3. Fourier Series
(following James 1.1 → 1.3)

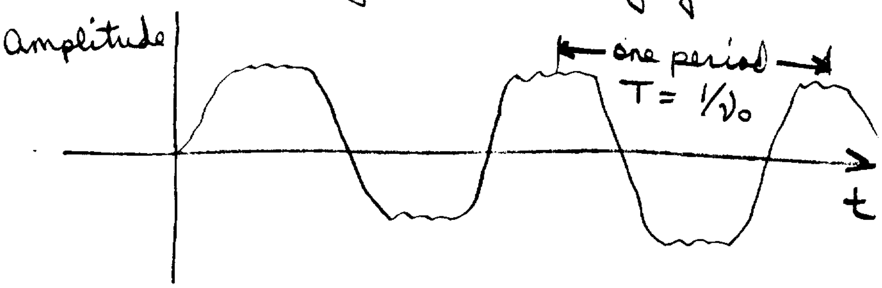
3.1 The Qualitative Approach

We first look at periodic signals (waves) and, in a later chapter, non-periodic signals (ie collections of impulses).

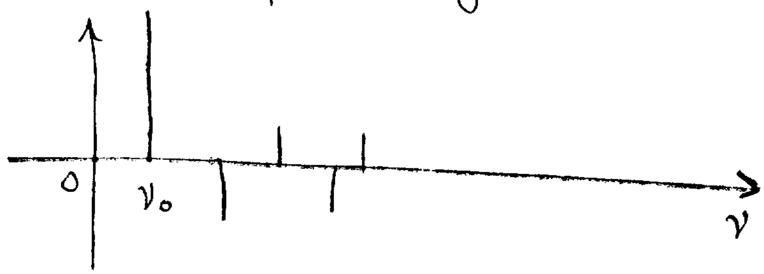
Periodic signals ⟶ Use Fourier Series

Non-periodic " ⟶ Use Fourier Transforms.

Take a steady note - say from a violin:



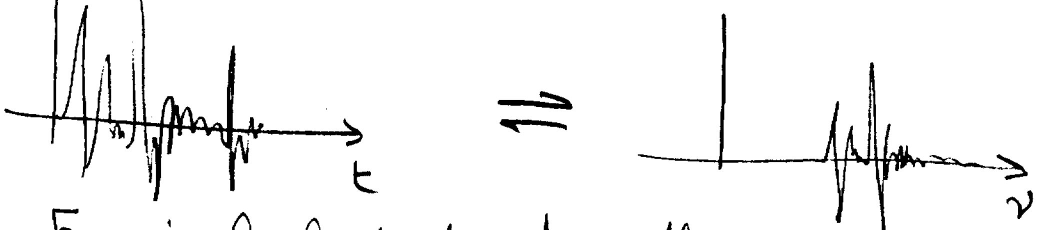
This is composed of a number of frequencies:



$$\nu = \frac{1}{T} = \frac{\omega}{2\pi} \ \sec^{-1}$$

$$\omega = 2\pi\nu \text{ radians/sec.}$$

Or take a non-periodic signal - like a crash:



We use Fourier Analysis to get another view (besides the time behaviour) of a signal. This is useful in studying defects (mechanical, electrical, biological .... any system).

We'll study the Fourier Series first since you are probably somewhat familiar with it & later we show how it leads to the Fourier Transform.

3.2 Fourier Series (following James 1.2)

We can approximate any periodic signal as:

$$f(t) = \sum_{n=-\infty}^{\infty} \left[a_n \cos(2\pi n \nu_0 t) + b_n \sin(2\pi n \nu_0 t) \right] \quad (3.1)$$

an even function (ie $\cos(x) = \cos(-x)$)

ν_0 = fundamental period

an odd function (ie $\sin(x) = -\sin(-x)$)

We can also write it as:

$$f(t) = a_0 + \sum_{n=1}^{\infty} \left[a_n \cos(2\pi n \nu_0 t) + b_n \sin(2\pi n \nu_0 t) \right]$$

$$+ \sum_{n=-1}^{-\infty} \left[a_n \cos(2\pi n \nu_0 t) + b_n \sin(2\pi n \nu_0 t) \right]$$

[note: b_0 does not contibute since $\sin(0) = 0$]

$$= \frac{A_0}{2} + \sum_{n=1}^{\infty} \left[A_n \cos(2\pi n \nu_0 t) + B_n \sin(2\pi n \nu_0 t) \right] \quad (3.2)$$

where $A_n = a_n + a_{-n}$ and $B_n = b_n - b_{-n}$

(we divide A_0 by 2 so we can use the same formula for A_0 as for A_n)

We can synthesize (ie put together) a periodic signal by combining harmonics.

3.3 The Amplitudes of the Harmonics
(following James 1.3)

If we are given f(t), we want to extract the A's + B's to find the relative (and absolute strengths) of the various frequency components. This is Fourier Analysis (breaking apart).

We use orthogonality:

$$\int_T \cos(2\pi n \nu_0 t)\cos(2\pi m \nu_0 t)\,dt = 0 \quad \text{if } n \neq m$$

T ← any complete cycle

$$= \frac{1}{2\nu_0} \quad \text{if } n = m$$

$$\int_T \sin(2\pi n \nu_0 t)\sin(2\pi m \nu_0 t)\,dt = 0 \quad \text{if } n \neq m$$

$$= \frac{1}{2\nu_0} \quad \text{if } n = m$$

Thus, multiplying 3.2 by $\sin(2\pi m \nu_0 t)$:

$$\int_T f(t)\sin(2\pi m \nu_0 t)\,dt = \int_T \sum_{n=1}^{\infty} A_n \cos(2\pi n \nu_0 t)\sin(2\pi m \nu_0 t)\,dt \rightarrow 0 \text{ for any } n + m.$$

$$+ \int_T \sum_{n=1}^{\infty} B_n \sin(2\pi n \nu_0 t)\sin(2\pi \nu_0 m t)\,dt \rightarrow \frac{B_m T}{2} = \frac{B_m}{2\nu_0}$$

$$+ \frac{A_0}{2}\int_T \sin(2\pi m \nu_0 t)\,dt \rightarrow 0$$

$$\therefore B_m = \frac{2}{T}\int_T f(t)\sin(2\pi m \nu_0 t)\,dt$$

+ multiplying 3.2 by $\cos(2\pi m \nu_0 t)$ gives:

$$\int_T f(t)\cos(2\pi m\nu_0 t)\,dt = \int_T \sum_{n=1}^{\infty} A_n \underbrace{\cos(2\pi n\nu_0 t)\cos(2\pi m\nu_0 t)\,dt}_{\to\ \frac{A_m T}{2} = A_m/2\nu_0}$$

$$+ \int_T \sum_{n=1}^{\infty} B_n \underbrace{\sin(2\pi n\nu_0 t)\cos(2\pi m\nu_0 t)\,dt}_{\to\ 0}$$

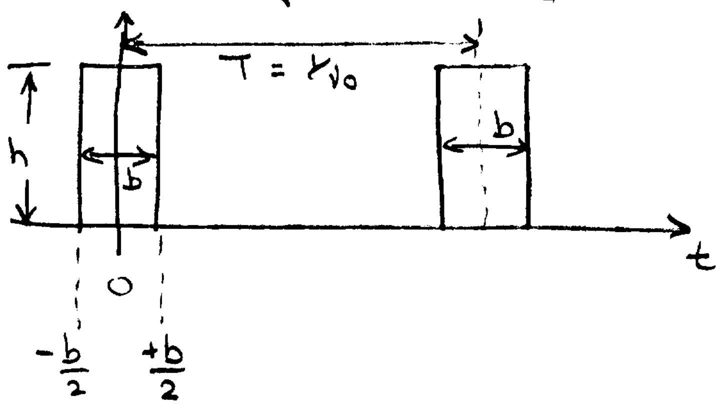
$$+ \frac{A_0}{2}\underbrace{\int_T \cos(2\pi m\nu_0 t)\,dt}_{\to\ 0 \text{ for } m\neq 0}$$

$$\therefore\ A_m = \frac{2}{T}\int_T f(t)\cos(2\pi m\nu_0 t)\,dt$$

When $m=0$, $A_0 = \frac{2}{T}\int_T f(t)\,dt = 2\times$ average of $f(t)$ over a period

$$\therefore\ \frac{A_0}{2} = \text{average over a period}$$
$$= \text{D.C. component}$$

Example: Square Wave



$$A_m = 2\nu_0 \int_{-\frac{1}{2}\nu_0}^{+\frac{1}{2}\nu_0} f(t)\cos(2\pi m \nu_0 t)\,dt$$

$$= 2h\nu_0 \int_{-b/2}^{b/2} \cos(2\pi m \nu_0 t)\,dt$$

$$= \frac{2h\nu_0}{2\pi m \nu_0}\left\{ \sin(\pi m \nu_0 b) - \sin(-\pi m \nu_0 b)\right\}$$

$$= \frac{2h}{\pi m}\sin(\pi m \nu_0 b) \quad \text{since } \sin(x) = -\sin(-x)$$

$$= 2h\nu_0 b \cdot \frac{\sin(\pi m \nu_0 b)}{\pi m \nu_0 b}$$

← $\frac{\sin x}{x}$ in form.

$$B_m = 2\nu_0 \int_{-b/2}^{+b/2} \sin(2\pi m \nu_0 t)\,dt = 0$$

$$A_0 = 2\nu_0 \int_{-b/2}^{+b/2} h\,dt = 2\nu_0 b h$$

$$\therefore f(t) = \underbrace{hb\nu_0}_{=\frac{hb}{T}} + \underbrace{2hb\nu_0}_{\frac{2hb}{T}} \sum_{m=1}^{\infty} \frac{\sin(\pi \nu_0 m b)}{\pi \nu_0 m b}\cos(2\pi m \nu_0 t)$$

↑ average height

↑ fit gets better as more terms are added.

Aside:

$$\frac{\sin x}{x} \equiv \operatorname{sinc} x \quad (\text{pronounced 'sink'})$$

and $\frac{\sin 0}{0} = \operatorname{sinc} 0 = 1$ by De l'Hôpital's Rule

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We can also write:

$$f(t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} R_n \cos(2\pi n \nu_0 t + \phi_n)$$

$\nwarrow$ $A_n = R_n \cos\phi_n$

$B_n = R_n \sin\phi_n$

$R_n = \sqrt{A_n^2 + B_n^2}$ + $\phi_n = \tan^{-1}(-A_n/B_n)$

This notation is not used much.

This arises by noting ~~that~~:

$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$

$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$

Using this notation we can think of the periodic function as a collection of cos waves with varying phase shifts and amplitudes.

Complex exponential notation:

We can use sin and cos but manipulation is easier if we make use of the identity:

$$e^{i\Theta} = \cos\Theta + i\sin\Theta$$ [I'll prove this in the next chapter]

The complex conjugate is:

$$e^{-i\Theta} = \cos\Theta - i\sin\Theta$$

The exponential notation also leads to Fourier Transforms (our ultimate goal).

From the above: $\cos\Theta = \dfrac{e^{i\Theta}+e^{-i\Theta}}{2}$, $\sin\Theta = \dfrac{e^{i\Theta}-e^{-i\Theta}}{2i}$

If we define $\Theta = 2\pi\nu_0 t$ we have:

$$f(t) = \frac{A_0}{2} + \sum_{m=1}^{\infty}\left(A_m\cos m\Theta + B_m\sin m\Theta\right)$$

$$= \frac{A_0}{2} + \sum_{m=1}^{\infty}\frac{A_m}{2}\left(e^{im\Theta}+e^{-im\Theta}\right) + \sum_{m=1}^{\infty}\frac{B_m}{2i}\left(e^{im\Theta}-e^{-im\Theta}\right)$$

$$= \frac{A_0}{2} + \sum_{m=1}^{\infty}\underbrace{\left(\frac{A_m}{2}+\frac{B_m}{2i}\right)}_{=\frac{A_m}{2}-\frac{iB_m}{2}\ \equiv\ D_m}e^{im\Theta} + \sum_{m=1}^{\infty}\underbrace{\left(\frac{A_m}{2}-\frac{B_m}{2i}\right)}_{=\frac{A_m}{2}+\frac{iB_m}{2}\ \equiv\ D_{-m}}e^{-im\Theta}$$

$$= \sum_{m=-\infty}^{\infty} D_m e^{im\Theta}$$

(Recall: $B_0 = 0$)

Note:

- Since $f(t)$ is real and both $a_m \cos m\Theta + b_m \sin m\Theta$ (or equally $A_m \cos m\Theta + B_m \sin m\Theta$) are real, then $D_m e^{im\Theta} + D_{-m} e^{-im\Theta}$ must be real even though D_{-m}, D_m and $e^{im\Theta}$ are individually complex.

From the definition of D_m and D_{-m} we have:

$$D_m = \frac{1}{T}\int_T f(t)\cos(2\pi m\nu t)\,dt - i\frac{1}{T}\int_T f(t)\sin(2\pi m\nu_0 t)\,dt$$

$$= \frac{1}{T}\int_T f(t)\,e^{-2\pi i m\nu_0 t}\,dt$$

This is handy.

- Note: James, Appendix 1.4 says

$$f(t) = \sum_{m=-\infty}^{\infty} e^{im\Theta}\left\{\frac{A_m - iB_m}{2}\right\} = \frac{A_0}{2} + \sum_{m=1}^{\infty} e^{im\Theta}(A_m - iB_m)$$

$$= \frac{A_0}{2} + \sum_{m=1}^{\infty} C_m e^{im\Theta}$$

I suspect this is a false claim. Beware.

Proof: $$\sum_{m=-\infty}^{\infty} e^{im\Theta}\left\{\frac{A_m - iB_m}{2}\right\} = \sum_{m=1}^{\infty} e^{im\Theta}\left\{\frac{A_m - iB_m}{2}\right\} \quad (A)$$

$$+ \sum_{m=-1}^{-\infty} e^{im\Theta}\left\{\frac{A_m - iB_m}{2}\right\} + \frac{A_0}{2}$$

$$= \sum_{m=1}^{\infty} e^{-im\Theta}\left\{\frac{A_{-m} - iB_{-m}}{2}\right\} = \sum_{m=1}^{\infty} e^{-im\Theta}\left\{\frac{A_m + iB_m}{2}\right\}$$

- $$= \sum_{m=1}^{\infty}\{\cos(m\Theta) - i\sin m\Theta\}\left\{\frac{A_m + iB_m}{2}\right\}$$

↑ need to show that this is the same as (A). I haven't been able to show this.

3.4 Recap

Fourier Analysis gives the frequency content of a signal.

$$f(t) = \sum_{n=-\infty}^{\infty}{}' [a_n \cos(2\pi n \nu_0 t) + b_0 \sin(2\pi n \nu_0 t)]$$

$$= \frac{A_0}{2} + \sum_{n=1}^{\infty}{}' [A_n \cos(2\pi n \nu_0 t) + B_n \sin(2\pi n \nu_0 t)]$$

$$= \frac{A_0}{2} + \sum_{n=1}^{\infty}{}' R_n \cos(2\pi n \nu_0 t + \phi_n)$$

$$= \sum_{n=-\infty}^{\infty}{}' D_n e^{in\theta} \quad , \quad \theta = 2\pi \nu_0 t$$

$$A_n = \frac{2}{T} \int_T f(t) \cos(2\pi n \nu_0)\, dt \qquad A_0 = \text{D.C. component}$$

$$B_n = \frac{2}{T} \int_T f(t) \sin(2\pi n \nu_0)\, dt$$

$$R_n = \sqrt{A_n^2 + B_n^2}$$

$$\phi_n = \tan^{-1}(A_{-n}/B_n)$$

$$D_n = \frac{1}{T} \int_T f(t) e^{-2\pi i n \nu_0 t}\, dt$$