1 PRINCIPLES OF HEAT TRANSFER

This chapter is intended to discuss different energy transport mechanisms which are usually classified as conduction, convection and radiation. From the second law of thermodynamics we know that the heat flows whenever there is a temperature difference, i.e., temperature gradient. The knowledge of the temperature distribution is essential to evaluate the heat flow. The temperature distribution and the heat flow constitute two basic elements in the design of thermal equipments such as boilers, heat exchangers, nuclear reactor cores, etc. Since in nuclear reactors, under normal operating conditions, radiation heat transfer has limited application, the present discussion will be mainly focused on conduction and convection heat transfers.

1.1 Mechanisms of Heat Transfer

I. Conduction

The conduction is defined as the transfer of energy from one point of a medium to an other under the influence of temperature differences. On the elementary particle level, the conduction is visualized as the exchange of kinetic energy between the particles in high and low temperature regions. Therefore, the conduction is attributed to the elastic collisions of molecules in gases and liquids, to the motion of free electrons in metals, and to the longitudinal oscillation of atoms in solid insulators of electricity. A distinguishing characteristic of conduction is that it takes place within the boundary of a medium, or across the boundary of a medium into an other medium in contact with the first, without an appreciable displacement of the matter.

On the microscopic level, the physical mechanisms of conduction are complex. Fortunately, we will consider the conduction heat transfer at a macroscopic level and use a phenomenological law based on experiments made Biot and formulated J.B. Fourier in 1882. This law can be illustrated by considering a simple case, a wall of thickness L, surface area A and whose faces are kept at temperatures t_1 and t_2 as shown in Fig. 3.1. t_1 is greater than t_2. Under these conditions, heat flows from the face of high temperature to the face of low temperature. According to Fourier's law of heat conduction, the rate of heat transfer in the x-direction through the wall element, dx, located at x is proportional to:

- the gradient of temperature in that direction, dt/dx, and
- to the surface area normal to the direction of heat transfer, A.

Therefore, the heat transfer rate is given by:

$$q_x = -kA\frac{dt}{dx} \qquad (3.1)$$

where k is the constant of proportionality and it is called the "thermal conductivity"; it is a property of the material. The minus sign appearing in Eq. 3.1 is due to the convention that the heat is taken to be positive in the direction of increasing x and also ensures that heat flows in the direction of decreasing temperature, thus satisfies the second law of thermodynamics. As sketched in Fig. 3.2a, if the temperature decreases with increasing x, then the gradient is a negative quantity

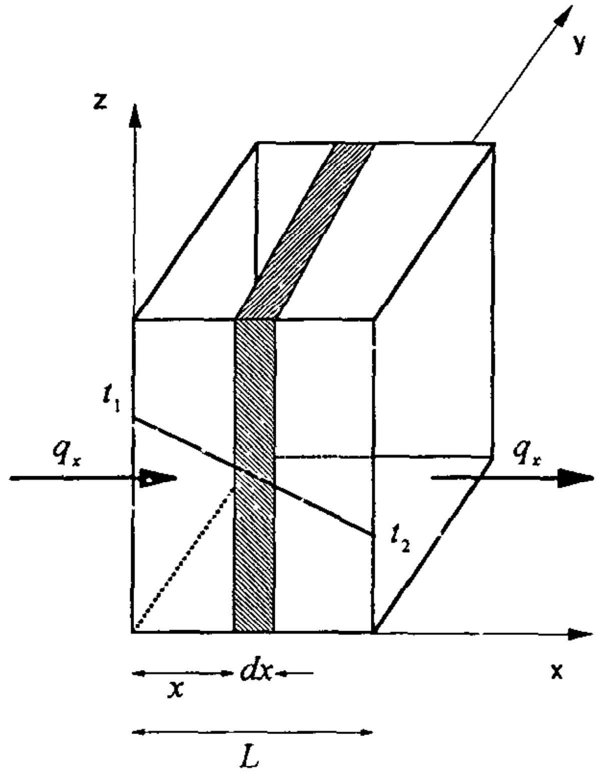


Figure 3.1 Heat flow across a plane wall.

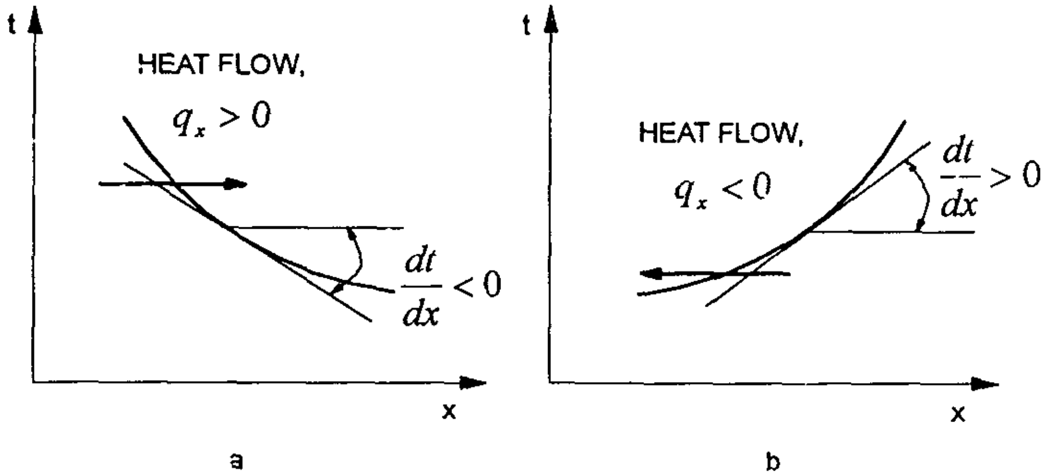


Figure 3.2 Sign convention for the direction of heat in the Fourier law.

and the minus sign of Eq. 3.1 ensures that q_x is positive. Conversely, if the temperature increases with increasing x (Fig. 3.2b) the gradient is positive and q_x is negative. In either case heat flows in the direction of decreasing temperature.

Dividing both sides of Eq. 3.1 we obtain:

$$q_x'' = \frac{q_x}{A} = -k\frac{dt}{dx} \tag{3.2}$$

where the quantity q''_x is called the *heat flux*. The dimension of heat flow rate is energy per unit time, i.e., J/s whereas the dimension of *dt* and *dx* are Kelvin (K) or degree Celsius (°C) and meter (m), respectively. Consequently, the unit of thermal conductivity is:

$$k \; : \; \frac{J}{s}\frac{1}{mK} \text{ or } \frac{W}{mK}$$

and the unit of heat flux is:

$$q''_x \; : \; \frac{J}{sm^2} \text{ or } \frac{W}{m^2} .$$

Assuming a linear temperature variation in the wall illustrated in Fig. 3.1, Eq. 3.1 can be easily integrated:

$$\int_0^L \frac{q_x}{A} dx = -\int_{t_1}^{t_2} k dt \qquad (3.3)$$

to obtain:

$$q''_x L = -k(t_2 - t_1) \qquad (3.4)$$

or

$$q''_x = k\frac{t_1 - t_2}{L} . \qquad (3.5)$$

Since $t_1 > t_2$, q''_x is a positive quantity. Therefore, it is in the positive direction. If $t_2 > t_1$, then q''_x would be negative and heat flow would be in the negative *x* direction.

Eq. 3.1 or 3.2 give one dimensional form of Fourier's law of heat conduction. In general, the temperature in a body may vary in all three coordinate directions, i.e.,

$$t = t(x, y, z, \tau) \qquad (3.6)$$

where τ is the time. Therefore, the general form of Fourier's law is:

$$\vec{q''} = -k \vec{\nabla} t \qquad (3.7)$$

where $\vec{q''}$ is the conduction heat flux vector and $\vec{\nabla}$ is the gradient of the scalar temperature field. According to Fig. 3.3, $\vec{q''}$ can be written as:

$$\vec{q''} = q''_x \vec{i} + q''_y \vec{j} + q''_z \vec{k} \qquad (3.8)$$

and $-k\vec{\nabla}$ as:

$$-k \vec{\nabla} t = -k \vec{i} \frac{\partial t}{\partial x} - k \vec{j} \frac{\partial t}{\partial y} - k \vec{k} \frac{\partial t}{\partial z} \qquad (3.9)$$

Comparing equations 3.8 and 3.9 we conclude that:

$$q_x'' = -k\frac{\partial t}{\partial x}; \quad q_y'' = -k\frac{\partial t}{\partial y}; \quad q_z'' = -k\frac{\partial t}{\partial z} \tag{3.10}$$

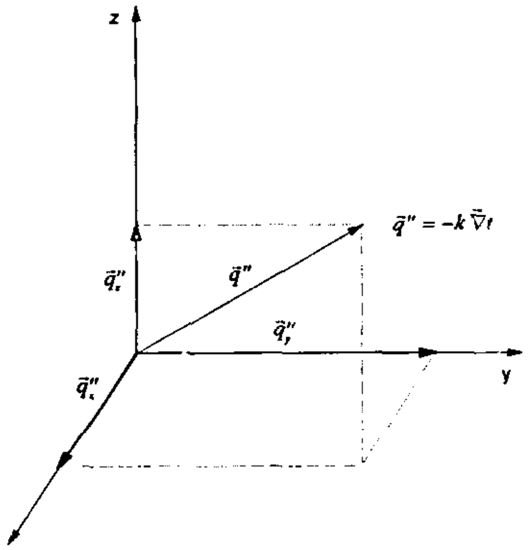


Figure 3.3 Three components of the heat flux.

$\vec{i}$, $\vec{j}$ and $\vec{k}$ are unit vectors in the x, y and z directions. In the above discussion, the medium is assumed to be isotropic.

The thermal conductivity defined with Eq. 3.1 is a property of a material and is determined experimentally. From gases to highly conducting metals, k varies by a factor of about $1.5x10^4$. The numerical value of the thermal conductivity is an indication of how fast heat is conducted through the material. Thermal conductivity varies with temperature. Only for limited number of materials, the thermal conductivity depends weakly on temperature. In many others, this dependence is rather strong. Table 3.1 gives the thermal conductivity of selected materials.

Table 3.1 Thermal conductivity of selected materials
(at 25 °C if not specified)

Material	k in W/mK	Material	k in W/mK
Copper	386	Uranium dioxide at 1200 °C	2.6
Aliminum	204	Uranium dioxide at 1800 °C	2.2
Steel	64	Water (light and heavy)	0.611
Stainless steel, 18-8	15	Air	0.027
Zirconium	13		
Uranium metal at 500 °C	30		
Uranium dioxide at 600 °C	4		

II. Convection

Convection is the term used for heat transfer mechanism which takes place in a fluid because of a combination of conduction due to the molecular interactions and energy transport due to the macroscopic (bulk) motion of the fluid itself. In the above definition the motion of the fluid is essential otherwise the heat transfer mechanism becomes a static conduction situation as illustrated in Fig. 3.1. When the term of convection is used, usually a solid surface is present next to the fluid. There are also cases of convection where only fluids are present, such as a hot jet entering into a cold reservoir. However, the most of the industrial applications involve a hot or cold surface transferring heat to the fluid or receiving heat from the fluid.

If the fluid motion is sustained by a difference of pressure created by an external device such as a pump or fan, the term of "forced convection" is used. On the other hand, if the fluid motion is predominantly sustained by the presence of a thermally induced density gradient, then the term of "natural convection" is used.

To understand better the heat exchange between a solid and fluid, consider a heated wall over which a fluid flows as sketched in Fig. 3.4. The temperature of the wall is t_w. The velocity and the temperature of the fluid far from the wall (free stream) are U_∞ and t_∞ , respectively. For a given

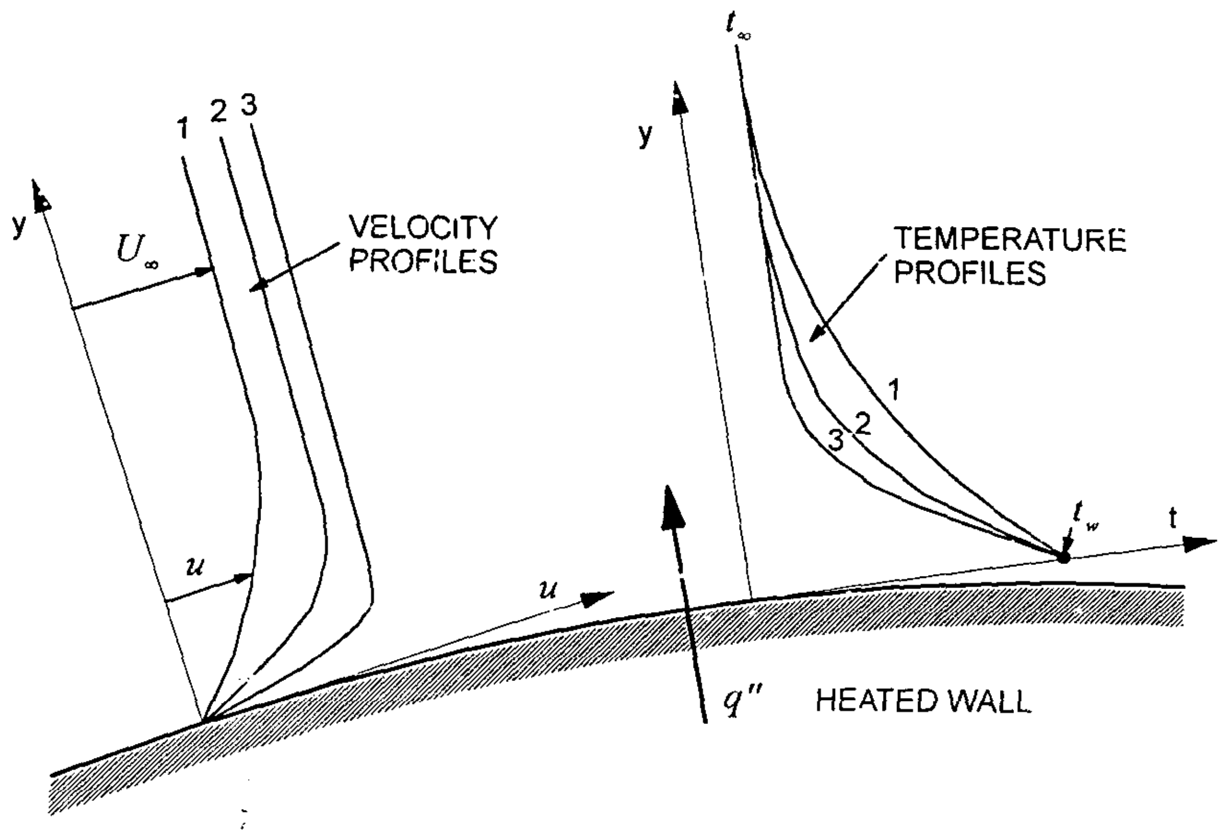


Figure 3.4 Convection heat transfer to a flow over a heated wall.

stream velocity, the velocity of the fluid decreases as we get closer to the wall. This is due to the viscous effects of the flowing fluid. On the wall, because of the adherence (nonslip) condition the velocity of the fluid is zero. The region in which the velocity of the fluid varies from the free stream value to zero is called "velocity boundary layer." Similarly the region in which the fluid temperature varies from its free-stream value to that on the wall is called the "thermal boundary layer." Since the velocity of the fluid at the wall is zero, the heat must be transferred by conduction at that point. Thus, we calculate the heat transfer by using the Fourier's heat conduction law (Eq. 3.1 or 3.2), with thermal conductivity of the fluid corresponding to the wall temperature and the fluid temperature gradient at the wall. The question at this point is that: since the heat flows by conduction in this layer, why do we speak of convection heat transfer and need to consider the velocity of the fluid. The short answer to this question is that the temperature gradient of the fluid on the wall is highly dependent on the flow velocity of the free-stream. As this velocity increases, the distance from the wall we travel to reach free stream temperature decreases. In other words, the thickness of velocity and thermal boundary layers on the wall decreases. The consequence of this decrease is to increase the temperature gradient of the fluid at the wall, i.e., an increase in the rate of heat transferred from the wall to the fluid. The effect of increasing free stream velocity on the fluid velocity and temperature profiles close to the wall is illustrated in Fig. 3.4. Note also that the temperature gradient of the fluid on the wall increases with increasing free stream velocity.

Sir Isaac Newton experimentally found that the heat flux on the wall is proportional to $(t_w - t_\infty)$:

$$\frac{q_c}{A} \sim (t_w - t_\infty) \tag{3.11}$$

Introducing a proportionality constant h, he proposed a law known as Newton's law of cooling:

$$q_c = hA(t_w - t_\infty) \tag{3.12}$$

where h is the convection heat transfer coefficient or the film conductance and A heat exchange surface. The unit of h is W/m^2 K or J/sm^2 K. Table 3.2 gives the orders of magnitude of convective heat transfer coefficients.

Table 3.2 Order of magnitude of convective heat transfer coefficients

Fluid and flow conditions	h W/m^2 K
Air, free convection	5-25
Water, free convection	15-100
Air or superheated steam, forced convection	30-300
Oil, forced convection	60-1,800
Water, forced convection	300-15,000
Liquid sodium, forced convection	10,000-100,000
Boiling water	3,000-60,000
Condensing steam	3,000-100,000

From the above discussion, we conclude that the basic laws of heat conduction must be coupled with those of fluid motion to describe, mathematically, the process of convection. The mathematical treatment of the resulting system of differential equations is very complex. Therefore, for engineering applications, the convection will be treated by an ingenuous combination of mathematical techniques, empiricism and experimentation.

III. Radiation

It has been experimentally observed that a body may loose or gain thermal energy in the absence of a physical transporting medium. For example, a hot object placed in a vacuum chamber with cooler walls is observed to loose thermal energy. This loss of energy is due to the electromagnetic waves emissions (or photons) known as thermal radiation. Regardless of the form of the matter (solid, liquid or gas) this emission is caused by the changes in the electrons configuration of the constituent atoms or molecules. In the above example, radiation heat transfer could also occur between the hot object and cold chamber walls even if the chamber was filled with a sufficiently transparent continuous medium such as air. The wavelength of the electromagnetic radiation is comprised between 10^{-1} μm and 10^{-2} μm. The maximum flux at which radiation may be emitted from a surface is given by the Stefan-Boltzmann law:

$$q_r'' = \frac{q_r}{A} = \sigma T_1^4 \qquad W/m^2 \tag{3.13}$$

where T_1 is the absolute temperature (in K) of the surface and σ is the Stefan-Boltzman constant ($\sigma = 5.57 \times 10^{-8}\ W/m^2K^4$) . Eq. 3.13 applies only to an ideal radiator or "Black body." In practice, the radiant surfaces do not emit thermal energy ideally. To take into account the "gray" nature of the real surfaces, a dimensionless factor, ε, called emissivity is introduced. Therefore heat flux emitted by the surface is written as:

$$q_r'' = \varepsilon\sigma T_1^4 \quad W/m^2 \tag{3.14}$$

with $0 < \varepsilon \leq 1$. If ε=1, we obtain an ideal radiator.

If heat is transferred by radiation between two gray surfaces of finite size, as illustrated in Fig. 3.5, the rate of heat flow will depend on temperatures T_1 and T_2, on emittances ε_1 and ε_2 , and on the

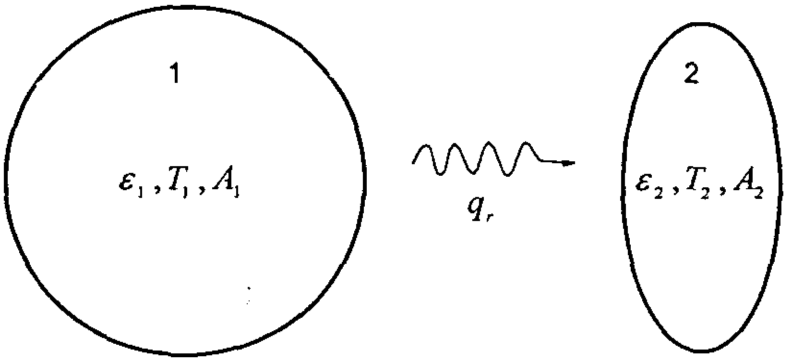


Figure 3.5 Radiation heat transfer between two finite gray surfaces.

geometry of the system. From Fig. 3.5 it is obvious that some radiation originating from object 1 will not be intercepted by object 2, and vice versa. In such a case, the determination of the heat flow rate is rather complicated. Usually we write that:

$$q_r = A_1 \mathbf{F}_{12} \sigma (T_1^4 - T_2^4) \tag{3.15}$$

where q_r is the net radiant energy interchange from object 1 to object 2 and $\mathbf{F}_{12}$ is a transfer factor which depends on emittances and geometry. For an annular space between two infinite cylinders or between two spheres $\mathbf{F}_{12}$ is given by:

$$\mathbf{F}_{12} = \frac{1}{\frac{1}{\varepsilon_1} + \frac{A_1}{A_2}\left(\frac{1}{\varepsilon_2} - 1\right)} \tag{3.16}$$

where ε_1 and ε_2 are the emissivities of objects 1 and 2, respectively. If $A_1 \cong A_2$, the radiant net energy exchange between concentric cylinders is given by:

$$q_r = A \frac{1}{\frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2} - 1} \sigma (T_1^4 - T_2^4) \tag{3.17}$$

and corresponding heat flux:

$$q_r'' = \frac{q_r}{A} = \frac{\sigma}{\frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2} - 1} (T_1^4 - T_2^4) \quad . \tag{3.18}$$

In many engineering applications, it is convenient to express the net energy exchange as:

$$q_r = h_r A (T_1 - T_2) \, . \tag{3.19}$$

Comparing Eqs. 3.17 and 3.19, we conclude that the "radiation heat transfer coefficient," h_r for concentric cylinders when $A_1 \cong A_2$ is given by:

$$h_r = \frac{\sigma}{\frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2} - 1} (T_1^2 + T_2^2)(T_1 + T_2) \quad . \tag{3.20}$$

With this approach, we have modeled the radiation heat transfer in a manner similar to convection heat transfer. It should be noted that h_r depends strongly on temperature, while the temperature dependence of the convection heat transfer coefficient is generally weak.

In many engineering problems we may consider simultaneously convection and radiation heat transfer. In such a case the total heat transfer from the surface is written as:

$$q = q_c + q_r \tag{3.21}$$

1.2 Conduction Heat Transfer

In this chapter, using the Fourier's heat conduction law, we will establish a general equation for the conduction of heat in solids. This equation will be presented in rectangular coordinates as well as in polar cylindrical and in spherical coordinates. We will also discuss the most frequently encountered boundary conditions. Given the introductory nature of this section, the application of the general conduction equation will only be limited to one dimensional steady state and transient problems.

1.2.1 General Conduction Equation

In studying heat conduction problems, the main objective is to determine the temperature distribution in a solid as a function of space and time, $t(x,y,z,\tau)$, for a given set of initial and boundary conditions. Once this distribution is known, the heat flux at any point of the solid or on its surface can easily be determined. In the following, using the energy conservation principle and the Fourier's heat conduction law we will establish the general heat conduction equation. The solution of this equation for a given set of initial and boundary conditions will allow us to determine the required temperature distribution. To derive the conduction equation, consider the solid medium shown in Fig. 3.6 and select within this solid a differential control volume in the shape of a

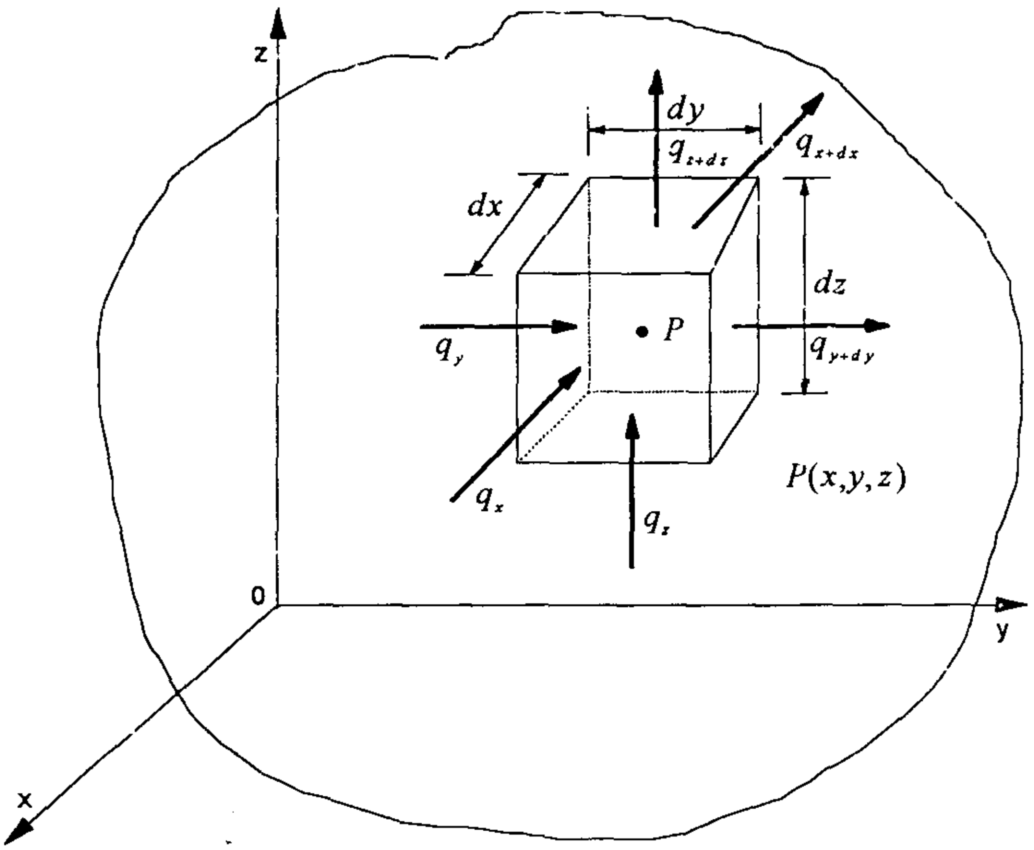


Figure 3.6 Control volume for conduction analysis in rectangular coordinate system.

parallelepiped of dimensions *dx, dy, dz* in the *x, y, z* directions as illustrated in the same figure. Indicating by:

q_x and q_{x+dx} heat entering and leaving the control volume in the *x*-direction,
q_y and q_{y+dy} heat entering and leaving the control volume in the *y*-direction,
q_z and q_{z+dz} heat entering and leaving the control volume in the *z*-direction,
Q_g heat generation in the control volume, and
U internal energy of the control volume

the energy conservation principle applied to the control volume can be written as:

$$q_x + q_y + q_z - q_{x+dx} - q_{y+dy} - q_{z+dz} + Q_g = \frac{\partial U}{\partial \tau} \quad . \tag{3.22}$$

Using the Fourier's heat conduction law, we can write that:

$$q_x = -\left(k\frac{\partial t}{\partial x}\right)_x dydzd\tau \tag{3.23}$$

$$q_y = -\left(k\frac{\partial t}{\partial y}\right)_y dxdzd\tau \tag{3.24}$$

$$q_z = -\left(k\frac{\partial t}{\partial z}\right)_z dxdzd\tau \tag{3.25}$$

$$q_{x+dx} = -\left(k\frac{\partial t}{\partial x}\right)_{x+dx} dydzd\tau \tag{3.26}$$

$$q_{y+dy} = -\left(k\frac{\partial t}{\partial y}\right)_{y+dy} dxdzd\tau \tag{3.27}$$

$$q_{z+dz} = -\left(k\frac{\partial t}{\partial z}\right)_{z+dz} dxdyd\tau \tag{3.28}$$

we can also write that:

$$Q_g = q'''(x, y, z, \tau)dxdydzd\tau \tag{3.29}$$

$$\frac{\partial U}{\partial \tau} = c\rho\frac{\partial t}{\partial \tau}dxdydzd\tau \tag{3.30}$$

where $q'''(x, y, z, \tau)$ is the heat generation rate per unit volume, and c and ρ are the specific heat and specific mass, respectively. Eqs. 3.26, 3.27, 3.28 , after using Taylor series expansion and neglecting the terms of second and higher orders, can be written as:

$$q_{x+dx} = -\left(k\frac{\partial t}{\partial x}\right)_x dydzd\tau - \frac{\partial}{\partial x}\left(k\frac{\partial t}{\partial y}\right)_x dxdydzd\tau\ , \tag{3.31}$$

$$q_{y+dy} = -\left(k\frac{\partial t}{\partial y}\right)_y dxdzd\tau - \frac{\partial}{\partial y}\left(k\frac{\partial t}{\partial y}\right)_y dxdydzd\tau\ , \tag{3.32}$$

$$q_{z+dz} = -\left(k\frac{\partial t}{\partial z}\right) dxdyd\tau - \frac{\partial}{\partial z}\left(k\frac{\partial t}{\partial z}\right)_z dxdydzd\tau\ . \tag{3.33}$$

Substituting Eqs. 3.23-3.25 and Eqs. 3.29-3.33 into Eq. 3.22 we obtain the general conduction equation:

$$\frac{\partial}{\partial x}\left(k\frac{\partial t}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial t}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial t}{\partial z}\right) + q''' = c\rho\frac{\partial t}{\partial \tau}. \tag{3.34}$$

The conductivity, k, can be a function of space and temperature. However, we will assume that the conducting medium is homogeneous and isotropic. Under this condition, the thermal conductivity depends only on temperature and because of this dependence in Eq. 3.34 it is left in the derivatives. If the conductivity is independent of temperature, i.e., position, Eq. 3.34 becomes:

$$k\left(\frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2}\right) + q''' = c\rho\frac{\partial t}{\partial \tau} \tag{3.35}$$

When there is no internal heat generation, the above equation reduces to:

$$\frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2} = \frac{1}{\alpha}\frac{\partial t}{\partial \tau} \tag{3.36}$$

where $\alpha = k/c\rho$ (m^2/s) is a thermophysical property of the material and it is called "the thermal diffusivity." This equation is called Fourier's equation.

For steady state conditions, Eq. 3.35 reduces to:

$$\frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2} = -\frac{q'''}{k} \tag{3.37}$$

which is known as Poisson's equation. Finally for steady state conditions without heat generation Eq. 3.35 becomes:

$$\frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2} = 0 \tag{3.38}$$

which is Laplace's equation.

Eq. 3.34 can also be obtained from general conservation equation given in Chapter 2 (Eq. 2.5) repeated here for convenience:

$$\frac{\partial \psi}{\partial \tau}+\vec{\nabla} \cdot \psi \vec{v}+\vec{\nabla} \cdot \bar{\bar{J}}_{\psi}-S_{\psi}=0 \tag{3.39}$$

where

ψ : property per unit volume of material,

$\bar{\bar{J}}_{\psi}$: flow of property per unit of area and time through the control surface bounding the control volume,

S_{ψ} : generation of property per unit volume and time,

$\vec{v}$: flow velocity.

In the present case, the body is at rest, i.e., $\vec{v}=0$. Considering Fig. 3.7 and interpreting in Eq. 3.39

ψ : as the internal energy, ρu,

$\bar{\bar{J}}_{\psi}$: as the heat flux, and

S_{ψ} : as the heat generation rate q''',

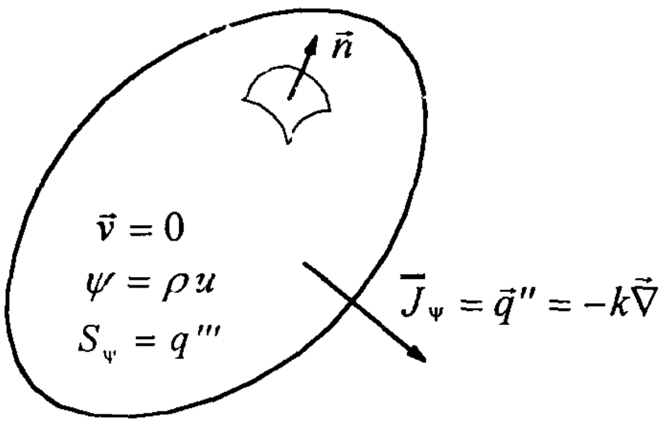


Figure 3.7 Control volume to be used with general local conservation equation.

we obtain:

$$\frac{\partial \rho u}{\partial \tau}+\vec{\nabla} \cdot \vec{q}''-q'''=0 \ . \tag{3.40}$$

Using the general form of Fourier's law (Eq. 3.7) and knowing that:

$$\frac{\partial \rho u}{\partial \tau} = c\rho \frac{\partial t}{\partial \tau} \tag{3.41}$$

Eq. 3.40 becomes:

$$\vec{\nabla} . k \vec{\nabla} . t + q''' = c\rho \frac{\partial t}{\partial \tau} . \tag{3.42}$$

This equation is the same as Eq. 3.34.

The derivation of the general conduction equation can also be carried out in cylindrical coordinate system (r, θ, z) defined in Fig. 3.8a and spherical coordinate system (r, ϕ, θ) defined in Fig. 3.8b. The resulting equations are:

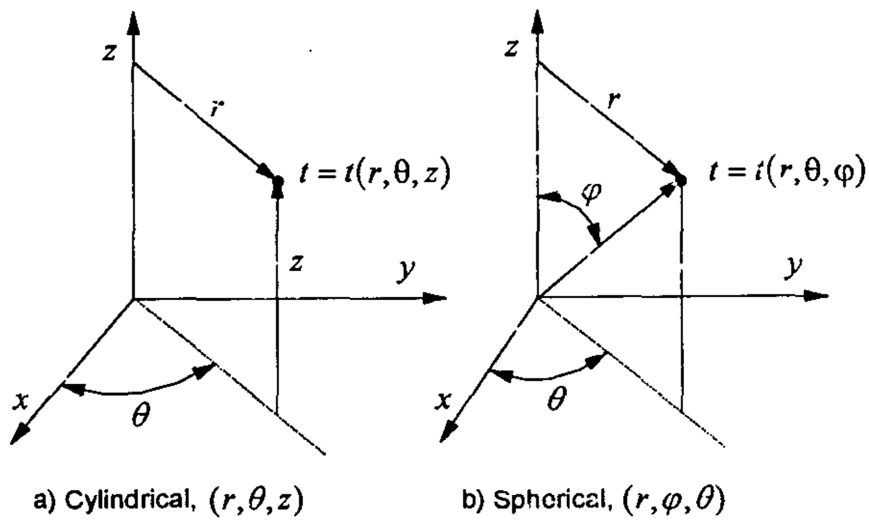


Figure 3.8 Different coordinate systems.

Cylindrical coordinates:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(kr\frac{\partial t}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial \theta}\left(k\frac{\partial t}{\partial \theta}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial t}{\partial z}\right) + q''' = c\rho\frac{\partial t}{\partial \tau} \tag{3.43}$$

Spherical coordinates:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(kr^2\frac{\partial t}{\partial r}\right) + \frac{1}{r^2 \sin\phi}\frac{\partial}{\partial \phi}\left(k\sin\phi\frac{\partial t}{\partial \phi}\right) + \frac{1}{r^2 \sin^2\phi}\frac{\partial}{\partial \theta}\left(k\frac{\partial t}{\partial \theta}\right) + q''' = c\rho\frac{\partial t}{\partial \tau} . \tag{3.44}$$

1.2.2 Initial and Boundary Conditions

The evaluation of the constants that appear in the solution of the heat conduction equation requires the use of boundary and initial conditions. In the following, we will discuss the most frequently encountered boundary and initial conditions.

I. Initial conditions

In transient heat conduction problems, the temperature distribution in the body under observation should be known prior to the initiation of the transient. For example we will specify that at $\tau = 0$, the temperature distribution in the body is given by $t(x,y,z)$.

II. Boundary conditions

The boundary conditions specifies the thermal conditions applied to the boundary surfaces of the body. For example, on the boundary surfaces we may specify the temperature, the heat flux or the heat transfer to a fluid by convection.

1. *Prescribed boundary temperature condition*

The temperature on the boundary surfaces of the body, t_s, is imposed as illustrated in Fig. 3.9. This temperature may be uniform and constant, a function of space and time or, a function of space only or time only.

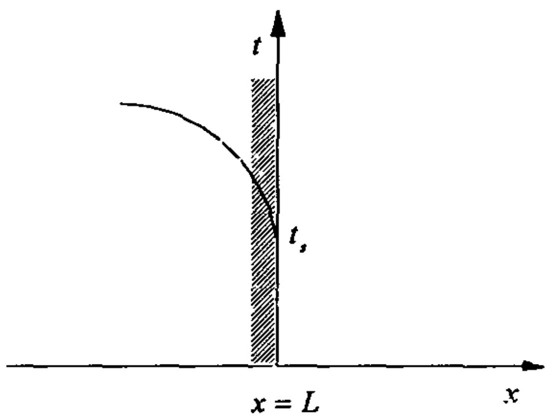


Figure 3.9 Prescribed boundary temperature.

2. *Prescribed boundary heat flux condition*

The heat flux across the boundaries is specified. This flux may be uniform and constant, a function of space and time or, a function of space or time only. The heat flux may be removed from the boundary surface (Fig. 3.10a) or supplied to the boundary surface (Fig. 3.10b).

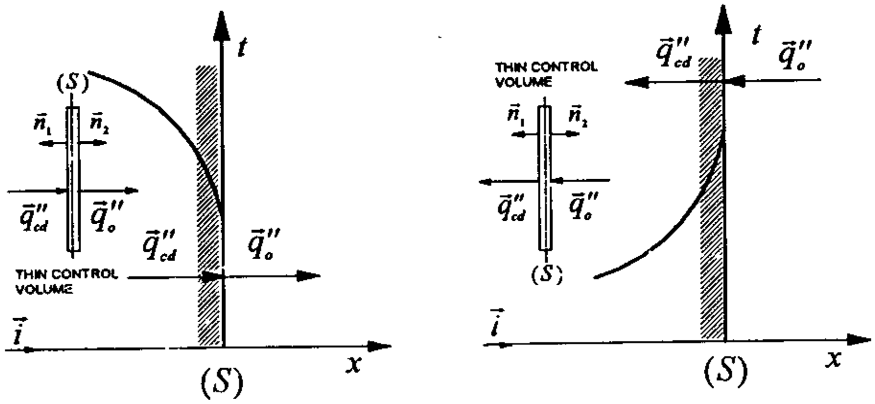
If heat is removed from the boundary (Fig. 3.10a), the application of the macroscopic energy conservation principle (Eq. 2.23) to a very thin layer at the boundary (see insert in Fig. 3.10a) yields:

$$\int_A \vec{q}''_{cd} \cdot \vec{n}\, dA = 0 \tag{3.45}$$

or

$$\vec{n}_1 \cdot \vec{q}''_{cd} + \vec{n}_2 \cdot \vec{q}''_o = 0 \tag{3.46}$$

where $\vec{q}''_{cd}$ is the conduction heat flux and $\vec{q}''_o$ is the prescribed heat flux. Using the Fourier's law of conduction, we obtain:



a) Heat removal from boundary

b) Heat addition to boundary

Figure 3.10 Prescribed heat flux at the boundary.

$$\vec{q}''_{cd} = -k\left(\frac{\partial t}{\partial x}\right)\vec{i} \tag{3.47}$$

we obtain:

$$\vec{n}_1 \cdot \left(-k\frac{\partial t}{\partial x}\right)\vec{i} + \vec{n} \cdot \vec{q}''_o = 0 \tag{3.48}$$

Since $\vec{n}_1 \cdot \vec{i} = -1$ *and* $\vec{n}_2 \cdot \vec{q}''_o = q''_o$, the above equation becomes:

$$-k\left(\frac{\partial t}{\partial x}\right)_S = q''_o. \tag{3.49}$$

If the heat is supplied to the boundary (Fig. 3.10b), the same reasoning as above yields:

$$k\left(\frac{\partial t}{\partial x}\right)_S = q''_o \tag{3.50}$$

If the boundary surfaces are well insulated, i.e., $q''_o = 0$, Eqs. 3.49 and 3.50 are reduced to:

$$\left(\frac{\partial t}{\partial x}\right)_S = 0 \tag{3.51}$$

3. *Convective boundary condition*

A frequently encountered situation is the one in which the bounding surfaces are in touch with a fluid where heat is transferred from surfaces to fluid or vice versa as illustrated in Fig. 3.11. If the heat is transferred from boundary surfaces to the fluid (Fig 3.11a), the application of the energy conservation principle to a very thin layer at the boundary yields:

$$\vec{n}_1 \cdot \vec{q}''_{cd} + \vec{n}_2 \cdot \vec{q}''_{cv} = 0 \tag{3.52}$$

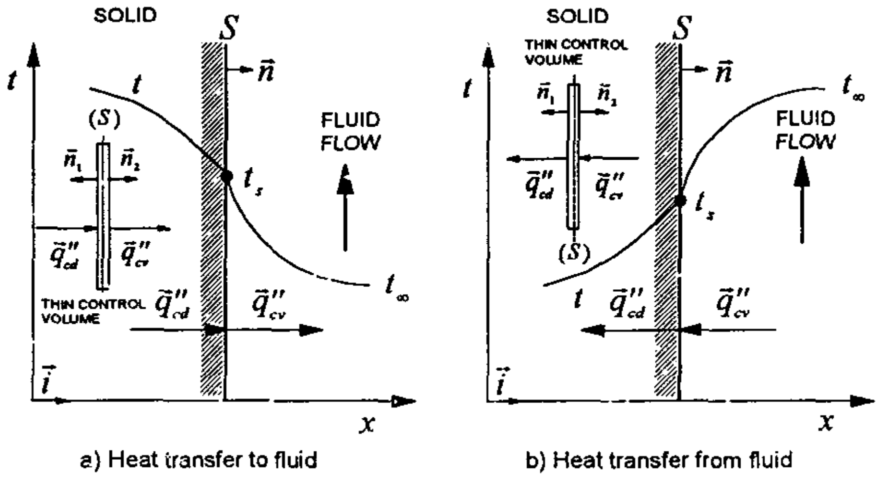


Figure 3.11 Convection at the boundary surfaces.

where $\vec{q}\,''_{cd}$ is the conduction heat flux and $\vec{q}\,''_{cv}$ convection heat flux. Using the Fourier law of conduction (Eq. 3.47) , Eq. 3.52 becomes:

$$\vec{n}_1 \, . \left(-k\frac{\partial t}{\partial x}\right)_S \vec{i} + \vec{n}_2 \, . \, \vec{q}\,''_{cv} = 0 \qquad (3.53)$$

or

$$-k\left(\frac{\partial t}{\partial x}\right)_S = q''_{cv} \qquad (3.54)$$

The above equation with theNewton's cooling law (Eq. 3.12) can be written as:

$$-k\left(\frac{\partial t}{\partial x}\right)_S = h_c(t_s - t_\infty) \qquad (3.55)$$

If the heat is transferred from fluid to the boundary surfaces (Fig. 3.10b), energy conservation principle gives:

$$k\left(\frac{\partial t}{\partial x}\right)_S = q''_{cv} \qquad (3.56)$$

In this case q''_{cv} is given by:

$$q''_{cv} = h_c(t_\infty - t_s)$$

Substituting the above equation into Eq. 3.56 we obtain:

$$k\left(\frac{\partial t}{\partial x}\right)_S = h(t_\infty - t_s) \, . \qquad (3.57)$$

4. *Interface of two medium with different conductivity*

When two media with conductivity k_1 and k_2 have a common interface as illustrated in Fig. 3.12, the heat flux at this interface for each medium should be equal, i.e.,

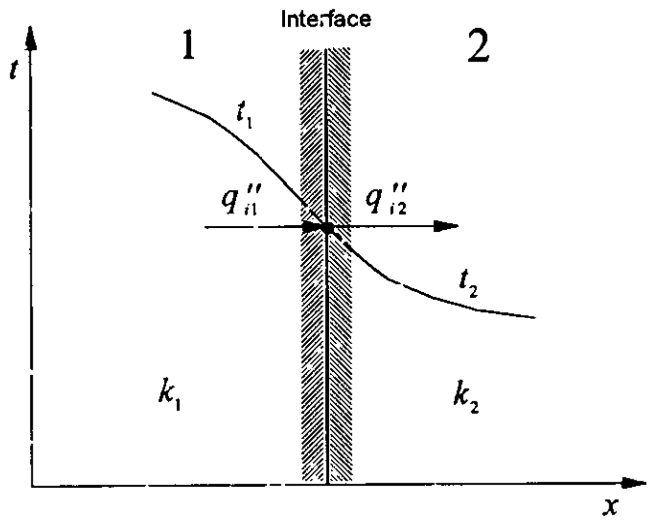


Figure 3.12 Interface of two medium with different conductivities.

$$q''_{i1} = q''_{i2} \tag{3.58}$$

or using the Fourier's law of conduction (Eq. 3.1) we obtain:

$$k_1\left(\frac{\partial t_1}{\partial x}\right)_i = k_2\left(\frac{\partial t_2}{\partial x}\right)_i \tag{3.59}$$

If the contact resistance between the two media is zero, then the temperatures on both sides of the interface are equal, i.e.,

$$t_{1i} = t_{2i} \ . \tag{3.60}$$

However, in practice the contact resistance is different from zero. In this case, representing the conductance at the interface by h_g, the temperatures on both sides of the interface are related by:

$$q''_{i1} = q''_{i2} = h_g(t_{1i} - t_{2i}) \ . \tag{3.61}$$

The contact conductance will be discussed in details in chapter on "Heat Removal from Nuclear Reactors."

1.2.3 One Dimensional Steady State Conduction

In this section, we will discuss heat conduction problems where only one dimension is enough to describe the temperature distribution. For example, the heat flow in a wall of finite thickness in x-direction but infinite extent in the y- and z-directions or heat flow in a long cylinder with angular symmetry constitute one dimensional heat transfer problems. For a one-dimensional steady state conduction heat transfer, the general conduction equations (Eqs. 3.34, 3.43 or 3.44) are written as:

Rectangular coordinates:

$$\frac{\partial}{\partial x}\left(k\frac{\partial t}{\partial x}\right)+q'''=0 \tag{3.62}$$

Cylindrical coordinates:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(kr\frac{\partial t}{\partial r}\right)+q'''=0 \tag{3.63}$$

Spherical coordinates:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(kr^2\frac{\partial t}{\partial r}\right)+q'''=0 \tag{3.64}$$

1.2.3.1 Conduction Heat Transfer in a Slab

I. Plane wall with prescribed boundary temperatures

As illustrated in Fig. 3.13 the wall has a finite thickness (L) but infinite extent. Both faces, located at $x = 0$ and $x = L$ are kept at specified temperatures t_1 and t_2, respectively. There is no heat sources within the slab. Under these conditions Eq. 3.62, for a constant conductivity, reduces to

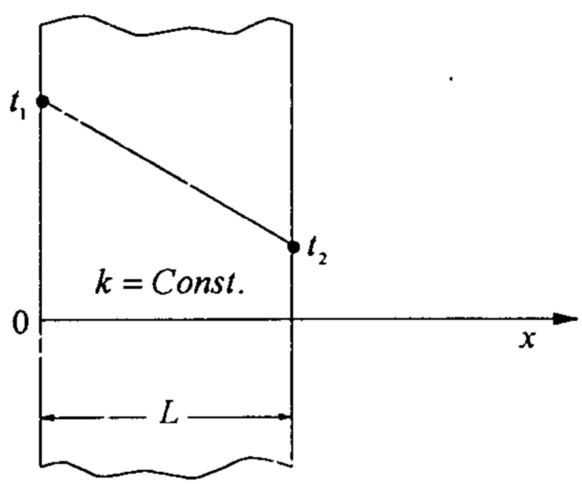


Figure 3.13 Slab with prescribed temperatures.

$$\frac{\partial^2 t}{\partial x^2}=0 \tag{3.65}$$

with boundary conditions:

$$x=0 \qquad t=t_1 \tag{3.66}$$

$$x=L \qquad t=t_2. \tag{3.67}$$

The solution of Eq. 3.65 is:

$$t = Ax + B \tag{3.68}$$

A and B are arbitrary constants. Application of the boundary conditions allows us to determine the values of the constants. The substitution of these values into Eq. 3.68 yields:

$$t(x) = t_1 + \frac{t_2 - t_1}{L} x . \tag{3.69}$$

The heat flux through any plane in the wall perpendicular to the x-axis can be determined by using the Fourier's law of conduction:

$$q'' = -k\frac{\partial t}{\partial x} = k\frac{t_1 - t_2}{L} . \tag{3.70}$$

II. Multilayer wall with prescribed boundary temperatures

Figure 3.14 illustrates a wall of two layers. The thickness of the walls are L_1 and L_2 and the con-

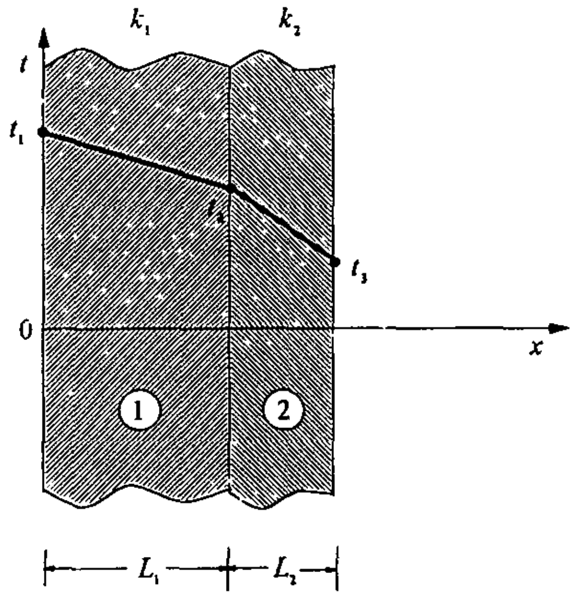


Figure 3.14 Multilayer wall.

ductivity are k_1 and k_2 , respectively. The outside temperatures are t_1 and t_3 , respectively. we wish to determine the heat flux through the wall.

Since the steady state conditions exist, the heat flux through the layers is constant. The application of Eq. 3.70 to layers 1 and 2 yield:

$$q'' = k_1\frac{t_1 - t_2}{L_1} \quad \text{or} \quad t_1 - t_2 = q''\frac{L_1}{k_1} \tag{3.71}$$

$$q'' = k_2\frac{t_2 - t_3}{L_2} \quad \text{or} \quad t_2 - t_3 = q''\frac{L_2}{k_2} \tag{3.72}$$

Upon addition of the above equations we obtain:

$$q'' = \frac{t_1 - t_3}{\frac{L_1}{k_1} + \frac{L_2}{k_2}}. \tag{3.73}$$

Calling:

$$K = \frac{1}{\frac{L_1}{k_1} + \frac{L_2}{k_2}} \tag{3.74}$$

Eq. 3.73 can then be written as:

$$q'' = K(t_1 - t_3) \tag{3.75}$$

where K is the overall heat transfer coefficient. The above discussion shows that the knowledge of the interface temperature, t_2, is not necessary to determine heat flux through the multilayer walls.

III. Multilayer wall bounded on each side by convecting fluids

Fig. 3.15 illustrates a multilayer wall bounded on each side by convecting fluids. The convection coefficients are respectively h_1 and h_2 and the temperature of the circulating fluids are t_{f1} and t_{f2}, respectively. We wish to determine the heat flux through the wall.

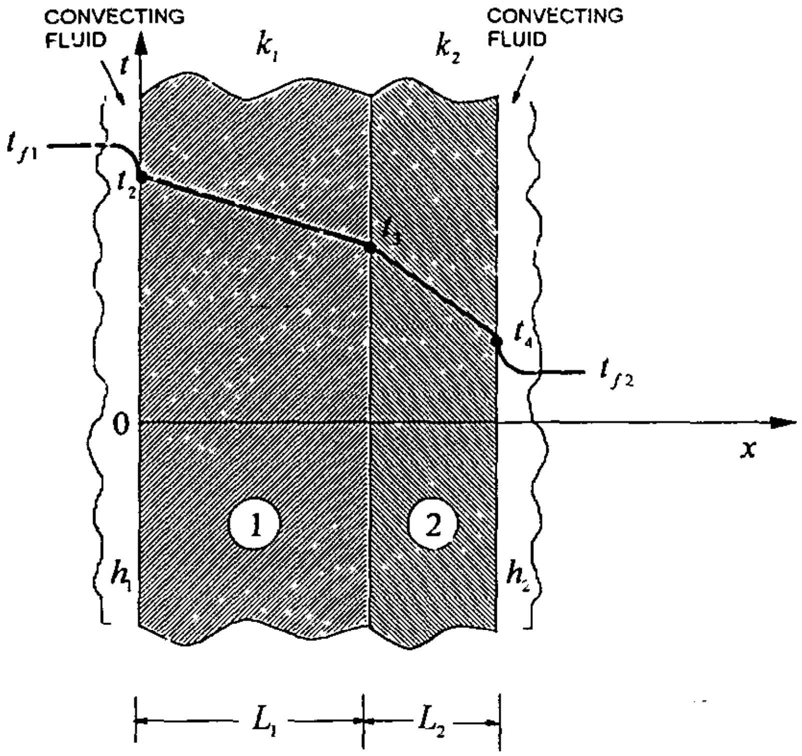


Figure 3.15 Multilayer wall with convection on both sides.

The heat flux through the layers is constant and can be written as:

$$q'' = h_1(t_{f1} - t_2) \quad \text{or} \quad t_{f1} - t_2 = \frac{q''}{h_1} \tag{3.76}$$

$$q'' = k_1 \frac{t_2 - t_3}{L_1} \quad \text{or} \quad t_2 - t_3 = \frac{q''}{k_1} L_1 \tag{3.77}$$

$$q'' = k_2 \frac{t_3 - t_4}{L_2} \quad \text{or} \quad t_3 - t_4 = \frac{q''}{k_2} L_2 \tag{3.78}$$

$$q'' = h_2 (t_4 - t_{f2}) \quad \text{or} \quad t_4 - t_{f2} = \frac{q''}{h_2} \tag{3.79}$$

Upon addition of the above equations we obtain:

$$q'' = \frac{t_{f1} - t_{f2}}{\frac{1}{h_1} + \frac{L_1}{k_1} + \frac{L_2}{k_2} + \frac{1}{h_2}} \tag{3.80}$$

or

$$q'' = K(t_{f1} - t_{f2}) \tag{3.81}$$

where

$$K = \frac{1}{\frac{1}{h_1} + \frac{L_1}{k_1} + \frac{L_2}{k_2} + \frac{1}{h_2}} . \tag{3.82}$$

If the wall consists of n layers. the overall heat transfer coefficient will have the following form:

$$K = \frac{1}{\frac{1}{h_1} + \sum_n \frac{L_n}{k_n} + \frac{1}{h_2}} \tag{3.83}$$

IV. Plane wall with heat generation and prescribed boundary temperatures

The only difference between this case and the case *I* is the heat generation in the slab. For a constant conductivity, Eq. 3.62 becomes:

$$k \frac{\partial^2 t}{\partial x^2} + q''' = 0 \tag{3.84}$$

with boundary conditions given by Eqs. 3.66 and 3.67. The solution of the above equation is:

$$t(x) = -\frac{q'''}{2k} x^2 + Ax + B \tag{3.85}$$

The application of the boundary conditions yields:

$$A = \frac{q'''}{2k} L + \frac{t_2 - t_1}{L} , \tag{3.86}$$

and

$$B = t_1 . \tag{3.87}$$

Knowing A and B, the temperature distribution in the slab is given by:

$$t(x) = \frac{q''' L^2}{2k}\left[\frac{x}{L} - \left(\frac{x}{L}\right)^2\right] + (t_2 - t_1)\frac{x}{L} + t_1 \ . \tag{3.88}$$

V. Plane wall with heat generation, one surface insulated the other subjected to convective heat transfer

This case is illustrated in Fig. 3.16. The temperature distribution is given by Eq. 3.85. In this case, the constants are determined by using the following boundary conditions:

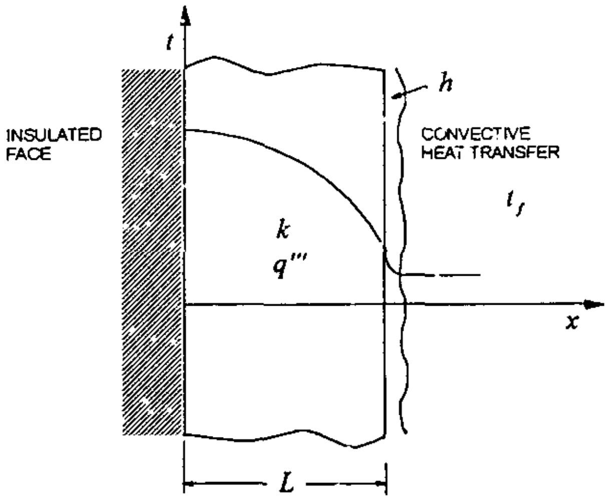


Figure 3.16 Plane wall with heat generation; one face insulated the other cooled by convection.

$$x = 0 \qquad \frac{\partial t(x)}{\partial x} = 0 \tag{3.89}$$

$$x = L \qquad -k\frac{\partial t(x)}{\partial x} = h[t(x) - t_f] \tag{3.90}$$

and are given by:

$$A = 0 \tag{3.91}$$

$$B = \frac{q'''}{2k}L^2 + \frac{q'''}{h}L + t_f . \tag{3.92}$$

The temperature distribution has, therefore, the following form:

$$t(x) = \frac{q''' L^2}{2k}\left[1 - \left(\frac{x}{L}\right)^2\right] + \frac{q''' L}{h} + t_f . \tag{3.93}$$

VI. Plane wall with heat generation and convective boundary conditions on both faces

As illustrated in Fig. 3.17, both faces of the plate are washed with a fluid at temperature t_f. The heat transfer coefficient is h. The faces 1 and 2 are located at $x = -L$ and $x = L$, respectively. Heat generation rate is q'''.

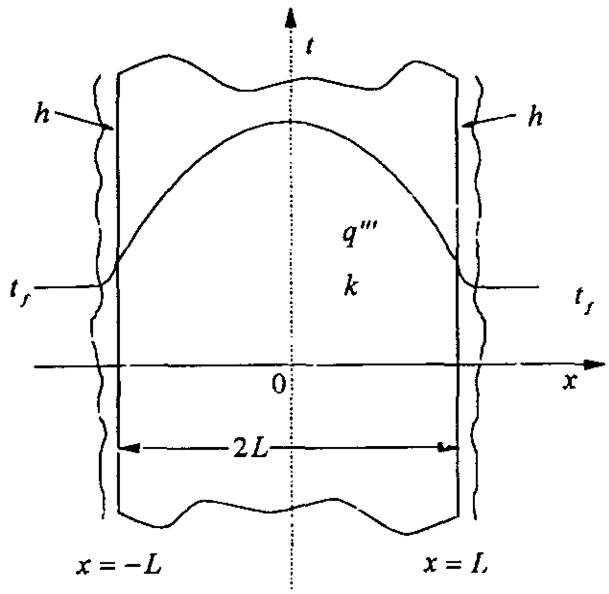


Figure 3.17 Plane wall with heat generation and convective boundaries.

The temperature distribution is again given by Eq. 3.85 subject to following boundary conditions:

$$x = L \qquad -k\left(\frac{\partial t(x)}{\partial x}\right) = h[t(x) - t_f] \tag{3.94}$$

$$x = -L \qquad k\left(\frac{\partial t(x)}{\partial x}\right) = h[t(x) - t_f] \tag{3.95}$$

Eq. 6.94 signify that the conduction heat transfer that arrives to the face at $x = L$, $-k(\partial t/\partial x)$, is equal to the convective heat flux that enters the fluid bulk, i.e., $h[t(L) - t_f]$. The same boundary condition at the face at $x = -L$ does not have a minus sign in front of the $k(\partial t/\partial x)$ term. This point was examined in details during the discussion of the boundary conditions in Section 1.2.2. Using boundary conditions 3.94 and 3.95, the constants A and B appearing in Eq. 3.85 are determined as:

$$A = 0 \tag{3.96}$$

$$B = q'''L\left(\frac{1}{h} + \frac{L}{2k}\right) + t_f . \tag{3.97}$$

The temperature distribution is given by:

$$t(x) = \frac{q'''L^2}{2k}\left[1 - \left(\frac{x}{L}\right)^2\right] + \frac{q'''L}{h} + t_f. \tag{3.98}$$

Comparing Eqs. 3.93 and 3.98 , we observe that the temperature distributions are the same. Eq. 3.98 shows that the maximum temperature occurs in the midplane of the slab, i.e. , $x = 0$. Therefore, at this point the temperature gradient is zero and there is no heat flux in either direction of x-axis. Eq. 3.98 also shows that the temperature distribution in the slab is symmetrical. When a given case has both geometrical and thermal symmetries about $x = 0$, it is more convenient to solve the conduction equation over the half region, i.e., for the slab under consideration between $x = 0$ and $x = L$ by using the following boundary conditions:

$$x = 0 \qquad \frac{\partial t(x)}{\partial x} = 0 \tag{3.99}$$

$$x = L \qquad -k\frac{\partial t}{\partial x} = h[t(x) - t_f] \tag{3.100}$$

instead of using boundary conditions given by Eqs. 3.94 and 3.95. This discussion also explains why the temperature distributions given by Eqs. 3.93 and 3.98 are the same.

Example:

The fuel element of a pool type reactor is composed of a plate of metallic uranium of thickness $2L_1$ placed in sandwich between two aluminum plates (cladding) of thickness$(L_2 - L_1)$. This fuel element is illustrated in Fig. 3.18. Heat energy, due to the fission of U_{235}, is generated in the fuel plate at a uniform rate q'''. The fission energy deposited in the cladding plates is negligible. The convection heat transfer coefficient and the temperature of the fluid washing the fuel element are h and t_f, respectively. Determine the temperature distribution in the fuel element.

This is a multiregion problem that involves two governing equations. As seen from Fig. 3.18, the problem has geometric and thermal symmetries with respect to the mid-plane of the fuel element. Under these conditions, it is more convenient to solve the problem over the half of the fuel element extending from $x = 0$ to $x = L_2$. Indicating by 1 the fuel region and by 2 the cladding region, the heat conduction equations are written as:

Fuel:

$$\frac{d^2 t_1}{dx^2} + \frac{q'''}{k_1} = 0 \qquad \text{for} \qquad 0 \le x \le L_1 \tag{3.101}$$

Cladding:

$$\frac{d^2 t_2}{dx^2} = 0 \qquad \text{for} \qquad L_1 \le x \le L_2 \tag{3.102}$$

with boundary conditions given by:

$$x = 0 \qquad \frac{dt_1(x)}{dx} = 0 \tag{3.103}$$

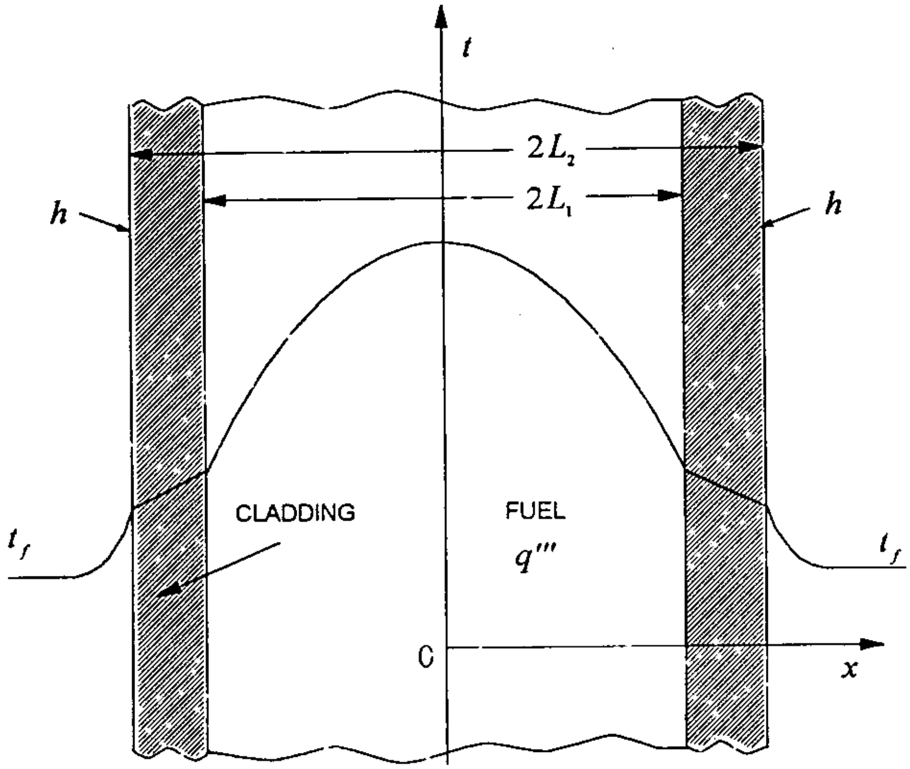


Figure 3.18 Fuel element of a pool type reactor.

$$x = L_1 \qquad t_1(x) = t_2(x) \tag{3.104}$$

$$x = L_1 \qquad k_1 \frac{dt_1(x)}{dx} = k_2 \frac{dt_2(x)}{dx} \tag{3.105}$$

$$x = L_2 \qquad -k_2 \frac{dt_2(x)}{dx} = h[t_2(x) - t_f] \tag{3.106}$$

Solutions of Eqs. 3.101 and 3.102 are given by:

$$t_1(x) = -\frac{q''' x^2}{2k_1} + Ax + B \ , \tag{3.107}$$

and

$$t_2(x) = Cx + D . \tag{3.108}$$

Combining Eqs. 6.103 through 6.109, we obtain four equations:

$$A = 0, \tag{3.109}$$

$$-\frac{q''' L_1^2}{2\,k_1} + B = CL_2 + D . \tag{3.110}$$

$$-q'''L_1 = k_2 C\ , \tag{3.111}$$

$$-k_2 C = h(CL_2 + D - t_f)\ . \tag{3.112}$$

The solution of these equations yields the values of A, B, C and D. The temperature distribution throughout the fuel element is then given by:

Fuel:

$$t_1(x) = \frac{q'''L_1^2}{k_2}\left[1 - \left(\frac{x}{L_1}\right)^2 - 2\left(\frac{k_1}{k_2}\right) + 2\left(\frac{L_2}{L_1}\right)\left(\frac{k_1}{k_2}\right)\left(1 + \frac{k_2}{hL_2}\right)\right] + t_f \tag{3.113}$$

Cladding:

$$t_2(x) = -\frac{q'''L_1^2}{k_2}\left[\frac{x}{L_1} - \frac{L_2}{L_1}\left(1 + \frac{k_2}{hL_2}\right)\right] + t_f\ . \tag{3.114}$$

1.2.3.2 Conduction in Cylindrical Geometry

I. Long hollow cylinder with prescribed temperature on the walls

Consider the long hollow cylinder illustrated in Fig. 3.19 with inner and outer radii r_1 and r_2 , respectively. The temperature of the inner wall is t_1 and that of the outer wall is t_2. There is no heat generation within the cylinder and the conductivity of the material is constant. We wish to determine the temperature variation in the cylinder wall.

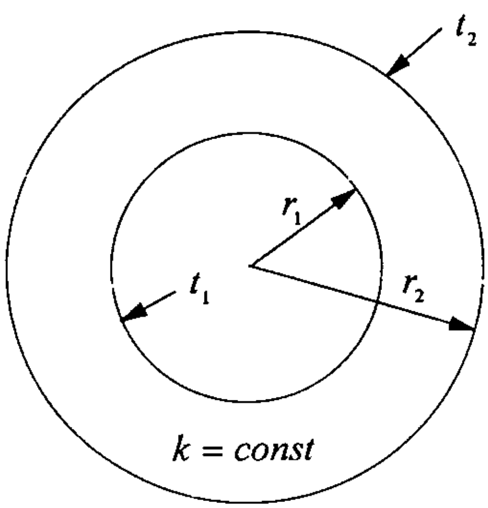


Figure 3.19 Long hallow cylinder.

The application of Eq. 3.63 to the present situation yields:

$$\frac{d}{dr}\left(r\frac{dt}{dr}\right) = 0 \tag{3.115}$$

with boundary conditions:

$$r = r_1 \qquad t = t_1 \tag{3.116}$$

$$r = r_2 \qquad t = t_2 \tag{3.117}$$

The integration of Eq. 3.115 gives:

$$t(r) = A \ln r + B \tag{3.118}$$

where A and B can be easily determined by using boundary conditions:

$$A = \frac{t_2 - t_1}{\ln(r_2/r_1)}, \tag{3.119}$$

$$B = t_1 - (t_2 - t_1)\frac{\ln r_1}{\ln(r_2/r_1)}. \tag{3.120}$$

The temperature distribution is then given by:

$$t(r) = t_1 + \frac{t_2 - t_1}{\ln(r_2/r_1)} \ln\frac{r}{r_1}. \tag{3.121}$$

Based on the above temperature distribution, the linear heat flux (or heat flux per unit length) through a surface located at r can be easily calculated:

$$q' = \frac{q}{L} = -k2\pi r\frac{dt}{dr} = -k2\pi r\frac{1}{r}\frac{t_2 - t_1}{\ln(r_2 - r_1)} = 2\pi k\frac{t_1 - t_2}{\ln(r_2/r_1)} \tag{3.122}$$

II. Hollow cylinder with convective boundaries on both walls

Fig. 3.20 is a sketch of a pipe in which a fluid at temperature t_{f1} circulates. Heat is transferred from this fluid to the pipe by convection, through the pipe wall by conduction then to the fluid outside again by convection. The temperature of the fluid outside is t_{f2} . What is the linear heat flux through the wall of the pipe.

Under steady state conditions, the linear heat flux is constant and we can write:
Inner surface of the pipe:

$$q' = 2\pi r_1 h_1 (t_{f1} - t_1) \quad \text{or} \quad t_{f1} - t_1 = \frac{q'}{2\pi r_1 h_1} \tag{3.123}$$

Through the wall (Eq. 3.127):

$$q' = 2\pi k\frac{t_1 - t_2}{\ln(r_2/r_1)} \quad \text{or} \quad t_1 - t_2 = \frac{q'}{2\pi k / \ln(r_2/r_1)} \tag{3.124}$$

Outer surface of the pipe:

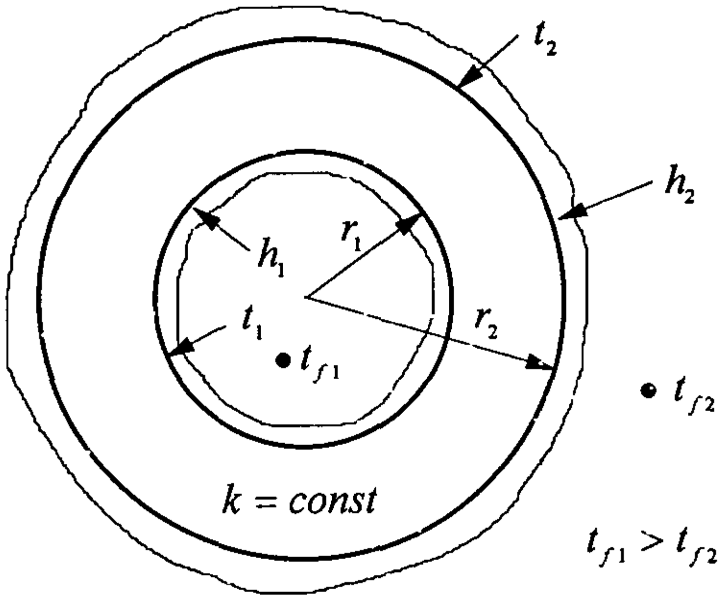


Figure 3.20 Pipe with convective boundaries.

$$q' = 2\pi r_2 h_2 (t_2 - t_{f2}) \quad \text{or} \quad t_2 - t_{f2} = \frac{q'}{2\pi r_2 h_2} \; . \tag{3.125}$$

Upon addition of Eq. 3.123, 3.124 and 3.125, we obtain:

$$q' = \frac{t_{f1} - t_{f2}}{\frac{1}{2\pi r_1 h_1} + \frac{\ln(r_2/r_1)}{2\pi k} + \frac{1}{2\pi r_2 h_2}} \tag{3.126}$$

or

$$q' = K(t_{f1} - t_{f2}) \tag{3.127}$$

where K is the overall heat transfer coefficient and has the following form:

$$K = \frac{1}{\frac{1}{2\pi r_1 h_1} + \frac{\ln(r_2/r_1)}{2\pi k} + \frac{1}{2\pi r_2 h_2}} \tag{3.128}$$

III. Long Solid cylinder with heat generation and prescribed boundary temperature

The mathematical formulation of the problem is given by Eq. 3.63. Assuming that the conductivity of the cylinder material is constant, this equation becomes:

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dt}{dr}\right) + \frac{q'''}{k} = 0 \; . \tag{3.129}$$

The boundary conditions are:

$$r = 0 \qquad \frac{dt(r)}{dr} = 0 \tag{3.130}$$

$$r = r_o \qquad t = t_w \tag{3.131}$$

The solution of Eq. 3.129 is:

$$t(r) = -\frac{q'''}{4k}r^2 + A\ln r + B \tag{3.132}$$

The application of boundary conditions shows that:

$$A = 0 \tag{3.133}$$

$$B = t_w + \frac{q'''}{4k}r_o^2 \ . \tag{3.134}$$

The temperature distribution is then given by:

$$t(r) = \frac{q''' r_c^2}{4k}\left[1 - \left(\frac{r}{r_o}\right)^2\right] + t_w \tag{3.135}$$

IV. Solid cylinder with heat generation and convective boundary condition

In this case the boundary conditions will be:

$$r = 0 \qquad \frac{dt(r)}{dr} = 0 \tag{3.136}$$

$$r = r_o \qquad -k\left(\frac{dt(r)}{dr}\right) = h[t(r) - t_f] \tag{3.137}$$

where t_f is the temperature of the convecting fluid. Using the above conditions we obtain for the integration constants A and B the following:

$$A = 0 \tag{3.138}$$

$$B = \frac{q'''}{4k}r_o^2 + \frac{q'''}{2h}r_o + t_f \tag{3.139}$$

The temperature distribution is given by:

The boundary conditions are:

$$r = 0 \qquad \frac{dt(r)}{dr} = 0 \tag{3.130}$$

$$r = r_o \qquad t = t_w \tag{3.131}$$

The solution of Eq. 3.129 is:

$$t(r) = -\frac{q'''}{4k}r^2 + A\ln r + B \tag{3.132}$$

The application of boundary conditions shows that:

$$A = 0 \tag{3.133}$$

$$B = t_w + \frac{q'''}{4k}r_o^2 \,. \tag{3.134}$$

The temperature distribution is then given by:

$$t(r) = \frac{q''' r_o^2}{4k}\left[1 - \left(\frac{r}{r_o}\right)^2\right] + t_w \tag{3.135}$$

IV. Solid cylinder with heat generation and convective boundary condition

In this case the boundary conditions will be:

$$r = 0 \qquad \frac{dt(r)}{dr} = 0 \tag{3.136}$$

$$r = r_o \qquad -k\left(\frac{dt(r)}{dr}\right) = h[t(r) - t_f] \tag{3.137}$$

where t_f is the temperature of the convecting fluid. Using the above conditions we obtain for the integration constants A and B the following:

$$A = 0 \tag{3.138}$$

$$B = \frac{q'''}{4k}r_o^2 + \frac{q'''}{2h}r_o + t_f \tag{3.139}$$

The temperature distribution is given by:

$$t(r) = \frac{q''' r_o^2}{4k}\left[1 - \left(\frac{r}{r_o}\right)^2\right] + \frac{q''' r_o}{2h} + t_f. \tag{3.140}$$

Example

Consider the long cylinder sketched in Fig. 3.21. The outer surface of the cylinder at $r = r_2$ is

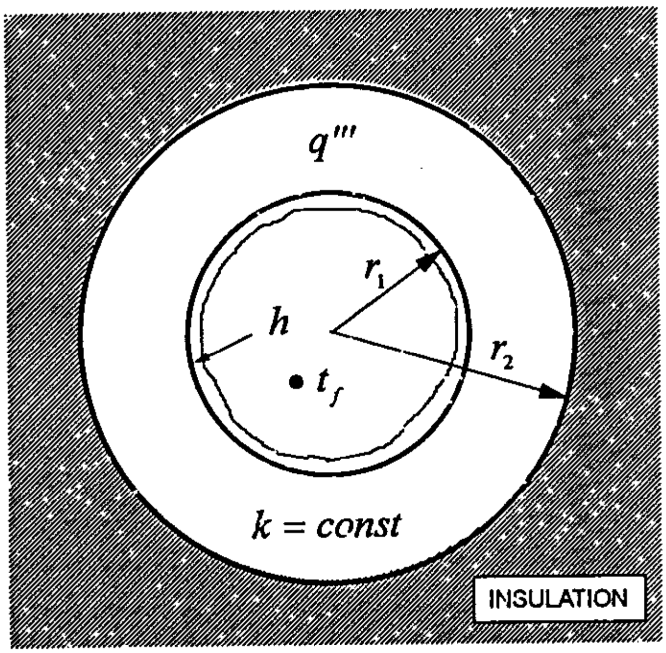


Figure 3.21 Long hollow cylinder with heat generation.

perfectly insulated whereas the inner surface at $r = r_1$ is cooled by convection. Heat is generated uniformly in the cylinder at a rate of q'''. Determine the temperature distribution in the cylinder at the point where the coolant temperature is t_f. The conductivity of the cylinder material is constant.

To determine the temperature distribution in the wall region, Eq. 3.63 should be solved subject to the following boundary conditions:

$$r = r_1 \qquad k\frac{dt(r)}{dr} = h[t(r) - t_f] \tag{3.141}$$

$$r = r_2 \qquad \frac{dt(r)}{dr} = 0 \tag{3.142}$$

The solution of Eq. 3.63 is given by Eq. 3.132. The integration constants are determined by using the above boundary conditions. The use of Eq. 3.142 gives:

$$-\frac{1}{2}\frac{q'''}{4k} r_2 + A\frac{1}{r_2} = 0 \tag{3.143}$$

or

$$A = \frac{q'''}{2k} r_2^2 \tag{3.144}$$

and Eq. 3.141 yields:

$$-\frac{q''' r_1}{2} + \frac{q'''}{2}\frac{r_2^2}{r_1} = h\left(-\frac{q'''}{4k}r_1^2 + \frac{q'''}{2k}r_2^2 \ln r_1 + B - t_f\right) \tag{3.145}$$

or

$$B = \frac{q''' r_1}{2h}\left(\frac{r_2^2}{r_1^2} - 1\right) + \frac{q''' r_1^2}{2k}\left(\frac{1}{2} - \frac{r_2^2}{r_1^2}\ln r_1\right) + t_f \tag{3.146}$$

Substituting Eqs. 3.144 and 3.146 into Eq. 3.132, we obtain the temperature distribution as:

$$t(r) = -\frac{q'''}{4k}r^2 + \frac{q'''}{4k}r_2^2 \ln r + \frac{q''' r_1}{2h}\left(\frac{r_2^2}{r_1^2} - 1\right) + \frac{q''' r_1^2}{2k}\left(\frac{1}{2} - \frac{r_2^2}{r_1^2}\ln r_1\right) + t_f \quad . \tag{3.147}$$

1.2.4 One Dimensional Time Dependent Conduction

In this section we will discuss transient conduction problems in a system. Transient heat transfer conditions are achieved when heat generation is suddenly started or stopped, or the boundary conditions of a heated body are suddenly changed. Under these conditions, the temperature at each point in the body will start changing. These changes will continue until a new equilibrium is reached between the energy created in the body and the energy removed from the body, or until an equilibrium temperature is reached between the hot body and the surrounding. To determine the temperature distribution within a solid during a transient process, we should solve general conduction equation (Eqs. 3.34 or 3.43, or 3.44) with appropriate boundary and initial conditions. For one dimensional geometry these equations reduce to:
Rectangular coordinates:

$$\frac{\partial}{\partial x}\left(k\frac{\partial t}{\partial x}\right) + q''' = c\rho\frac{\partial t}{\partial \tau} \tag{3.148}$$

Cylindrical coordinates:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(kr\frac{\partial t}{\partial r}\right) + q''' = c\rho\frac{\partial t}{\partial \tau} \tag{3.149}$$

Spherical coordinates:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(kr^2\frac{\partial t}{\partial r}\right) + q''' = c\rho\frac{\partial t}{\partial \tau}\,. \tag{3.150}$$

In certain class of problems, the spatial distribution of the temperature in solid body stays nearly uniform during the transient. Under this condition, we may assume that the temperature in the body is independent of space and varies only with time. The analysis of heat transfer with such an assumption is called the "*lumped system analysis.*" Since the temperature is a function of time only, the heat transfer analysis can be easily conducted. Because its simplicity, in this section the discussion of the transient heat transfer will start with lumped system analysis.

1.2.4.1 Lumped System Approach- System with High Conductivity

Lumped system approach assumes that the thermal conductivity of the solid object is so great that during a transient heat transfer process the temperature gradient within the object is small, i.e., the temperature, for all practical purposes, can be considered as uniform at any instant. To illustrate this approach, two examples that consist the immersion of a hot object in a quenching bath of infinite extent, i.e. , constant temperature and a solid object in which heat is suddenly generated and placed in a constant temperature surrounding will be considered. Both cases will be discussed by using macroscopic energy equation (2.23) which for the present case is written as:

$$\frac{d}{d\tau}\int_V \rho u dV = -\int_A \vec{n} \cdot \vec{q}\,'' dA + \int_V q''' dV \qquad (3.151)$$

where

u	:	internal energy per unit mass
ρ	:	density
$\vec{n}$	:	unit normal vector to the bounding surface
$\vec{q}\,''$	:	heat flux applied to the bounding surface
q'''	:	heat generation rate

Assuming that ρ, u and q''' are constant throughout the solid body , q'' is constant over the bounding surface of the body and knowing that:

$$\rho V \frac{du}{d\tau} = c\rho V \frac{dt}{d\tau} \qquad (3.152)$$

Eq. 3.151 becomes:

$$c\rho V \frac{dt}{d\tau} = -A\,\vec{n} \cdot \vec{q}\,'' + Vq''' \qquad (3.153)$$

where V and A are the volume and bounding surface area of the solid object, respectively. If $\vec{n} \cdot \vec{q}\,''$ is positive, heat flows out of the object, if it is negative, heat flows into the object.

I. Immersion of a high thermal conductivity solid body in a quenching bath

Consider a solid body at an initial temperature t_i immersed suddenly in a quenching bath of infinite extent, i.e. , at a constant temperature t_f as sketched in Fig. 3.22. Assuming that the material of the body has a high thermal conductivity, the gradient within the body will be small, consequently, the temperature distribution will be uniform and almost equal to the surface temperature. The heat transfer from the body to the surrounding is controlled by convection. Under these conditions, the term $A\,\vec{n} \cdot \vec{q}\,''$ Eq. 6.153 is positive and given by:

$$A\,\vec{n} \cdot \vec{q}\,'' = Aq''_{cv} = Ah_c(t - t_f). \qquad (3.154)$$

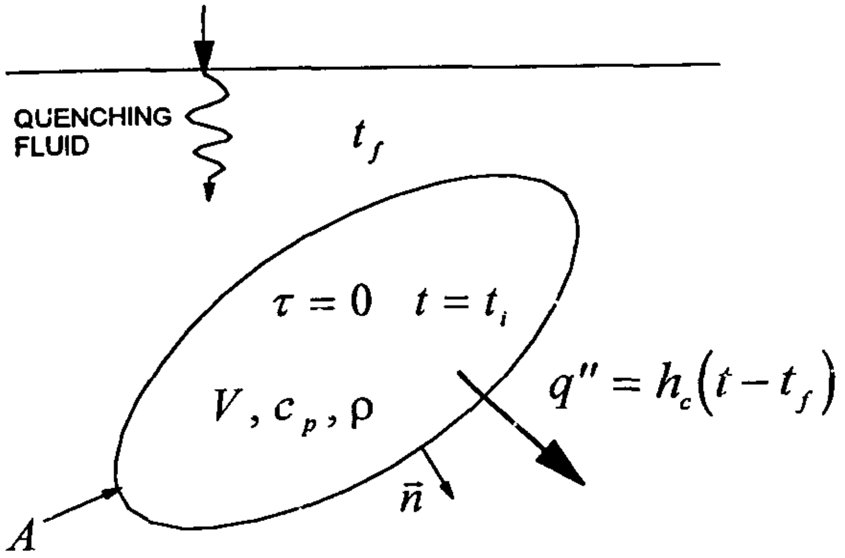


Figure 3.22 Solid object in an infinite quenching bath.

There is no heat generation in the body. Therefore, Eq. 6.153 becomes:

$$\frac{dt}{d\tau} = -\frac{Ah_c}{c\rho V}(t - t_f) \tag{3.155}$$

with initial condition:

$$\tau = 0 \qquad t = t_i . \tag{3.156}$$

Introducing the following variable change:

$$\theta = t - t_f \tag{3.157}$$

Eq. 3.155 can be written as:

$$\frac{d\theta}{d\tau} = -\frac{Ah_c}{c\rho V}\theta \tag{3.158}$$

with boundary condition:

$$\theta_i = t_i - t_f . \tag{3.159}$$

The solution of Eq. 3.158 subject to initial condition is given by:

$$\theta = \theta_i \exp\left(-\frac{Ah_c}{c\rho V}\tau\right) \tag{3.160}$$

or

$$t - t_f = (t_i - t_f)\exp\left(-\frac{Ah_c}{c\rho V}\tau\right) . \tag{3.161}$$

The quantity $c\rho V/Ah_c$ the "thermal time constant" for the geometry under consideration and has the dimension of time. The numerator of the time constant, $c\rho V$, is called is called the "lumped thermal capacitance," and $1/Ah_c$ is known as the convective resistance.

Let us give a closer look the exponent of Eq. 3.161 and rearrange it as follows:

$$\frac{h_c A \tau}{c\rho V} = \frac{h_c V}{k}\frac{k}{A}\frac{\tau A^2}{c\rho\, V^2} = \frac{h_c V}{kA}\frac{\alpha \tau A^2}{V^2} \tag{3.162}$$

with $\alpha = k/c\rho$. We observe that the time constant does not contain the thermal conductivity; we introduced it by multiplying the numerator and the denominator of Eq. 3.162 by the thermal conductivity, k. The ratio of volume to bounding surface area of the body is called the "characteristic length," i.e.,

$$L_c = \frac{V}{A}. \tag{3.163}$$

With this definition, Eq. 3.162 becomes:

$$\frac{h_c A \tau}{c\rho V} = \frac{h_c L_c}{k}\frac{\alpha \tau}{L_c^2}. \tag{3.164}$$

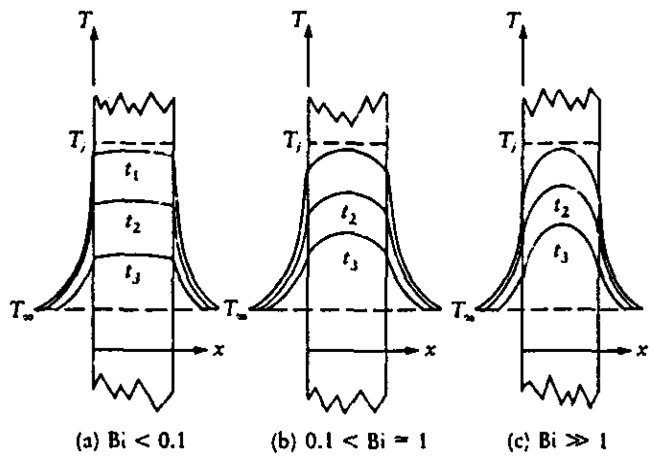


Figure 3.23 Relationship between the Biot number and the temperature profile.

The term $h_c L_c/k\ (= h_c V/kA)$ is known as "Biot number." The term $\alpha\tau/L_c^2\ (= \alpha\tau A^2/V^2)$ is known as "Fourier number." The Biot number is a dimensionless ratio of convection coefficient to thermal conductivity and gives an indication of the temperature drop within the solid body compared to the temperature difference between the solid surface and the fluid. If the Biot number:

$$Bi = \frac{h_c}{k} L_c = \frac{h_c V}{kA} \leq 0.1 \tag{3.165}$$

then Eq. 6.161 can be used with little error. Therefore, the criterion for the use of lumped system approach is appropriate when Biot number is less than 0.1. The effect of the Biot number on the temperature distribution in the solid body is illustrated in Fig. 3.23. Fig. 3.23a shows that when $Bi < 0.1$, the temperature distribution is nearly flat and the convection heat transfer coefficient is the controlling parameter. Fig. 3.23c shows that $Bi >> 1$, the conduction process controls the heat transfer. In turn, Fig. 3.23b shows that for $0.1 < Bi < 1$ both conduction and convection should be accounted for.

The Fourier number is a dimensionless time parameter. It represents the ratio of heat transfer by conduction to the energy storage rate within the body. In terms of dimensionless numbers, Eq. 3.161 is written as:

$$t - t_f = (t - t_i)\exp(-Bi.Fo). \tag{3.166}$$

II . Sudden heat generation in a solid body

Consider the solid body sketched in Fig. 3.24. Initially the body is in equilibrium with the surrounding which has an infinite extent. The temperature of the surrounding is t_f and it is constant. At time zero, heat is suddenly generated in the body at a rate of $q^{'''}$ W/m^3 . The conductivity of the material is great and heat transfer from the body to the surrounding is controlled by convection only. We wish to determine the variation of the body temperature with time.

Lumped system approach can also be used in this case to determine the temperature history of the body. The application of Eq. 3.153 to the present case yields:

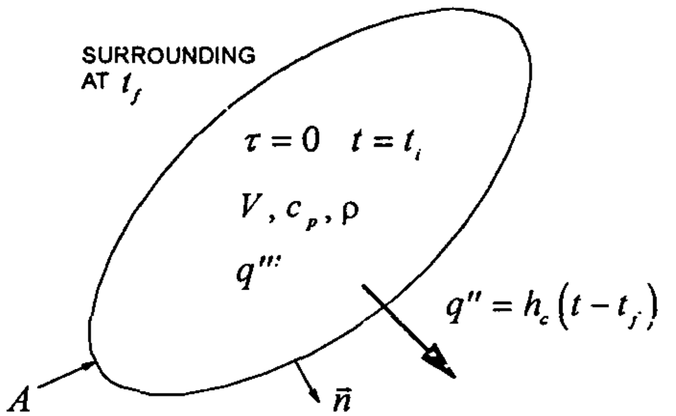


Figure 3.24 Solid body with heat generation.

$$\frac{dt}{d\tau} = -\frac{Ah_c}{c\rho V}(t - t_f) + \frac{q^{'''}}{c\rho} \tag{3.167}$$

or

$$\frac{d\theta}{d\tau} = -\frac{Ah_c}{c\rho V}\theta + \frac{q'''}{c\rho} \tag{3.168}$$

where $\theta = t - t_f$. The initial condition is:

$$\tau = 0 \qquad t = t_f \quad \text{or} \quad \theta = 0 \tag{3.169}$$

The solution of Eq. 3.168 is given by:

$$\theta = B\exp\left(-\frac{Ah_c}{c\rho V}\tau\right) + \frac{q'''V}{Ah_c}. \tag{3.170}$$

Using the initial condition, the constant, B, is determined as:

$$B = -\frac{q'''V}{Ah_c}. \tag{3.171}$$

Substituting Eq. 3.171 into Eq. 3.170 and knowing that $\theta = t - t_f$, we obtain for the variation of the temperature with time the following expression:

$$t - t_f = \frac{q'''V}{Ah_c}\left[1 - \exp\left(-\frac{Ah_c}{c\rho V}\tau\right)\right]. \tag{3.172}$$

In the above solution, if $\tau = 0$, $t = t_f$; if $\tau \to \infty$, the temperature of the body becomes:

$$t = t_f + \frac{Vq'''}{A\,h_c}. \tag{3.173}$$

1.2.4.2 Systems with High Surface Conductance

We will discuss now the systems where the convection heat transfer coefficient (film conductance) is very high. Therefore, the surface temperature of the object, for all practical purposes, is equal to the temperature of the surrounding fluid. Because of the shape of the fuel rods used in nuclear reactors, we will only consider transient conduction in solid cylinders. We will assume that the cylinder is infinitely long and axial symmetry exists. Under these conditions, the transient problem will have two independent variables: radial coordinate, r, and time, τ. In the absence of heat sources and constant conductivity, the heat conduction equation (Eq. 3.148) reduces to:

$$\frac{\partial t}{\partial \tau} = \alpha\left(\frac{\partial^2 t}{\partial r^2} + \frac{1}{r}\frac{\partial t}{\partial r}\right). \tag{3.174}$$

The solution of this equation can be obtained by using the method of separation of variables. Therefore, a solution in the following form will be sought:

$$t(r,\tau) = \mathbf{R}(r)\mathbf{T}(\tau). \tag{3.175}$$

Substituting Eq. 3.175 into Eq. 3.174, we obtain:

$$\frac{1}{\alpha \mathbf{T}}\frac{\partial \mathbf{T}}{\partial \tau} = \frac{1}{\mathbf{R}}\left(\frac{\partial^2 \mathbf{R}}{\partial r^2} + \frac{1}{r}\frac{\partial \mathbf{R}}{\partial r}\right) = -\lambda^2. \tag{3.176}$$

λ^2 is a separation constant introduced because of the fact that each member of this equation is a function of only one of the variable and that the equality between the members is only possible when both of them are equal to the same constant. The separation constant is taken to be negative to obtain a negative exponential solution in time.

Eq. 3.175 yields two ordinary differential equations:

$$\frac{d\mathbf{T}}{d\tau} + \lambda^2 \alpha \mathbf{T} = 0 \tag{3.177}$$

and

$$\frac{d^2\mathbf{R}}{dr^2} + \frac{1}{r}\frac{d\mathbf{R}}{dr} + \lambda^2 \mathbf{R} = 0. \tag{3.178}$$

The solution of these equations are:

$$\mathbf{T}(\tau) = A\exp(-\lambda^2 \alpha \tau) \tag{3.179}$$

and

$$\mathbf{R}(r) = CJ_o(\lambda r) + DY_o(\lambda r), \tag{3.180}$$

respectively. J_o and Y_o are zero order Bessel functions of the first and second kind, respectively. Since the cylinder is solid, Y_o is undefined when $r \to 0$. Consequently, to obtain a meaningful solution to a physical problem, the constant D should be equal to zero. The solution of Eq. 3.174 is then given by:

$$t(r,\tau) = B\exp(-\lambda^2 \alpha \tau) J_o(\lambda r) \tag{3.181}$$

where $B = AC$. The constants B and λ are to be determined by initial and boundary conditions.

Consider now a solid cylinder of radius r_o subject to an initial temperature distribution $f(r)$ which is symmetrical with respect to the axis of the cylinder. Assume that the temperature of the surface of the cylinder is suddenly reduced to 0 °C (or to any other constant temperature) and maintained at that value for all subsequent times. This is equivalent to immerse the cylinder in an infinite surrounding at temperature t_f with very high heat transfer coefficient such as seen, for example, under boiling liquid conditions. What is the temperature distribution in the cylinder as a function of space and time.

The temperature distribution in the solid cylinder is given by Eq. 3.181 subject the following initial and boundary conditions:

$$\tau = 0 \qquad t = f(r) \tag{3.182}$$

$$\tau \geq 0 \qquad t = 0 \quad \text{at} \quad r = r_o \,. \tag{3.183}$$

The application of the boundary condition given by Eq. 3. 183 to Eq. 3.181 results in:

$$J_o(\lambda\, r_o) = 0\,. \tag{3.184}$$

This equation has infinite number of roots (λ_n with $n = 1, 2, 3, ...\infty$) and each root correspond to a particular solution of Eq. 3.174. The general solution of Eq. 3.174 is then given by:

$$t(r,\tau) = \sum_{n=1}^{\infty} B_n \exp\left(-\lambda_n^2 \alpha\tau\right) J_o(\lambda_n r). \tag{3.185}$$

The application of initial condition given by Eq. 3.183 to the above solution leads to:

$$f(r) = \sum_{n=1}^{\infty} B_n J_o(\lambda_n r)\,. \tag{3.186}$$

Since λ_n's are defined as the roots of Eq. 3.184, the set of functions:

$$\{J_o(\lambda_n r)\} \qquad n = 1, 2, 3, ...\infty \tag{3.187}$$

as discussed in Appendix III, constitutes a set of orthogonal functions. The constants B_n's appearing in Eq. 3.186 can, therefore, be determined by using the properties of the orthogonal functions. According to Appendix III, these constants are given by:

$$B_n = \frac{\int_o^{r_o} r f(r) J_o(\lambda_n r) dr}{\frac{r_o^2}{2} J_1^2(\lambda_n r_o)} \tag{3.188}$$

where J_1 is first order Bessel function of the first kind. The final solution is obtained by substituting Eq. 3.188 into Eq. 3.186:

$$t(r,\tau) = \frac{2}{r_o^2} \sum_{n=1}^{\infty} \exp\left(-\lambda_n^2 \alpha\tau\right) \frac{J_o(\lambda_n r)}{J_1^2(\lambda_n r_o)} \int_o^{r_o} r f(r) J_o(\lambda_n r) dr\,. \tag{3.189}$$

1.2.4.3 System with Finite Internal Conductivity and Surface Conductance

In this case both conductances (internal and surface) have finite values. The long cylinder discussed in the previous section is now immersed in a fluid of finite heat transfer coefficient h. The temperature distribution is still given by Eq. 3.181. The only difference is in the boundary conditions:

$$\tau = 0 \qquad t = f(r) \tag{3.190}$$

$$\tau \geq 0 \qquad \frac{\partial t}{\partial r} = \frac{h}{k} t \quad \text{at } r = r_o \tag{3.191}$$

In Eq. 3.191 it is assumed that $t_f = 0$. This assumption does not affect the generality of the solution. If t_f were different from zero, we would simply change the reference temperature and write: $\theta = t - t_f$. This change would give the same result as $t_f = 0$. The application of boundary condition given by Eq. 3.191 leads to:

$$B\exp(-\lambda^2\alpha\tau)\left[\frac{\partial J_o(\lambda r)}{\partial r}\right]_{r=r_o} = -\frac{h}{k}[B\exp(-\lambda^2\alpha\tau)J_o(\lambda r)]_{r=r_o} \tag{3.192}$$

Knowing that:

$$\frac{\partial J_o(\lambda r)}{\partial r} = -\lambda J_1(\lambda r) \tag{3.193}$$

Eq. 3.192 becomes:

$$\lambda r_o \frac{J_1(\lambda r_o)}{J_o(\lambda r_o)} = \frac{hr_o}{k}. \tag{3.194}$$

The above equation has infinite number of roots (λ_n with $n = 1, 2, 3, \ldots\ldots\infty$) and each root corresponds to a particular solution of Eq. 3.174. The general solution is given by:

$$t(r,\tau) = \sum_{n=1}^{\infty} B_n \exp(-\lambda_n^2\alpha\tau) J_o(\lambda_n r) \tag{3.195}$$

where B_n are constants to be determined. Upon application of initial condition given by Eq. 3.190, we obtain:

$$f(r) = \sum_{n=1}^{\infty} B_n J_o(\lambda_n r) \ . \tag{3.196}$$

Referring to Appendix III and comparing Eqs. 3.195 and III.31, we conclude that the set:

$$\{J_o(\lambda_n r)\} \tag{3.197}$$

constitutes a set of orthogonal functions. The constants B_n in Eq. 3.196 can then be determined by using the properties of orthogonal functions and according to III.32 in Appendix III have the following form:

$$B_n = \frac{\frac{2}{r_o^2}\int_o^{r_o} rf(r) J_o(\lambda_n r)dr}{J_o^2(\lambda_n r_o) + J_1^2(\lambda_n r_o)}. \tag{3.198}$$

Finally the temperature distribution is given by:

$$t(r,\tau) = \frac{2}{r_o^2}\sum_{n=1}^{\infty} \exp(-\lambda_n^2\alpha\tau)\frac{J_o(\lambda_n r)}{J_o^2(\lambda_n r_o) + J_1^2(\lambda_n r_o)}\int_o^{r_o} rf(r) J_o(\lambda_n r)\, dr \tag{3.199}$$

1.2.4.4 System with Finite Internal Conductivity and Surface Conductance and Subject to Sudden Heat Generation

A long solid cylinder of radius r_o has an initial temperature distribution $f^*(r)$ which is symmetrical with respect to the axis of the cylinder. For times $t \geq 0$ heat is generated in this cylinder at a constant rate of $q''' \; Watt/m^3$. The boundary surface of the cylinder is subject to convection with an infinite surrounding at temperature $t_f = 0$ °C. The convection heat transfer coefficient is constant and equal to h_c. Determine the temperature distribution as a function of space and time in the cylinder.

The difference between this case and the two cases studied above is the sudden heat generation in the solid cylinder. Under this condition, the mathematical formulation of the problem is written as:

$$\frac{\partial^2 t}{\partial r^2} + \frac{1}{r}\frac{\partial t}{\partial r} + \frac{q'''}{k} = \frac{1}{\alpha}\frac{\partial t}{\partial \tau}. \tag{3.200}$$

The initial and boundary conditions are specified as:

$$\tau = 0 \qquad t = f^*(r) \quad \text{for} \quad 0 \leq r \leq r_o, \tag{3.201}$$

$$\tau \geq 0 \qquad -k\frac{\partial t(r)}{\partial r} = h_c\, t(r) \quad \text{for} \quad r = r_o. \tag{3.202}$$

Moreover, the temperature should have a finite value at $r = 0$.

Because of the presence of the term q'''/k, Eq. 3.200 is a nonhomogeneous differential equation and its solution can not be obtained by the method of separation of variables. To get around of this difficulty, we will assume that the solution of this equation has the following form:

$$t(r,\tau) = t_h(r,\tau) + t_s(r). \tag{3.203}$$

Substituting Eq. 3.203 into Eq. 3.200, we obtain:

$$\frac{\partial^2 t_h}{\partial r^2} + \frac{1}{r}\frac{\partial t_h}{\partial r} - \frac{1}{\alpha}\frac{\partial t_h}{\partial \tau} = -\frac{\partial^2 t_s}{\partial r^2} - \frac{1}{r}\frac{\partial t_s}{\partial r} - \frac{q'''}{k}. \tag{3.204}$$

Since the left hand side of this equation is a function of r and τ and the right hand side is a function of r only, the equality of both sides is only possible if they are equal to the same constant. If this constant is different from zero, we still obtain a nonhomogeneous equation. The only possibility toward a solution is that this constant be equal to zero. Therefore, Eq. 3.204 yields two differential equations:

$$\frac{\partial^2 t_s}{\partial r^2} + \frac{1}{r}\frac{\partial t_s}{\partial r} + \frac{q'''}{k} = 0 \tag{3.205}$$

and

$$\frac{\partial^2 t_h}{\partial r^2}+\frac{1}{r}\frac{\partial t_h}{\partial r}-\frac{1}{\alpha}\frac{\partial t_h}{\partial \tau}=0\,. \qquad (3.206)$$

An examination of the above equations shows that the problem is split into a steady state problem for $t_s(r)$, (Eq. 3.205), and into a homogeneous transient problem for $t_h(r,\tau)$, (Eq. 3.206). The nonhomogeneity q'''/k is included in the steady state problem whereas the transient nature of the problem is included in the homogeneous equation. The initial and boundary conditions for Eqs. 3.205 and 3.206 are obtained by combining Eqs. 3.203, and Eqs. 3.201 and 3.202:

$$\tau=0 \qquad t(r,0)=t_h(r,0)+t_s(r) \qquad (3.207)$$

$$\tau\ge 0 \qquad -k\frac{\partial t_h(r,\tau)}{\partial r}-k\frac{\partial t_s(r)}{\partial r}=ht_h(r,\tau)+ht_s(r) \quad \text{at } r=r_o\,. \qquad (3.208)$$

Since

$$t(r,0)=f^*(r) \qquad (3.209)$$

Eq. 3.207 becomes:

$$\tau=0 \qquad t_h(r,0)=f^*(r)-t_s(r)=f(r)\,. \qquad (3.210)$$

Eq. 3.208 can be written as:

$$\tau\ge 0 \qquad -k\frac{\partial t_h(r,\tau)}{\partial r}-ht_h(r,\tau)=k\frac{\partial t_s(r)}{\partial r}+ht_s(r) \quad \text{at } r=r_o \qquad (3.211)$$

We can easily see that the above equality is only possible if both sides are equal to the same constant and this constant can not be anything else but zero. Consequently, the boundary condition given by Eq. 3.211 becomes:

$$-k\frac{\partial t_s(r)}{\partial r}=ht_s(r) \qquad \text{at } r=r_o \qquad (3.212)$$

and

$$\tau\ge 0 \qquad -k\frac{\partial t_h(r,\tau)}{\partial r}=h\,t_h(r,\tau) \qquad \text{at } r=r_o \qquad (3.213)$$

Eq. 3.212 constitutes the boundary condition for Eq. 3.205, and Eqs. 3.210 and 3.213 constitute the initial and boundary conditions for Eq. 3.206.

The solution of Eq. 3.205, subject to boundary condition specified by Eq. 3.212, is given by Eq. 3.140 with $t_f=0$:

$$t_s(r)=\frac{q'''}{4k}\left[1-\left(\frac{r}{r_o}\right)^2\right]+\frac{q'''r_o}{2h} \qquad (3.214)$$

In turn, the solution of Eq. 3.206, subject to initial and boundary conditions specified with Eqs. 3.210 and 3.213, is given by Eq. 3.199. Therefore the final solution is:

$$t(r,\tau) = t_s(r) + t_h(r,\tau) = \frac{q'''}{4k}\left[1-\left(\frac{r}{r_o}\right)^2\right] + \frac{q''' r_o}{2h}$$

$$+\frac{2}{r_o^2}\sum_{n=1}^{\infty}\exp(-\lambda_n^2\alpha\tau)\frac{J_o(\lambda_n r)}{J_o^2(\lambda_n r_o)+J_1^2(\lambda_n r_o)}\int_o^{r_o} rf(r)J_o(\lambda_n r)dr \qquad (3.215)$$

where $f(r)$ is defined with Eq. 3.210.

If initially, the cylinder were in equilibrium with surrounding, $f^*(r)$ would be zero and $f(r)$ would be:

$$f(r) = -t_s(r). \qquad (3.216)$$

Under this condition the temperature distribution is given by:

$$t(r,\tau) = t_s(r) + t_h(r,\tau) = \frac{q'''}{4k}\left[1-\left(\frac{r}{r_o}\right)^2\right] + \frac{q''' r_o}{2h}$$

$$-\frac{2}{r_o^2}\sum_{n=1}^{\infty}\exp(-\lambda_n^2\alpha\tau)\frac{J_o(\lambda_n r)}{J_o^2(\lambda_n r_o)+J_1^2(\lambda_n r_o)}\int_o^{r_o} r\,t_s(r)J_o(\lambda_n r)dr\,. \qquad (3.217)$$

In the above temperature distribution when $\tau \to \infty$, the solution tends toward the steady state temperature distribution. When $\tau = 0$, the second term of the equation is nothing else but the development in series of the first term, $t_s(r)$. Therefore $t(r,0) = 0$; this is the initial condition.

Fig. 3.25 compares for a given time, τ, the transient temperatures with steady state temperatures.This figure shows that at each time, τ, the steady state temperatures, $t_s(r)$, are subtracted by an amount of $t_h(r,\tau)$ to obtain the transient temperature distribution, $t(r,\tau)$.

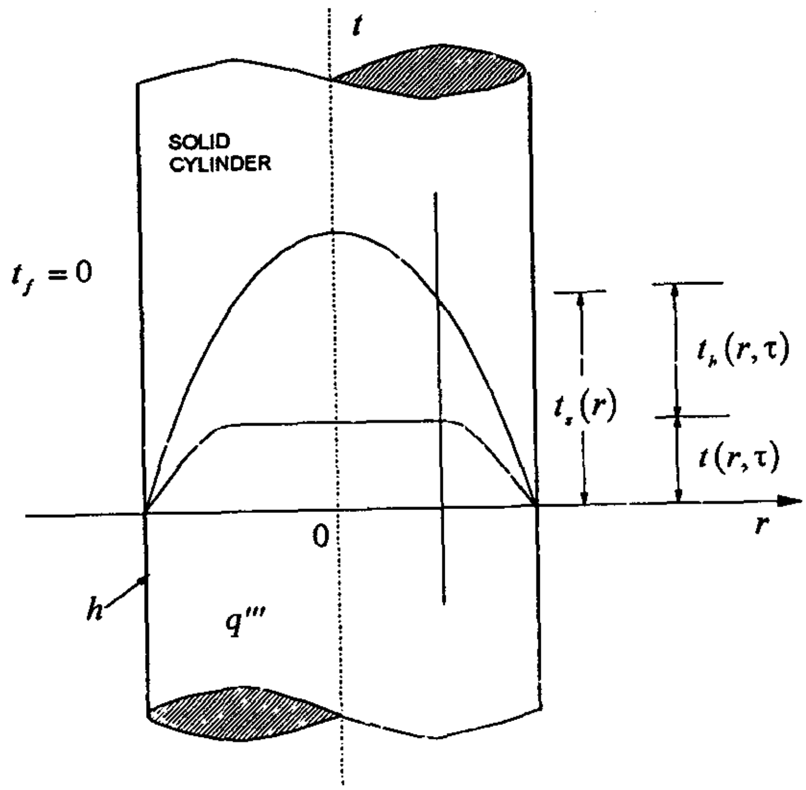


Figure 3.25 Comparison of the transient and steady state temperatures for a given time.