CHAPTER 1

FUNDAMENTAL CONCEPTS

This chapter deals with some basic definitions and mathematical tools we will use in the derivation of local and macroscopic fluid equations.

1.1 Definitions: Continuum, System and Control volume

Continuum

The continuum is a medium in which the smallest volume under consideration contains enough molecules to permit the statistically averaged characteristics to adequately describe the medium. A continuum can also be defined as a substance that is physically continuous. Therefore, it possesses properties that, if not constant, vary in a continuous manner. By a property of a substance we mean any observable quantity that describes its state. The concept of a continuum cannot, obviously, extended to a non homogeneous fluid. A pure or impure substance in which two phases are present cannot be considered as a continuum, although each of the phases may be so considered.

System

System is a part of a continuum that is separated from the rest for convenience in the formulation of a problem. The boundaries of a system may expend or contract, but it is assumed that the rest of the continuum does not cross them during any change of the system. However, energy and/or momentum can cross these boundaries.

Control volume

The same as a system, except that the rest of the continuum may cross the fixed or deformable boundaries of the control volume at one or more places.

1.2 Material and Spatial Coordinates

The study of the motion of fluids treated as continuum can be carried out in two different ways:

1. by inquiring how the positions in space, velocities, accelerations and state of <u>individual fluid elements</u> vary with time, or
2. by inquiring about the distribution of the flow velocity, acceleration and state of a fluid in a <u>given region of space</u> and the variation of these distributions with time.

The description of fluid motion concerned with the motion of every fluid element is known as the <u>material description</u>, while that concerned with the state of motion at each spatial position as the <u>spatial description</u>.

At some instant, let us look at the fluid and identify a fluid particle located at a given position. This position can be uniquely described by a position vector $\vec{R}$ that originates from an arbitrarily chosen fix point O (Fig. 1.1). If we also chose a coordinate system in which the motion of the fluid is to be studied and set its origin to the fix point O, then this position vector can be described by a unique coordinate triple X, Y, Z. Later, the same particle is located at a position $\vec{r}$ that

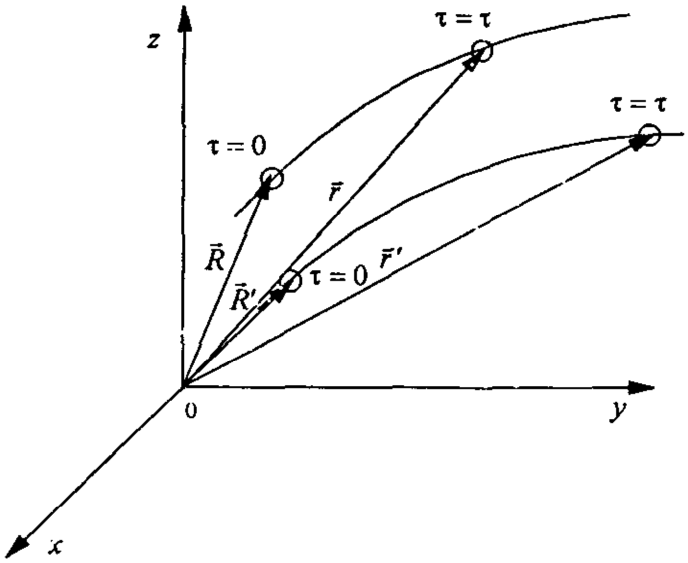

Figure 1.1 Motion of a fluid element at times 0 and τ.

originates from the same fixed point *O*. Without loss of generality, we can take the first instant to be time 0 and later instant to be the time τ. Similarly, if a particle was initially at $\vec{R}'$ at time 0 it would be at $\vec{r}'$ at time τ. Consequently, the instantaneous position of a fluid particle corresponding to $\vec{r}$ is a function of the initial position $\vec{R}$ and time τ. This statement, in the mathematical form, can be presented with the following equation:

$$\vec{r}=\vec{r}\,(\vec{R},\tau) \qquad \text{or} \qquad \begin{aligned} x &= x(X,Y,Z,\tau) \\ y &= y(X,Y,Z,\tau) \\ z &= z(X,Y,Z,\tau) \end{aligned} \tag{1.1}$$

x,y,z are the coordinate triple corresponding to the vector $\vec{r}$. The coordinate triple (X,Y,Z) which specify the initial position vector $\vec{R}$ of the fluid particle is referred to as the "material coordinates" of the particle. In a general manner, we can consider that every fluid particle has, corresponding to it, an initial position vector $\vec{R}$ and hence an initial coordinate triple (X,Y,Z). The coordinate triple (x,y,z) which specify the position vector $\vec{r}$ of the fluid particle is known as "spatial coordinates." It will be assumed that the motion is always continuous, single valued and that Eq. (1.1) can be inverted to give the initial position or material coordinates of the particle that is at any position $\vec{r}$ at time τ; that is:

$$\vec{R}=\vec{R}\,(\vec{r},\tau) \qquad \text{or} \qquad \begin{aligned} X &= X(x,y,z,\tau), \\ Y &= Y(x,y,z,\tau), \\ Z &= Z(x,y,z,\tau). \end{aligned} \tag{1.2}$$

These equations are continuous and single valued. Physically this means that a continuous arc of fluid particles does not break up during the motion or that the particles in the neighborhood of a given particle continue in its neighborhood during the motion. The single valuedness of the equations means that a particle can not split up and occupy two places nor can two distinct particles occupy the same place. Assumptions must also be made about the continuity of the derivatives. The necessary condition for the inverse function to exits is the Jacobian be different from 0:

$$J=\frac{\partial(x,y,z)}{\partial(X,Y,Z)}=\begin{pmatrix} \frac{\partial x}{\partial X} & \frac{\partial x}{\partial Y} & \frac{\partial x}{\partial Z} \\ \frac{\partial y}{\partial X} & \frac{\partial y}{\partial Y} & \frac{\partial y}{\partial Z} \\ \frac{\partial z}{\partial X} & \frac{\partial z}{\partial Y} & \frac{\partial z}{\partial Z} \end{pmatrix} \neq 0. \tag{1.3}$$

We can easily see that, for a given $\vec{R}$, Eq. 1.1 describes the path of a fluid particle that initially occupies this position. In turn, for a given time τ, the same equation describes the transformation of the region in space occupied by a fluid particle at time 0 into a region occupied by the same particle at time τ.

1.3 Material and Spatial Derivatives

The description of a flow parameter such as specific mass, velocity, acceleration, temperature, etc., can be done using material or spatial coordinates. The material description in which the observer travels with the labeled fluid particle and records the changes in flow parameters is given by:

$$\Psi=\Psi(\vec{R},\tau). \tag{1.4}$$

In turn, the spatial description in which the observer monitors a given point in the space and records the flow parameters of a succession of fluid particles is given by:

$$\Psi=\Psi(\vec{r},\tau). \tag{1.5}$$

Using Eq. (1.2) the material description of the flow parameters can changed into spatial description as:

$$\Psi=\Psi\left[\vec{R}(\vec{r},\tau),\tau\right] \tag{1.6}$$

and conversely, the spatial description into material description by using Eq. (1.1):

$$\Psi = \Psi\left[\vec{r}\,(\,\vec{R},\tau),\tau\right] \tag{1.7}$$

Associated with material and spatial descriptions, there are two derivatives with respect to time:

$\dfrac{d}{d\tau}$ or $(\dfrac{\partial}{\partial\tau})_{\vec{R}}$: derivative with respect time while keeping $\vec{R}$ constant,

$\dfrac{\partial}{\partial\tau}$ or $(\dfrac{\partial}{\partial\tau})_{\vec{r}}$: derivative with respect time while keeping $\vec{r}$ constant.

We must distinguishes between the time derivative of a scalar vector field function ,Ψ , as we fallow the fluid particle which has been initially at the location $\vec{R}$ and the time derivative of the same function at a fixed position in space corresponding to $\vec{r}$. The first derivative, $d\Psi/d\tau$, is the rate of change of Ψ an observer will record when moving with the particle and it is called the "*material derivative*." The second derivative, $\partial\Psi/\partial\tau$, is the rate of change an observer will record at a fixed point $\vec{r}$ and is called the "*spatial derivative*."

Knowing that the derivative of the position of a particle is its velocity:

$$\vec{v} = \frac{d\,\vec{r}}{d\tau} \tag{1.8}$$

we can establish a connection between the *material and spatial derivatives*. Let us take the derivative of Eq. 1.7 by keeping $\vec{R}$ constant:

$$\frac{d\Psi}{d\tau} = \frac{\partial}{\partial\tau}\Psi\left[\vec{r}\,(\vec{R},\tau),\tau\right] = \frac{\partial\Psi}{\partial x}\left(\frac{\partial x}{\partial\tau}\right)_X + \frac{\partial\Psi}{\partial y}\left(\frac{\partial y}{\partial\tau}\right)_Y + \frac{\partial\Psi}{\partial z}\left(\frac{\partial z}{\partial\tau}\right)_Z + \left(\frac{\partial\Psi}{\partial\tau}\right), \tag{1.9}$$

since $\partial x/\partial\tau, \partial y/\partial\tau, \partial z/\partial\tau$ are the components of the fluid particle velocity and calling them u, v, w we obtain:

$$\frac{d\Psi}{d\tau} = u\frac{\partial\Psi}{\partial x} + v\frac{\partial\Psi}{\partial y} + w\frac{\partial\Psi}{\partial z} + \frac{\partial\Psi}{\partial\tau}. \tag{1.10}$$

It is convenient to write the above equation as:

$$\frac{d\Psi}{d\tau} = \frac{\partial\Psi}{\partial\tau} + \vec{v}\,.(\vec{\nabla}\,\Psi) \tag{3.11}$$

Ψ may a scalar or vector field function. This equation can be interpreted as follows: at a given point in space, the function Ψ can change with time while at a given time, the same function changes, in general, from one point to another in space. Since the material time derivative represents the rate of change of Ψ as the observer follows a moving fluid element, it must represent the sum of rates of both these changes.

If Ψ is interpreted as the velocity , $\vec{v}$, of the fluid particle, we obtain:

$$\vec{a}=\frac{d}{d\tau}\vec{v}=\frac{\partial}{\partial\tau}\vec{v}+\vec{v}\,.\,\vec{\nabla}\vec{v} \quad . \tag{1.12}$$

Note that $\vec{\nabla}\vec{v}$ is a dyadic product of two vectors (Appendix I). This equation is the acceleration of a fluid particle which represents the rate of change of velocity experienced by a fluid particle in its motion. The term $\partial\vec{v}/\partial\tau$ is called the *local acceleration* , since it represents the acceleration felt by a fix observer at a given location in space. The local acceleration exists only in an unsteady flow. The term $\vec{v}\,.\,\vec{\nabla}\vec{v}$, which represents the acceleration experienced at a given instant by an observer moving in space, is known as the *convective acceleration*. The components of the acceleration vector are as follows:

$$\frac{du}{d\tau}=u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}+\frac{\partial u}{\partial\tau}\,, \tag{1.13}$$

$$\frac{dv}{d\tau}=u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}+\frac{\partial v}{\partial\tau}\,, \tag{1.14}$$

$$\frac{dw}{d\tau}=u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}+\frac{\partial w}{\partial\tau}\,. \tag{1.15}$$

1.4 Dilatation (Euler's Expansion Formula)

In Cartesian coordinates, the volume of a volume element dV with sides dx, dy, dz is given by:

$$dV_{x,y,z}=dx\,dy\,dz \tag{1.16}$$

However, for convenience we may decide to work with an other coordinates system which is designated by ξ,η,ζ and the relationship between the two coordinates systems is defined by the following equations:

$$\begin{aligned} x&=x(\xi,\eta,\zeta),\\ y&=y(\xi,\eta,\zeta),\\ z&=z(\xi,\eta,\zeta). \end{aligned} \tag{1.17}$$

The above equations can be regarded as a mapping from the ξ,η,ζ space to the x,y,z space. The relationship between the elementary volume $dV_{x,y,z}$ and the elementary volume in the new system of coordinates $dV_{\xi,\eta,\zeta}(=d\xi\,d\eta\,d\zeta)$ according to the general mapping theory (Kaplan, 1972) is given by:

$$dV_{x,y,z}=J\,dV_{\xi,\eta,\zeta} \tag{1.18}$$

where J is the Jacobian determinant of the mapping. From the above equation we can conclude that the Jacobian measures the ratio of the elementary volumes $dV_{x,y,z}$ and $dV_{\xi,\eta,\zeta}$.

Taking ξ, η, ζ as the material coordinates X, Y, Z , they are, thus, Cartesian coordinates at time 0 and $dXdYdZ$ is the initial volume dV_o of an elementary parallelepiped about the point $\vec{R}$ as shown in Fig. 1.2. Because of the motion, this parallelepiped is moved and distorted but since the motion is continuous it can not break up and at time τ, it is at the neighborhood of the point $\vec{r}=\vec{r}\,(\vec{R},\tau)$ (Fig. 1.2) . Since, on the other hand, for a given time τ , the equation:

$$x = x(X, Y, Z, \tau)$$

$$y = y(X, Y, Z, \tau) \tag{1.19}$$

$$z = z(X, Y, Z, \tau)$$

describes a transformation of a region which was occupied by a fluid at time 0 into a region which is occupied by the same fluid at time τ. The new elementary material volume is, therefore, given

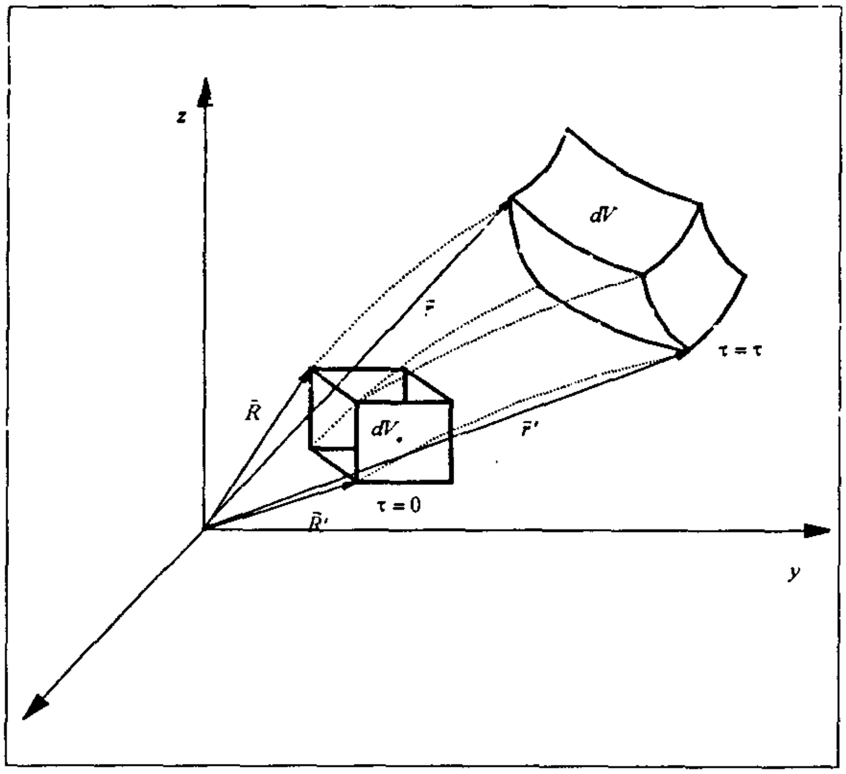

Figure 1.2 Volume element following the motion of the fluid
a) Time $\tau = 0$
b) Time $\tau = \tau$.

by:

$$dV = JdV_o\,. \tag{1.20}$$

The above equation represents the dilatation, or expansion, of an element of volume as it follows the motion of the fluid. The Jacobian must be finite, i.e., different from zero and infinite.

We may also ask how the dilatation changes as we follow the motion. To answer this, the material derivative of J ,i.e., $dJ/d\tau$ should be calculated. The derivation of this derivative is given by Aris (1962) and by Owczarek (1968). Here we will only reproduce the result:

$$\frac{dJ}{d\tau}=\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)J \tag{1.21}$$

or

$$\frac{1}{J}\frac{d}{d\tau}J=\vec{\nabla}\cdot\vec{v}. \tag{1.22}$$

Therefore, the divergence of the flow field represents the relative rate of expansion of the fluid at a given point. Eq. 1.22 is known as Euler's expansion formula.

1.5 Reynolds Transport Theorem

In Section 1.3, we established a relation between the material and spatial time derivative of a scalar or vector field function. In this section, we will derive a relation between the material and spatial derivative of volume integrals of scalar or vector fields. This relation is known as *Reynolds' Transport Theorem.*

Let $\Psi(\vec{r},\tau)$ be a scalar or vector field function and $V(\tau)$ a material volume moving with the fluid. The integral of this function over the material volume is:

$$\Psi(\tau)=\int_{V(\tau)}\Psi(\vec{r},\tau)\,dV \tag{1.23}$$

We are interested in determining the derivative of $\Psi(\tau)$. Since the material volume changes in time, we cannot take the differentiation through the integral sign. However, using Eqs. 1.1 and 1.20 we can write the above equation as:

$$\frac{d}{d\tau}\int_{V(\tau)}\Psi(\vec{r},\tau)\,dV=\frac{d}{d\tau}\int_{V_o}\Psi\left[\vec{r}\,(\vec{R},\tau),\tau\right]J\,dV_o\ . \tag{1.24}$$

Since V_o is a fixed volume at time 0, we can now take the differentiation through the integral :

$$\frac{d}{d\tau}\int_{V(\tau)}\Psi(\vec{r},\tau)\,dV=\int_{V_o}\left[\frac{d\Psi}{d\tau}J+\Psi\frac{dJ}{d\tau}\right]dV_o\ , \tag{1.25}$$

and using Euler's expansion formula (Eq. 1.22), write:

$$\frac{d}{d\tau}\int_{V(\tau)}\Psi(\vec{r},\tau)dV=\int_{V_o}\left[\frac{d\Psi}{d\tau}+\Psi(\vec{\nabla}\cdot\vec{v})\right]J\,dV_o\ . \tag{1.26}$$

Finally, with Eq. 1.20 the above equation becomes:

$$\frac{d}{d\tau}\int_{V(\tau)}\Psi(\vec{r},\tau)dV=\int_{V(\tau)}\left[\frac{d\Psi}{d\tau}+\Psi(\vec{\nabla}\cdot\vec{v})\right]dV\ . \tag{1.27}$$

Taking into account the relation between the material and spatial derivative (Eq. 1.11), Eq. 1.27 becomes:

$$\frac{d}{d\tau}\int_{V(\tau)} \Psi(\vec{r},\tau)\,dV = \int_{V(\tau)} \left[\frac{\partial \Psi}{\partial \tau} + \vec{v}\,.\,\vec{\nabla}\,\Psi + \Psi\,\vec{\nabla}\,.\,\vec{v}\right] dV \quad . \tag{1.28}$$

From vector analysis, we know that the dot product of the dyadic product of two vectors with $\vec{\nabla}$ can be written as:

$$\vec{\nabla}\,.\,\vec{a}\vec{b} = \vec{a}\,.\,\vec{\nabla}\vec{b} + \vec{b}\vec{\nabla}\,.\,\vec{a} \quad . \tag{1.29}$$

Calling $\vec{a}=\vec{v}$ and $\vec{b}=\Psi$ we obtain:

$$\vec{\nabla}\,.\Psi\,\vec{v} = \vec{v}\,.\,\vec{\nabla}\,\Psi + \Psi\,\vec{\nabla}\,.\,\vec{v}\;. \tag{1.30}$$

Therefore, Eq. 1.28 becomes:

$$\frac{d}{d\tau}\int_{V(\tau)} \Psi(\vec{r},\tau)\,dV = \int_{V(\tau)} \left[\frac{\partial \Psi}{\partial \tau} + \vec{\nabla}\,.\,\Psi\,\vec{v}\right] dV\;. \tag{1.31}$$

Using the divergence theorem of Gauss:

$$\int_V \vec{\nabla}\,.\,\vec{B}\,dV = \int_A \vec{B}\,.d\,\vec{A} = \int_A \vec{B}\,.\,\vec{n}\,dA \tag{1.32}$$

and interpreting $\vec{B}=\Psi\,\vec{v}$, Eq. 1.30 takes the following form:

$$\frac{d}{d\tau}\int_{V(\tau)} \Psi(\vec{r}\,,\tau)\,dV = \int_{V(\tau)} \frac{\partial \Psi}{\partial \tau}\,dV + \int_{A(\tau)} \Psi\,\vec{v}\,.d\,\vec{A} \tag{1.33}$$

with $d\vec{A}=\vec{n}\,dA$.

Eq. 1.32 states that the rate of change of the integral of a scalar or vector field function Ψ taken over a material volume V is equal to the integral of the rate of change taken over the volume fixed in space (control volume) which instantaneously coincides with the material volume plus the net flow of Ψ over the bounding surface (Fig. 1.3).

If we consider $V(\tau)$ as an arbitrary volume moving and deforming in space, and $\vec{\omega}$ as the velocity of the surface of this volume (different from the velocity of the fluid, $\vec{v}$), the Reynolds' transport theorem becomes the *general transport theorem or the Leibnitz rule:*

$$\frac{d}{d\tau}\int_{V(\tau)} \Psi(\vec{r},\tau)\,dV = \int_{V(\tau)} \frac{\partial \Psi(\vec{r},\tau)}{\partial \tau}\,dV + \int_{A(\tau)} \vec{\omega}\,\Psi(\vec{r},\tau)\,d\vec{A} \quad . \tag{1.34}$$

In this case, the volume $V(\tau)$ is not a material volume but a geometric volume bounded by a surface $A(\tau)$ which moves with a velocity $\vec{\omega}$

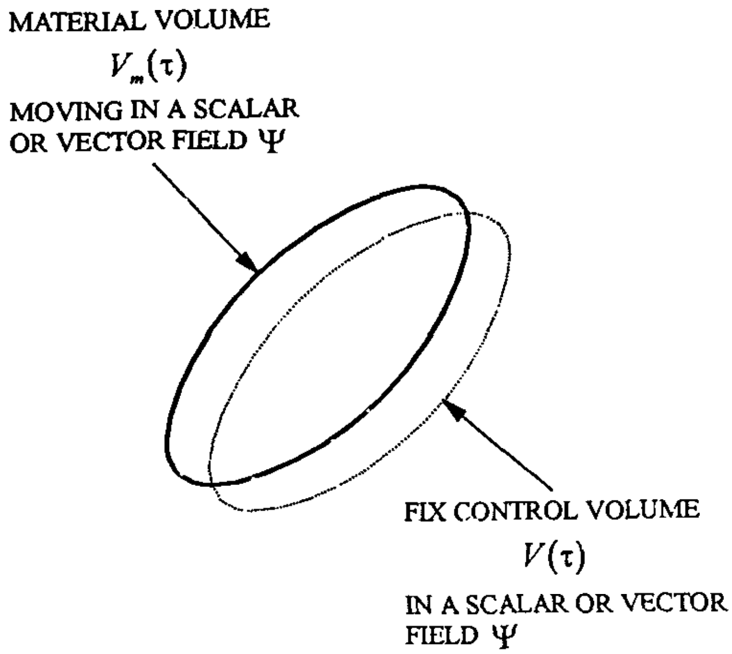

Figure 1.3 Material and control volumes.