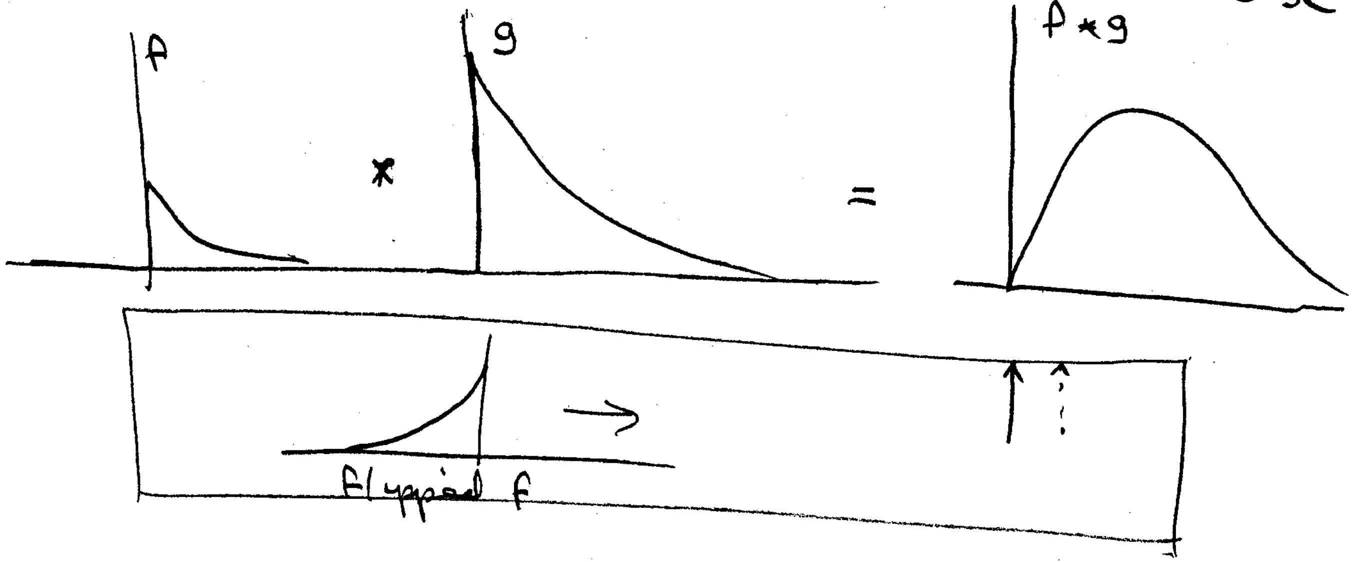
Visualizing a Convolution

(Based on Bracewell chapter 3)

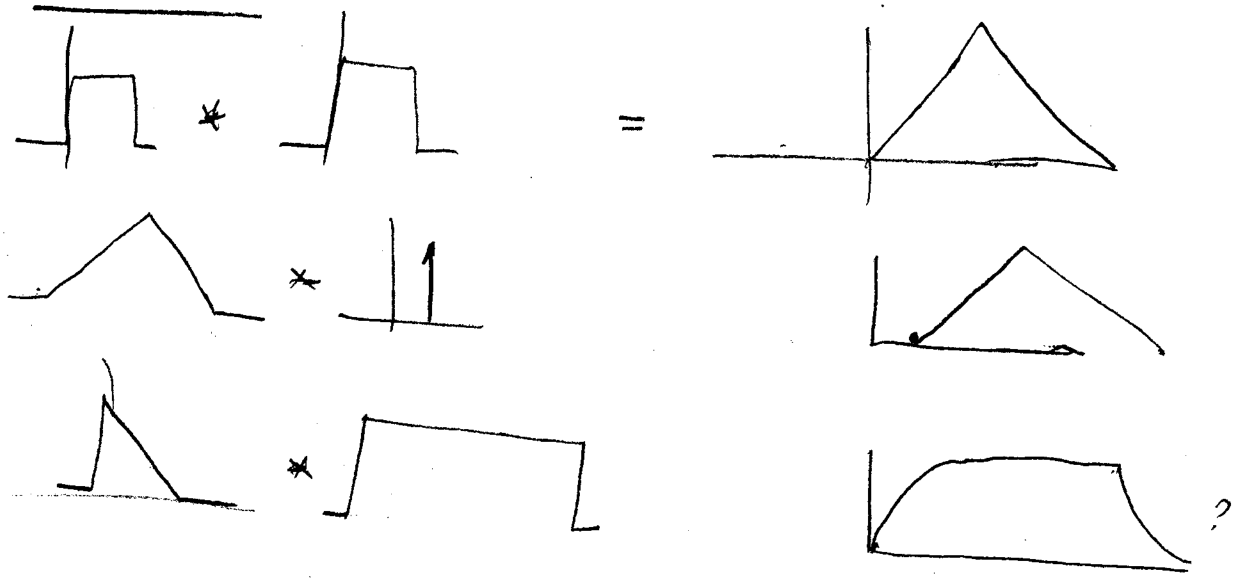


$$h(t) = \int_{-\infty}^{\infty} f(\tau)\, g(t-\tau)\, d\tau = f * g$$

Note! h(t) is smoother & more spread out than f(t).



Practice on:



6.3.1 The Convolution Theorem

We met this one at the very beginning. It is the reason why we are so interested in Fourier Transforms:

The Convolution Theorem states:

$$F_1(t) * F_2(t) \rightleftharpoons \Phi_1(\nu)\,\Phi_2(\nu)$$

In words: The F.T. of the convolution of two functions is just the <u>product</u> of the F.T. of the two functions.

$$\underbrace{\int_{-\infty}^{\infty} \underbrace{\int_{-\infty}^{\infty} F_1(\tau) F_2(t-\tau)\, d\tau}_{\text{a convolution}} \; e^{-2\pi i \nu t}\, dt}_{\text{The transform of a convolution}}$$

$$= \int_{-\infty}^{\infty} F_1(\tau) \Big[\underbrace{\int_{-\infty}^{\infty} F_2(t-\tau)\, e^{-2\pi i \nu (t-\tau)}\, dt}_{\Phi_2(\nu)} \Big] e^{-2\pi i \nu \tau}\, d\tau$$

replace $t-\tau$ with z
$dt = dz$

$$= \int_{-\infty}^{\infty} F_1(\tau)\, e^{-2\pi i \nu \tau}\, d\tau \cdot \Phi_2(\nu)$$

$$= \Phi_1(\nu)\, \Phi_2(\nu)$$

Q.E.D.

6.3.2. Examples of Convolutions

Let's see how we can manipulate functions using the Convolution Theorem to generate new transforms.

The δ-function

From the properties of the δ function

$$\int_{-\infty}^{\infty} F(\tau)\,\delta(\tau - a)\,d\tau = F(a)$$

Hence $\int_{-\infty}^{\infty} F(t-\tau)\,\delta(\tau - a)\,d\tau = F(t-a)$

which is just $F(t) * \delta(t-a) = F(t-a)$

Now, if $F(t) \rightleftharpoons \Phi(\nu)$ + $\delta(t-a) \rightleftharpoons e^{-2\pi i a}$ (we showed this twice before)

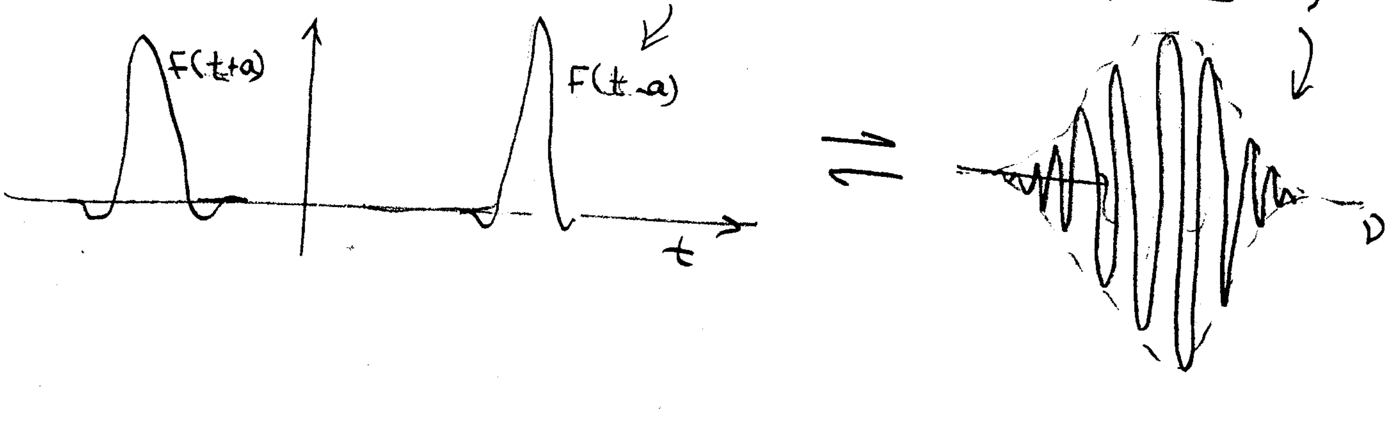
Then we immediately have:

$F(t-a) \rightleftharpoons \Phi(\nu) e^{-2\pi i a}$ from the convolution theorem.

Another example:

We already had $\delta(t-a) + \delta(t+a) \rightleftharpoons 2\cos\pi\nu a$

Thus $[\delta(t-a) + \delta(t+a)] * F(t) \rightleftharpoons 2\cos\pi\nu a\,\Phi(\nu)$



Gaussian example

We had already shown:

$$e^{-t^2/a^2} \rightleftharpoons a\sqrt{\pi}\, e^{-\pi^2\nu^2 a^2}$$

$$\therefore \quad e^{-t^2/a^2} * e^{-t^2/b^2} \rightleftharpoons ab\pi\, e^{-\pi^2\nu^2(a^2+b^2)}$$

This is certainly a lot easier than grinding through the calculus.

The R.H.S. is just another Gaussian, which we can transform back to t space by noting:

$$ab\pi\, e^{-\pi^2\nu^2(a^2+b^2)} = \frac{ab\sqrt{\pi}}{c} \cdot c\sqrt{\pi}\, e^{-\pi^2\nu^2 c^2}$$

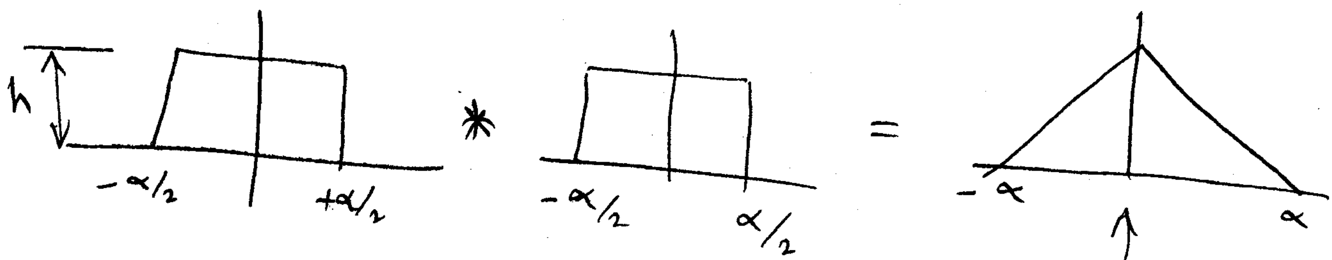
where $c = \sqrt{a^2+b^2}$

+ since $e^{-t^2/c^2} \rightleftharpoons c\sqrt{\pi}\, e^{-\pi^2\nu^2 c^2}$

we have $e^{-t^2/a^2} * e^{-t^2/b^2} = \dfrac{ab\sqrt{\pi}}{\sqrt{a^2+b^2}}\, e^{-t^2/(a^2+b^2)}$

ie the convolution of two Gaussians of width parameter a & b is another Gaussian of width parameter $\sqrt{a^2+b^2}$

Convolution of 2 'top-hat' functions



we did this graphically before to yield

The 'triangle-function' is denoted $\Lambda_a(t)$ with base $2a$

So $\quad h\Pi_a(t) * h\Pi_a(t) = h^2\Lambda_a(t)$

We can use this to get the F.T. of $\Lambda_a(t)$:

$$h\,\Pi_a(t) \rightleftharpoons ah\,\text{sinc}(\pi\nu a)$$

$$\therefore \quad h\Pi_a(t) * h\Pi_a(t) \rightleftharpoons a^2h^2\,\text{sinc}^2(\pi\nu a)$$

$$\therefore \quad h^2\Lambda_a(t) \rightleftharpoons a^2h^2\,\text{sinc}^2(\pi\nu a)$$

This is surely quicker and easier than direct integration.

The Autocorrelation Theorem

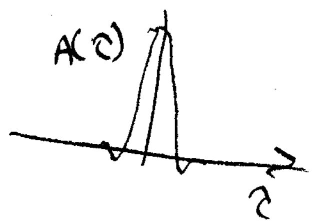
The autocorrelation function is defined as:

$$A(\tau) = \int_{-\infty}^{\infty} F(t)\,F(t+\tau)\,dt$$

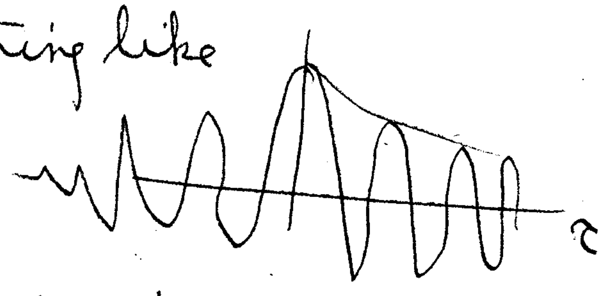
It is like a convolution but it, physically, is a measure of how much a given point of F, ie $F(t)$, is correlated to F at another time, $t+\tau$. τ is like a time delay.

If there is no correlation (ie a random signal), you'd get a δ-function

More likely, you'd get a bit of a spread for a real signal



If F was a 'narrow band' signal (ie contained frequencies that fell within a narrow band), you'd expect something like



(Just imagine how $F(t)$ relates to $F(t+\tau)$ if the signal were a pure sine wave).

If we take the Fourier Transform of $A(\tau)$ we get

$$\Gamma(\nu) = \int_{-\infty}^{\infty} A(\tau)\, e^{-2\pi i \nu \tau} d\tau$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(t)\, F(t+\tau)\, e^{-2\pi i \nu \tau} dt\, d\tau$$

$$= \int_{-\infty}^{\infty} F(t)\, e^{+2\pi i \nu t} dt \underbrace{\int_{-\infty}^{\infty} F(t+\tau)\, e^{-2\pi i \nu (t+\tau)} d\tau}_{\int_{-\infty}^{\infty} F(y)\, e^{-2\pi i \nu y} dy}$$

where $t+\tau \equiv y$

$$= \Phi^*(\nu)\ \Phi(\nu)$$

$$= |\Phi(\nu)|^2$$

(we showed that early on in our discussion of complex numbers)

Spectral Power Density

Thus the autocorrelation is (apart from a constant of proportionality) the F.T. of its power spectrum, $|\Phi(\nu)|^2$

6.4 The algebra of convolutions

$$[a(t) * b(t)] \rightleftharpoons [A(\nu) \cdot B(\nu)]$$

(we've shown this already)

But since the only difference in Fourier Transforms forward and back is a -ve sign in the exponent, we expect

$$[a(t) \cdot b(t)] \rightleftharpoons [A(\nu) * B(\nu)]$$

Thus

$$[a(t) * b(t)] \cdot [c(t) * d(t)] \rightleftharpoons [A(\nu) \cdot B(\nu)] * [C(\nu) \cdot D(\nu)]$$

Also, addition works in the traditional way:

$$[a(t) + b(t)] \rightleftharpoons [A(\nu) + B(\nu)]$$

So $[a(t) + b(t)] * [c(t) + d(t)] \rightleftharpoons [A(\nu) + B(\nu)] \cdot [C(\nu) + D(\nu)]$

Example

$$[a(t) * b(t) + c(t) \cdot d(t)] \cdot e(t) \rightleftharpoons [A(\nu) \cdot B(\nu) + C(\nu) * D(\nu)] * E(\nu)$$

easy!

6.5 Other theorems

6.5.1 The Derivative Theorem

If $F(t) \rightleftharpoons \Phi(\nu)$ then $\frac{dF}{dt} \rightleftharpoons +2\pi i \nu \, \Phi(\nu)$

Proof: F.T of $\frac{dF}{dt} = \int_{-\infty}^{\infty} \frac{dF(t)}{dt} e^{-2\pi i \nu t} dt$ (Ref ?

(I prefer this to the treatment in James)

$$= \lim_{\Delta t \to 0} \int_{-\infty}^{\infty} \frac{F(t+\Delta t) - F(t)}{\Delta t} e^{-2\pi i \nu t} dt$$

$$= \lim_{\Delta t \to 0} \int_{-\infty}^{\infty} \frac{F(t+\Delta t)\, e^{-2\pi i \nu t}}{\Delta t} dt - \lim_{\Delta t \to \infty} \int_{-\infty}^{\infty} \frac{F(t)}{\Delta t} e^{-2\pi i \nu t} dt$$

$$= \lim_{\Delta t \to 0} \int_{-\infty}^{\infty} \frac{F(t+\Delta t)\, e^{-2\pi i \nu (t+\Delta t)}}{\Delta t} e^{2\pi i \nu \Delta t} dt \quad - \quad ''$$

$$= \lim_{\Delta t \to 0} \left(\frac{\Phi(\nu)\, e^{2\pi i \nu \Delta t} - \Phi(\nu)}{\Delta t} \right) = \Phi(\nu) \left[\lim_{\Delta t \to \infty} \left(\frac{e^{2\pi i \nu \Delta t} - 1}{\Delta t} \right) \right]$$

Now $e^{2\pi i \nu \Delta t} = 1 + 2\pi i \nu \Delta t + \frac{(2\pi i \nu \Delta t)^2}{2} + \ldots$

$$\therefore \lim_{\Delta t \to 0} \left(\frac{e^{2\pi i \nu \Delta t} - 1}{\Delta t} \right) = 2\pi i \nu$$

Thus F.T. of $\frac{dF}{dt}$ is $\underline{2\pi i \nu \, \Phi(\nu)}$ Q.E.D.

By extension: $\frac{d^n F}{dt^n} \rightleftharpoons (2\pi i \nu)^n \, \Phi(\nu)$

This result is used in solving differential equations (a huge use of Fourier Transforms that we won't be exploring in this course).

Example: Simple harmonic motion:

$$m \frac{d^2F(t)}{dt^2} + kF(t) = 0$$

Taking F.T.:

$$m(2\pi i \nu)^2 \Phi(\nu) + k\Phi(\nu) = 0$$

$$\Rightarrow \left(\frac{k}{m} - 4\pi^2\nu^2\right) \Phi(\nu) = 0$$

& we see, without actually solving the equation that the frequency of the oscillator is $\nu = \pm 2\pi\sqrt{k/m}$.

6.5.2 The Convolution Derivative Theorem

$$\frac{d}{dt}\left[F_1(t) * F_2(t)\right] = F_1(t) * \frac{dF_2(t)}{dt} = \frac{dF_1(t)}{dt} * F_2(t)$$

Proof:

If we let $F_1(t) \rightleftharpoons \Phi_1(\nu)$

$F_2(t) \rightleftharpoons \Phi_2(\nu)$

then $\frac{d}{dt}\left[F_1(t) * F_2(t)\right] \rightleftharpoons (2\pi i\nu)\,\Phi_1(\nu)\cdot\Phi_2(\nu)$

$$\downarrow$$

$$\left[\frac{d}{dt}F_1(t)\right] * F_2(t) \longleftarrow \left(2\pi i\nu\,\Phi_1(\nu)\right)\cdot\Phi_2(\nu)$$

$$\downarrow$$

$$F_1(t) * \left[\frac{d}{dt}F_2 t\right] \longleftarrow \Phi_1(\nu)\cdot\left(2\pi i\nu\,\Phi_2(\nu)\right)$$

QED.

Here we have exploited the fact that convolutions transform to a product. So we do the transform to get rid of the messy convolution, do some simple algebra and then transform back to recover the convolutions.

Neat!

6.5.3 Parseval's / Rayleigh's / Power Theorem

$$\int_{-\infty}^{\infty} f(t)\, g^*(t)\, dt = \int_{-\infty}^{\infty} F(\nu)\, G^*(\nu)\, d\nu$$

$*$ denotes the complex conjugate.

Proof:

If $g(t) \rightleftharpoons G(\nu)$, ie $G(\nu) = \int_{-\infty}^{\infty} g(t) e^{-2\pi i \nu t} d\nu$

$+ \; g(t) = \int_{-\infty}^{\infty} G(\nu) e^{+2\pi i \nu t} dt$

then $G^*(\nu) = \int_{-\infty}^{\infty} g^*(t)\, e^{2\pi i \nu t}\, dt$ $+ \; g^*(t) = \int_{-\infty}^{\infty} G^*(\nu) e^{-2\pi i \nu t} dt$

(where we just took the complex conjugate of everything, ie $i \to -i$)

Thus

$$\int_{-\infty}^{\infty} f(t)\, g^*(t)\, dt = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} F(\nu)\, e^{+2\pi i \nu t} d\nu \right] \cdot g^*(t)\, dt$$

$$= \int_{-\infty}^{\infty} F(\nu) \underbrace{\int_{-\infty}^{\infty} g^*(t)\, e^{+2\pi i \nu t} dt}_{G^*(\nu)} \; d\nu$$

$$= \int_{-\infty}^{\infty} F(\nu)\, G^*(\nu)\, d\nu$$

QED

When $f = g$

$$\int_{-\infty}^{\infty} f(t)\, f^*(t)\, dt = \int_{-\infty}^{\infty} F(\nu)\, F^*(\nu)\, d\nu$$

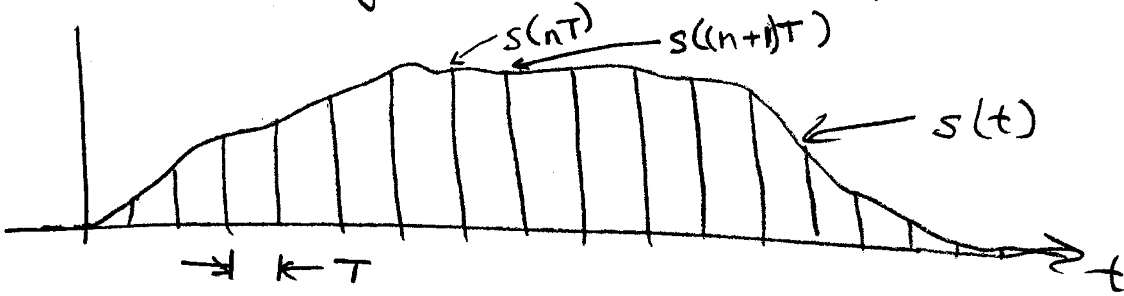
ie $$\int_{-\infty}^{\infty} |f(t)|^2\, dt = \int_{-\infty}^{\infty} |F(\nu)|^2\, d\nu$$ (Rayleigh's or Power Theorem)

Physically, we can interpret this as <u>the</u> total Power in the time domain
$=$ " " " freq. "

6.5.4 The Sampling Theorem

Fante does a better treatment of this topic than James so I'll follow Fante, Section 4.2, pp 128–131

Let's take a signal $s(t)$ and sample it every T seconds



The sampled signal, $\hat{s}(t)$ is just $s(t)$ multiplied by a train of δ-functions:

$$\hat{s}(t) = s(t) \sum_{n=-\infty}^{\infty}{}' \delta(t-nT)$$

(We reflect the signal to -ve n to make it symmetric, as usual).

The original signal, $s(t)$ has a Fourier Transform $S(\nu)$.

Let's take the F.T. of the sample, $\hat{s}(t)$:

$$\hat{S}(\nu) = \sum_{n=-\infty}^{\infty}{}' \int_{-\infty}^{\infty} s(t)\,\delta(t-nT)\, e^{-2\pi i\nu t}\, dt$$

$$= \sum_{n=-\infty}^{\infty}{}' s(nT)\, e^{-2\pi i\nu nT}$$

We can ~~show that~~ this is periodic as follows:

$$\hat{S}\left(\nu+\frac{m}{T}\right) = \sum_{n=-\infty}^{\infty}{}' s(nT)\, e^{-2\pi i\nu nT \underbrace{- 2\pi i m}_{\text{a multiple of } 2\pi}}$$

$$= \sum_{n=-\infty}^{\infty}{}' s(nT)\, e^{-2\pi i\nu nT}$$

$$= \hat{S}(\nu).$$

Now, how does $\hat{S}(\nu)$ relate to $S(\nu)$?

Recall $\quad s(t) = \int_{-\infty}^{\infty} S(\nu')\, e^{2\pi i\nu' t}\, d\nu' \quad$ (def^n of F.T)

$$\therefore \quad s(nT) = \int_{-\infty}^{\infty} S(\nu')\, e^{2\pi i\nu' nT}\, d\nu'$$

$$\therefore \quad \hat{S}(\nu) = \sum_{n=-\infty}^{\infty}{}' s(nT)\, e^{-2\pi i\nu nT}$$

$$= \sum_{n=-\infty}^{\infty}{}' \int_{-\infty}^{\infty} S(\nu')\, e^{2\pi i\nu' nT}\, d\nu'\, e^{-2\pi i\nu nT}$$

$$= \int_{-\infty}^{\infty} S(\nu') \sum_{n=-\infty}^{\infty}{}' e^{-2\pi i(\nu-\nu')nT}\, d\nu'$$

Now, it can be shown ~~that~~: $\displaystyle\sum_{n=-\infty}^{\infty} \delta(\nu - n/T) = \frac{1}{T}\sum_{n=-\infty}^{\infty}{}' e^{2\pi i n\nu T}$

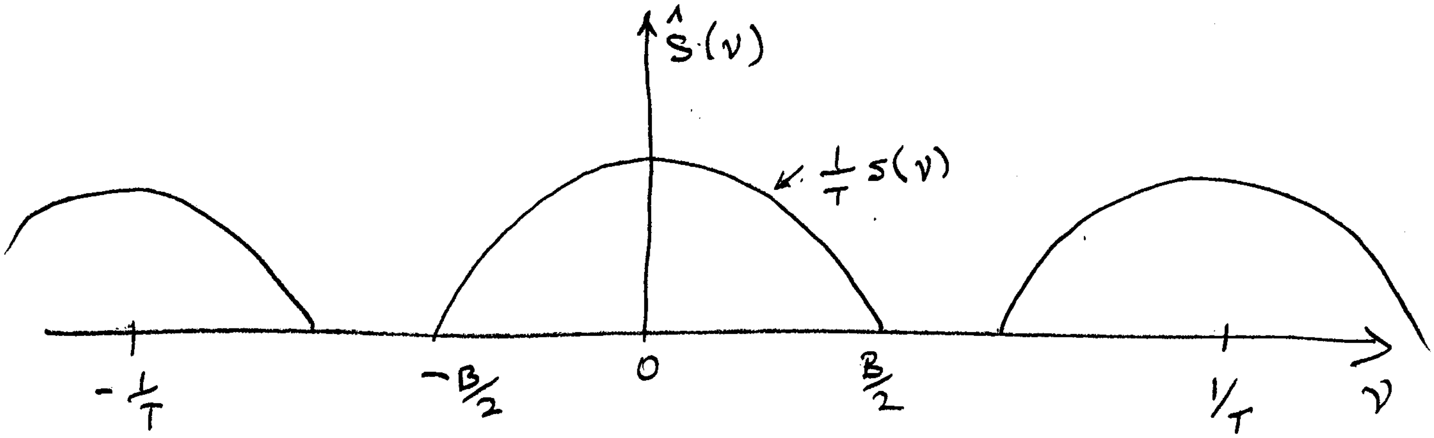
So we have:

$$\hat{S}(\nu) = \int_{-\infty}^{\infty} S(\nu')\, \frac{1}{T}\sum_{n=-\infty}^{\infty}{}' \delta\left(\nu' - \nu - \frac{n}{T}\right) d\nu'$$

$$= \frac{1}{T}\sum_{n=-\infty}^{\infty}{}' \int_{-\infty}^{\infty} S(\nu')\, \delta(\nu' - \nu - n/T)\, d\nu'$$

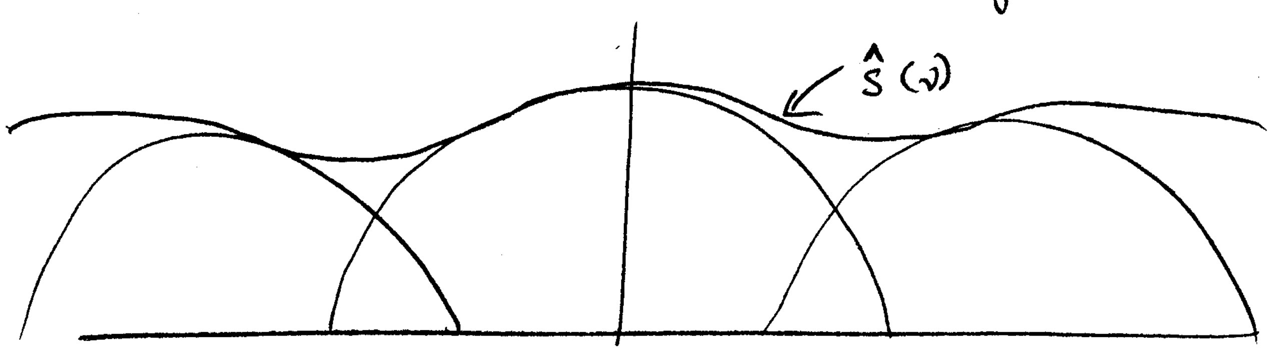
$$= \frac{1}{T}\sum_{n=-\infty}^{\infty}{}' S(\nu + n/T) = \frac{1}{T}\sum_{m=-\infty}^{\infty}{}' S(\nu - m/T) \quad \text{if you prefer}$$

So $\hat{S}(\nu)$ is just $S(\nu)$ scaled by $1/T$ and repeated with a frequency shift $\frac{n}{T}$ where $n = 0, \pm 1, \pm 2 \ldots\ldots$



In the above graph, the sampling rate, $1/T$, is chosen to be bigger than the bandwidth, B, of $S(\nu)$. This is called <u>oversampling</u>.

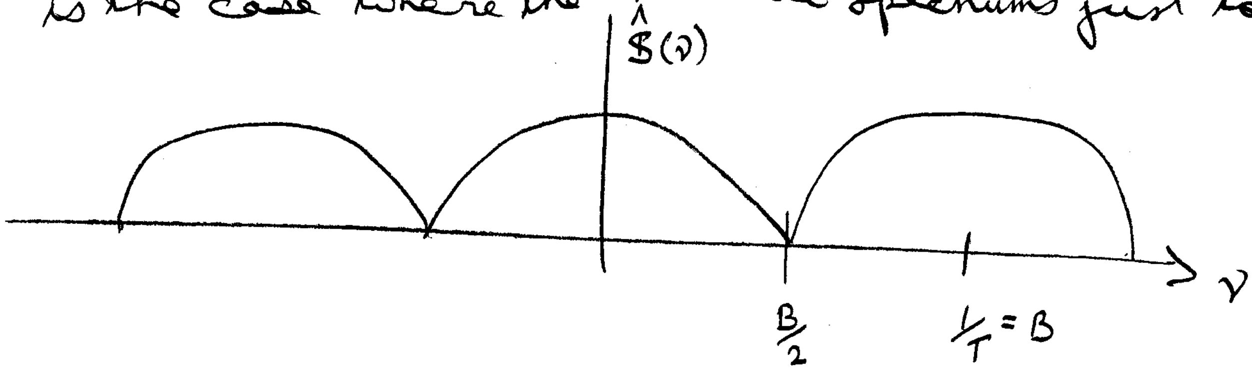
So you can see that as long as $\frac{1}{T} \geq B$, then we can reconstruct $S(\nu)$ from $\hat{S}(\nu)$, ie the periodic parts of $\hat{S}(\nu)$ don't overlap and distort $\hat{S}(\nu)$. If $\frac{1}{T} < B$, we'd get



and we couldn't get $S(\nu)$ out of $\hat{S}(\nu)$.
This case is called undersampled and the signal "foldsback" on itself.

This distortion of $\hat{S}(\nu)$ when $\frac{1}{T} < B$ is called <u>aliasing</u>.

The case of $\frac{1}{T} = B$ is called the <u>critically sampled</u> rate or the Nyquist sampling rate. It is the case where the periodic spectrums just touch.



We see that we must sample at <u>twice</u> the highest frequency in the signal ($\frac{B}{2}$).

To avoid aliasing we can either increase our sampling rate or filter S so that it contains no frequencies $> \frac{1}{2T}$, eg if we sample once per millisecond ($\nu = 1$ kHz), then the signal should not have any significant power in frequency components above 500 Hz.

(James 2.6 + 2.6.1 omitted)

6.6 The Similarity Theorem

If $f(t) \rightleftharpoons F(\nu)$

then $f(at) \rightleftharpoons \left|\frac{1}{a}\right| F(\nu/a)$

Proof:

$$F(\nu) = \int_{-\infty}^{\infty} f(t)\, e^{2\pi i \nu t}\, d\nu$$

if $a > 0$

$$\text{F.T. of } f(at) = \int_{-\infty}^{\infty} f(at)\, e^{2\pi i \nu t}\, dt$$

$$= \int_{-\infty}^{\infty} f(t')\, e^{2\pi i \frac{\nu}{a} t'}\, dt \quad (\text{where } t' = at)$$

$$= \frac{1}{a} \int_{-\infty}^{\infty} f(t')\, e^{2\pi i \nu' t'}\, dt' \quad (\text{where } \nu' = \nu/a)$$

$$= \frac{1}{a} F(\nu') = \frac{1}{a} F(\nu/a)$$

if $a < 0$, then $t' = at$ ($= -3t$, say, to illustrate)

$$\text{\& F.T. of } f(at) = \int_{\infty}^{-\infty} f(t')\, e^{-2\pi i \frac{\nu}{3} t'}\, dt'/(-3)$$

$$= \frac{1}{3} \int_{-\infty}^{\infty} f(t')\, e^{+2\pi i \nu' t'}\, dt' \quad \left(\text{where } \nu' = -\frac{\nu}{3} = \frac{\nu}{a}\right)$$

$$= \frac{1}{|a|} F(\nu/a)$$

$$\therefore f(at) \rightleftharpoons \left|\frac{1}{a}\right| F(\nu/a)$$

Note: Stretching the time scale shortens the freq. scale

6.7 Recap

Addition: $f_1(t) + f_2(t) \rightleftharpoons F_1(\nu) + F_2(\nu)$

Shift: $f_1(t+a) \rightleftharpoons F_1(\nu)\, e^{2\pi i \nu a}$

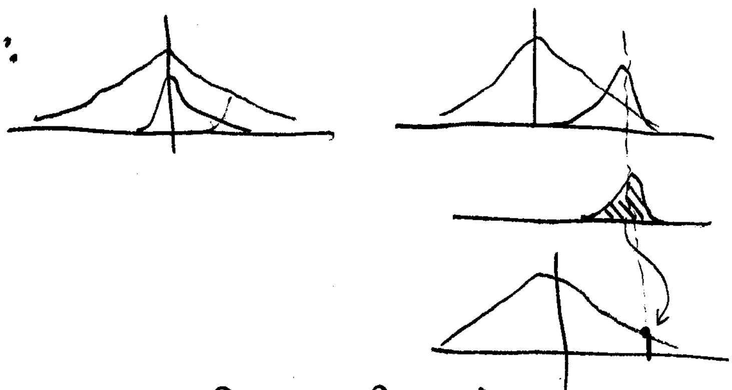
Convolution: $y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau \equiv x * h$

Commutative: $f_1(t) * f_2(t) = f_2(t) * f_1(t)$

Distributive: $f_1(t) * [f_2(t) + f_3(t)] = f_1(t) * f_2(t) + f_1(t) * f_3(t)$

Associative: $f_1(t) * [f_2(t) * f_3(t)] = [f_1(t) * f_2(t)] * f_3(t)$

Visualization:



Convolution Theorem: $f_1(t) * f_2(t) \rightleftharpoons F_1(\nu)\, F_2(\nu)$

Examples:

$$f(t) * \delta(t-a) = f(t-a) \rightleftharpoons F(\nu)\, e^{-2\pi i a}$$

$$[\delta(t-a) + \delta(t+a)] * f(t) \rightleftharpoons 2 \cos \pi \nu a\, F(\nu)$$

$$e^{-t^2/a^2} * e^{-t^2/b^2} \rightleftharpoons ab\pi\, e^{-\pi^2 \nu^2 (a^2+b^2)}$$

$$\Pi_a(t) * \Pi_a(t) = \Lambda_a(t) \rightleftharpoons a^2 \operatorname{sinc}^2(\pi \nu a)$$

Autocorrelation: $A(\tau) = \int_{-\infty}^{\infty} f(t)\, f(t+\tau)\, dt$

$$\text{F.T.}(A(\tau)) = F^*(\nu)\, F(\nu) = |F(\nu)|^2$$

$a(t) * b(t) \rightleftharpoons A(\nu)\, B(\nu)$ + $a(t)\, b(t) \rightleftharpoons A(\nu) * B(\nu)$

$$[a(t) * b(t)] \cdot [c(t) * d(t)] \rightleftharpoons [A(\nu) \cdot B(\nu)] * [C(\nu) \cdot D(\nu)]$$

$$\frac{d^n f}{dt^n} \rightleftharpoons (2\pi i \nu)^n F(\nu)$$

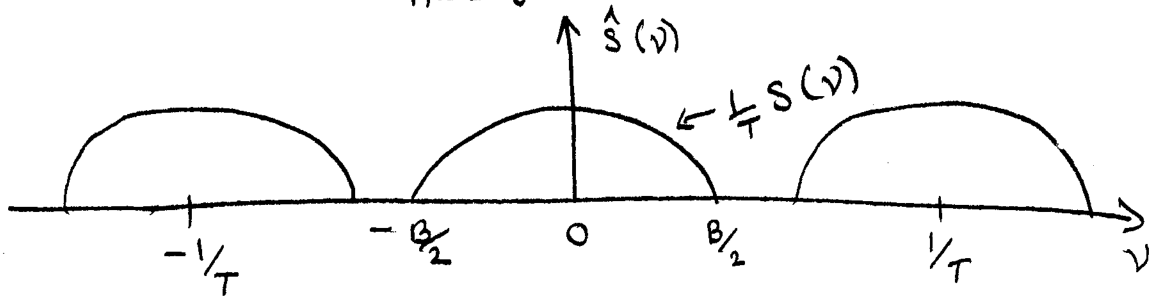
$$\frac{d}{dt}[f_1(t) * F_2(t)] = f_1(t) * \frac{d f_2(t)}{dt} = \frac{d f_1(t)}{dt} * F_2(t)$$

Power Theorem: $\int_{-\infty}^{\infty} f(t)\, g^*(t)\, dt = \int_{-\infty}^{\infty} F(\nu)^* G(\nu)\, d\nu$

$+ \int_{-\infty}^{\infty} f(t)\, f^*(t)\, dt = \int_{-\infty}^{\infty} |f(t)|^2 dt = \int_{-\infty}^{\infty} |F(\nu)|^2 d\nu$

Sampling:

$$\hat{S}(\nu) = \frac{1}{T} \sum_{n=-\infty}^{\infty} S(\nu + n/T)$$



$1/T > B$ oversampled

$= B$ critically sampled

$< B$ undersampled $\longrightarrow$ aliasing (distortion)

Need to sample at 2x highest frequency ($B/2$).

Similarity: $f(at) \rightleftharpoons \left|\frac{1}{a}\right| F(\nu/a)$