ROLPHTON

NUCLEAR TRAINING CENTRE

COURSE 221

NUCLEAR TRAINING COURSE

COURSE 221

2 - Level
2 - Science Fundamentals
1 - MATHEMATICS

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Mathematics - Course 221

OBJECTIVES

221.10-1 Basic Reliability Concepts

1. Given P(A) and P(B), the probabilities of independent events A and B, respectively, calculate P(A and B) and P(A or B), using the formulas:

   $P(A \text{ and } B) = P(A)P(B)$, and

   $P(A \text{ or } B) = P(A) + P(B) - P(A)P(B)$.

2. Define (a) independent events
   - (b) reliability
   - (c) unreliability
   - (d) unavailability of a safety system.

3. Given reliability R, calculate unreliability, Q, and vice versa.

4. State two methods of improving reliability of safety systems.

5. Calculate component failure rate, λ, given a total number of failures amongst a given number of components during a given time interval.

6. Calculate the test interval, T, in years, given the test frequency in tests per shift, day, week, month, or year.

7. Given information determining any two of the variables Q, λ, T, calculate the third variable for a tested safety system.

8. Given information determining the failure rate of the regulating system and the unavailability of the protective system, calculate the annual risk of a reactor power excursion.

9. Apply the above principles to calculate the unreliability of a network of components, given information determining the unreliabilities of the network components.

221.20-1 The Straight Line

1. Define:
   - (a) slope of a line
   - (b) rise of a line segment
   - (c) run of a line segment
   - (d) angle of inclination of a line

2. Write down the relationship between
   - (a) slope m, rise, and run
   - (b) slope m, and angle of inclination, θ

3. State the significance to orientation of a line if the line slope is
   - (a) positive
   - (b) negative
   - (c) zero
   - (d) undefined

4. Calculate the slope of a line, given
   - (a) two points on the line
   - (b) the slope of a parallel line
   - (c) the slope of a perpendicular line
   - (d) the equation of the line
   - (e) the rise and the run of a segment of the line
   - (f) the angle of inclination of the line

5. Given the slope of a line, calculate the change in y corresponding to a given change in x, and vice versa.

6. Identify whether the equation of a line is given in general or slope-intercept form, and convert from the one form to the other.

7. Find the equation of a line, given
   - (a) two points on the line
   - (b) one point on the line and the slope
   - (c) the slope of the line and the y-intercept

8. Graph a line given its equation.

221.20-2 The Derivative

1. State that for a linear function f(x) the following are equivalent:

   (a) the slope
   (b) the 'instantaneous' rate of change of f with respect to x at any point on the graph, $y = f(x)$.
   (c) the average rate of change of f with respect to x over any x-interval.

2. Define the derivative of a function f(x).

3. Recognize and use the notation:

   (a) $\frac{dy}{dx}$ (b) $f'(x)$

4. State that the graphical significance of $f'(x)$ is that $f'(x)$ is the slope of the tangent to the curve $y = f(x)$ at $(x, f(x))$.

5. State and apply the rules for differentiating the following:

   (a) x^n
   (b) $cf(x)$
   (c) c
   (d) $f(x) \pm g(x)$

221.20-3 Simple Applications of Derivatives

1. Given the function f(x), find

   (a) the slope, and
   (b) the equation of the tangent and normal to the curve $y = f(x)$ at any given point (x_1, y_1) on the curve.

2. Differentiate a given polynominal displacement function to obtain the corresponding velocity function.

3. Differentiate a given polynominal velocity function to obtain the corresponding acceleration function.

221.20-4 Differentiating Exponential Functions

1. Differentiate functions of the form

   (a) $f(x) = ke^{g(x)}$
   (b) $f(x) = P(x) \pm ke^{g(x)}$

   where k is a constant, and g(x) and P(x) are both polynominals.

2. Given the nuclear decay formula, $N(t) = N_0e^{-\lambda t}$, prove that

   (a) $\frac{dN}{dt} = -\lambda N$

   (b) $A(t) = A_0e^{-\lambda t}$, where $A = -\frac{dN}{dt}$

3. Given any two of the variables A, λ, N (activity, decay constant, number of radioactive nuclei, respectively), calculate the third variable.

4. Given any three of the following variables, calculate the fourth variable:

   (a) N, N_0, λ, t (see nuclear decay formula above)
   (b) A, A_0, λ, t (see activity decay formula above)
   (c) P, P_0, T, t (see power growth formula below)

5. Given the reactor power growth formula $P(t) = P_0e^{t/T}$ prove that

   (a) $\frac{dP}{dt} = \frac{P}{T}$

   (b) $\frac{d}{dt} \ln P = \frac{1}{T}$

6. State the advantage of

   (a) a log power signal (over a linear power signal) for power indication and control
   (b) a rate log power signal for reactor protection.

221.20-5 The Derivative in Science and Technology

1. Translate a given verbal rate-of-change statement into a differential equation, and vice versa.

2. Given a sketch showing the fluctuation of a controlled parameter about set point, sketch on the same time axis, typical corresponding proportional component, derivative component, and total response of a proportional-derivative controller.

3. For the case of tank level control via regulation of inflow, sketch typical level fluctuations following a step change in outflow for

   (a) proportional only control
   (b) proportional plus derivative control

4. State two advantages of adding a derivative component to proportional control.

221.30-1 The Integral

1. State that integration is the opposite of differentiation.

2. Recognize and use the integral notation.

3. Integrate functions of the following forms:

   (a) $f(x) = 0$

   (b) $f(x) = x^n$

   (c) $f(x) = e^{f(x)}f'(x)$
   (d) $f(x) = g(x) \pm h(x)$

4. Given an acceleration function, obtain the corresponding velocity and displacement functions by integration.

5. Given a velocity function, integrate to obtain the corresponding displacement function.

6. Given the equation of a curve, $y = f(x)$, find the area under the curve in the interval $x = a$ to $x = b$ by evaluating the appropriate definite integral.

221.30-2 Applications of The Integral as an Infinite Sum

1. Find the area between two curves (one of which could be an axis) by applying the 'slice technique', including a diagram showing representative slice.

2. Given force F as a function of displacement x, calculate the work done by this force acting through x = a to x = b.

3. Given a sketch showing the fluctuation of a controlled parameter about set point, sketch on the same time axis typical corresponding proportional component, reset component, and total response of a proportional-integral controller.

4. For the case of tank level control via regulation of inflow, sketch typical level fluctuations following a step change in outflow for

   (a) proportional only control
   (b) proportional plus reset control.

5. State the advantage of adding a reset component to proportional control.

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Mathematics - Course 221

INTRODUCTION

I COURSE CONTENT AND ORGANIZATION

(a) Content

This course provides an introduction to two topics:

(1) *equipment reliability evaluation*, and

(2) *calculus*.

Reliability is a specific engineering topic, whereas calculus is a basic mathematical tool. Although slightly more advanced reliability theory involves the use of calculus, as does practically every branch of science and technology, reliability and calculus appear in this course as unrelated topics.

Reliability calculations provide quantitative (ie, numerical) answers to such questions as the following:

- How reliable is this equipment?
- What is the annual risk of a reactor accident?
- How frequently should this safety system be tested?

Calculus provides the notation and techniques for solving two general classes of problems:

(1) How to find the true, or 'instantaneous' rate of change of one variable with respect to another, given the one as a function of the other (differential calculus), and:

(2) The inverse of problem (1) - How to find one variable as a function of another, given the rate of change of the one with respect to the other (integral calculus).

These techniques are applied first to the familiar quantities of velocity and acceleration, and subsequently to the time-dependent phenomena of reactor power growth, nuclear decay, water purification by ion exchange, and negative feedback in control loops.

This text makes no pretense at rigor. The least possible content and formalism have been introduced to reach the goal of treating the above applications.

(b) Organization

Six lessons reprinted from levels 4 and 3 Mathematics have been placed in front of the level 2 lessons. These six lessons provide essential background for the level 2 lessons. (Trainees will be checked out on the skills of these level 4 and 3 lessons only as these skills are involved in doing level 2 test items.) The text concludes with four appendices containing review exercises, selected AECB examination questions, methods of solving quadratic equations, and assignment answers, respectively.

II SUGGESTIONS REGARDING USE OF THIS TEXT

(1) Before becoming engrossed in the details of any lesson, scan its entire contents, paying particular attention to headings. Try to formulate a general impression of what you are expected to learn.

(2) Work through examples written into the text, referring to the text as necessary. Persist until you can work examples unaided.

(3) Do ALL assignments at the conclusions of the lessons.

(4) Practice your skills on the numerous review exercises provided in Appendix I.

III WHY CALCULUS AT LEVEL 2?

Calculus has formerly been reserved for level 1 in the NGD training program. Calculus is now being introduced at level 2, but this course is far more introductory and narrower in scope than the old level 1 course. Whereas the old level 1 course was first year university level, this course is sub-Ontario Grade 13 level.

Any discussion of training course content must be held in the light of the prevailing philosphy of training. To choose control room operator as an example of a position for which level 2 mathematics is prerequisite, two possible training philosophies are as follows:

(1) The operator needs to know nothing more than the appropriate response to each possible annunciation or sequence of annunciations, ie, he is 'programmed' to respond to every eventuality. Thus his training should consist entirely of rote memorization of procedures.

(2) Some operating procedures and emergency procedures must be learned and practised to the point where they can be performed without first having to think them through, but the operator should understand the plant systems well enough that he can make reasoned responses to such other plant situations as may arise.

Some companies lean towards philosophy (1) above, but there are problems with it. For one thing, the number of possible combinations and permutations of annunciations in a CANDU control room is so large, that to memorize detailed procedures for each one of them is impractical. Secondly, playing the role of a programmed robot could be demoralizing - people generally like to feel that they know what they are doing, and perform better when this is the case.

In any event, Ontario Hydro leans to philosphy (2). So does the AECB. Consequently, the prospective operator gets his level 2 training courses. And writes his AECB's.

Calculus provides the concepts, notation, and techniques necessary to a quantitative analysis and description of science and technology. Introductory calculus is therefore relevant background to other level 2 training, which concerns various aspects of nuclear science and technology. For example, the background knowledge of exponential and logarithmic functions, and of rates of change and integration, provided by this course, facilitates a quantitative or semi-quantitative discussion of reactor power changes and nuclear decay phenomena in the level 2 Nuclear Theory course, and of derivative and integral control in Instrumentation & Control courses.

A perusal of Appendix I confirms the relevance of 221 course content to the AECB examinations sat by operators. The point here is not that the AECB requires quantitative analyses with formal applications of calculus, but rather that the trainee is examined on subjects whose quantitative analysis certainly does involve calculus, and that the trainee with the background fundamental to understanding such subjects on the (higher) quantitative level is better able to understand and discuss them at the (lower) qualitative level. In fact, one of the best arguments for presenting calculus at level 2 is to ensure that the trainee can do it.

What of the job relevance of this course (aside from licensing requirements)? Continuing with the example of control room operator, let one concede at the outset that the operator will probably never be required to differentiate or integrate a function in the control room. Neither will he be required to recite Science Fundamentals nor even Equipment & Systems Principles. ALL of this training provides the operator with the conceptual framework and background knowledge necessary to 'evaluate the board', and make reasoned responses based on such evaluations, ie, this training is a consequence of implementing philosphy #2 on the previous page. In the parlance of the training theorist, calculus skills are "mediating skills" - skills not practised directly on the job, but facilitating job performance indirectly.

Of two people with similar native abilities examining the same control panel, the same faulty circuit, the same AECB examination, the same design manual, etc, the one with the richer web of relevant concepts and more extensive relevant knowledge in his background will, on the average, absorb what he sees faster, and analyze, apply, or synthesize the input more readily, because his brain has more reference data, more familiar stimuli with which to associate the new stimulus. In short, the richer one's relevant background, the higher is his potential job performance. Note the key word "relevant" in the foregoing - this argument cannot be used to justify NGD training courses in Babylonian architecture or ancient Near-Eastern literature, but it does vindicate introducing a control room operator, who interacts intimately with CANDU technology, to the mathematics which enables a quantitative description of that technology.

The foregoing argues generally in favour of level 2 calculus; the following are two specific examples of where Mathematics 221 content impinges on control room operation:

(a) Understanding the significance of linear power, log power, linear rate power and rate log power (cf Appendix 2), and:

(b) Interpretation of graphical representations of various physical parameters.

This brief apologetic concludes with a few comments on the following red herring: "I'll forget how to differentiate and integrate within days of writing the check-out, so why study calculus at all?" To begin with, memory-fade is a universal fact of life, true for all courses. If it were a legitimate basis for abolishing this course, it would be an equally legitimate basis for abolishing most courses. But it is not a legitimate basis for abilishing any course, because there is a useful residual to instruction/learning, which exists apart from the specific details of mathematics, history, literature, science, etc. This residual of one's general education consists of such things as the facility of critical analysis and an appreciation of the significance of the terms "objective" and "subjective". This residual remains long after the student's memory of specific details is all smudge and blur.

The useful residual of this course is envisaged to be concepts of function, rate of change (curve slope, derivative), and summation (area, integral), plus an ability to think quantitatively about time-dependent quantities, an ability which depends largely on exposure to the mathematics introduced in this text.

Mathematics - Course 421

STANDARD NOTATION

Introduction to Powers of 10

A *power of 10* consists of the *base* 10 raised to some *exponent:*

$$\left.\begin{array}{ll} 10^{n} & \leftarrow \text{exponent} \\ & \leftarrow \text{base} \end{array}\right\} \quad \text{power}$$

10^n stands for n factors of 10. For example,

$$10^5 = 10 \times 10 \times 10 \times 10 \times 10$$

Definitions:

$$\boxed{\begin{array}{l} 10^{-n} = \dfrac{1}{10^{n}} \\ \\ 10^{0} \;\; = 1 \end{array}}$$

Thus:

$$\begin{array}{l} \vdots \\ 10^{3} = 1000 \\ 10^{2} = 100 \\ 10^{1} = 10 \\ 10^{0} = 1 \\ 10^{-1} = .1 \\ 10^{-2} = .01 \\ 10^{-3} = .001 \\ \vdots \end{array}$$

Powers of 10 are <u>multiplied</u> according to the format,

$$\boxed{10^{n} \times 10^{m} = 10^{n+m}},$$

since (n factors of 10) x (m factors of 10) = (m+n) factors of 10.

Powers of 10 are <u>divided</u> according to the format,

$$\boxed{\frac{10^{n}}{10^{m}} = 10^{n-m}}$$

Example 1: $10^5 \times 10^8 = 10^{5+8} = 10^{13}$

Example 2: $10^8 \times 10^{-5} = 10^{8+(-5)} = 10^3$

Example 3: $\dfrac{10^5}{10^8} = 10^{5-8} = 10^{-3}$

Example 4: $\dfrac{10^8}{10^{-5}} = 10^{8-(-5)} = 10^{13}$

Example 5:

$$\begin{aligned} \frac{10^5 \times 10^{-7} \times 10^3}{10^{-11} \times 10^3} &= \frac{10^{5+(-7)+3}}{10^{-11+3}} \\ &= \frac{10^1}{10^{-8}} \\ &= 10^{1-(-8)} \\ &= 10^9 \end{aligned}$$

Combining Powers of 10 with Decimal Coefficients

A power of 10 can be combined with a decimal *coefficient*,

eg, $$\underset{\text{coefficient}}{\underset{\nearrow}{4.1}} \times \underset{\text{power of 10}}{\underset{\uparrow}{10^6}}$$

Recall that shifting the decimal point left one place decreases a number by a factor of 10. Thus the decimal may be shifted left n places in a number if it is multiplied by 10^n to compensate.

eg, $$\begin{aligned} 4 &= \underset{\curvearrowleft}{.4} \times 10^1 \\ &= \underset{\curvearrowleft\curvearrowleft}{.04} \times 10^2 \\ &= \underset{\curvearrowleft\curvearrowleft\curvearrowleft}{.004} \times 10^3 \end{aligned}$$

etc.

Similarly, shifting the decimal point right one place increases a number by a factor of 10. Thus the decimal may be shifted right n places if the number is multiplied by 10^{-n} to compensate.

eg, $4. = 4\underrightarrow{0}. \times 10^{-1}$

$= 4\underrightarrow{00}. \times 10^{-2}$

$= 4\underrightarrow{000}. \times 10^{-3}$

etc.

<u>Example 1</u>: $5280 = 5.\underleftarrow{280} \times 10^{3}$

<u>Example 2</u>: $0.\underrightarrow{004}3 = 4.3 \times 10^{-3}$

<u>Example 3</u>: $65.4 \times 10^{2} = 6\underleftarrow{.5}4 \times 10^{2} \times 10$

$= 6.54 \times 10^{3}$

(1 move left ⇒ 1 additional factor of 10
⇒ exponent increases by 1)

<u>Example 4</u>: $0.\underrightarrow{05}71 \times 10^{-6} = 5.71 \times 10^{-6} \times 10^{-2}$

$= 5.71 \times 10^{-8}$

(2 moves right ⇒ exponent decreases by 2)

<u>Standard Notation</u>

To express a number in *standard notation (S.N.)* rewrite the number with one nonzero digit left of the decimal point, and multiply by a power of 10 to compensate.

<u>Example 1</u>: Distance travelled by light in one year, ie, one light year is

9,460,000,000,000,000 = 9.46×10^{15} meters

<u>Example 2</u>: Fission cross section of U^{235} nucleus, for thermal neutrons is

0.000,000,000,000,000,000,000,58 = 5.8×10^{-22} cm^{2}

<u>Example 3</u>: $613 \times 10^{4} = 6.13 \times 10^{6}$

<u>Advantages of Standard Notation</u>

(1) Convenient notation for very large or very small numbers (cf Examples 1 and 2 above), for both ease of writing and ease of comparison.

(2) Facilitates rapid mental calculation.

(3) Shows number of significant figures explicitly, where ambiguity might exist in ordinary decimal notation (cf lesson 421.10-2).

The Four Basic Operations with Numbers in Standard Notation

1. Add numbers in standard notation according to the format,

$$\boxed{a \times 10^n + b \times 10^n = (a + b) \times 10^n}$$

Note that both numbers must have the same power of 10, and that the power of 10 does not change in the addition (similarly for subtraction).

Example 1: $2 \times 10^3 + 3 \times 10^3 = (2 + 3) \times 10^3$

$= 5 \times 10^3$

Example 2: $4.73 \times 10^{-5} + 2.18 \times 10^{-5} = 6.91 \times 10^{-5}$

Example 3: $6.93 \times 10^8 + 4.51 \times 10^6$

$= 6.93 \times 10^8 + .0451 \times 10^8$ (convert to same powers)

$= 6.98 \times 10^8$ (Sum justified to 2 D.P.)

Example 4: $9.78 \times 10^{12} + 5.14 \times 10^{11}$

$= 9.78 \times 10^{12} + .514 \times 10^{12}$ (convert to same powers)

$= 10.29 \times 10^{12}$ (Sum justified to 2 D.P.)

$= 1.029 \times 10^{13}$ (Adjust decimal, power to recover answer in S.N.)

2. Subtract numbers in standard notation according to the format,

$$\boxed{a \times 10^n - b \times 10^n = (a - b) \times 10^n}$$

Example 1: $7 \times 10^5 - 3 \times 10^5 = 4 \times 10^5$

<u>Example 2:</u> $4.65 \times 10^{-8} - 9.24 \times 10^{-10}$

$= 4.65 \times 10^{-8} - 0.0924 \times 10^{-8}$ (convert to same powers)

$= 4.56 \times 10^{-8}$ (difference justified to 2 D.P.)

<u>Example 3:</u> $6.25 \times 10^{12} - 11.3 \times 10^{13}$

$= 0.625 \times 10^{13} - 11.3 \times 10^{13}$ (convert to same powers)

$= -10.7 \times 10^{13}$ (difference justified to 1 D.P.)

$= -1.07 \times 10^{14}$ (adjust decimal, power to recover answer in S.N.)

3. <u>Multiply</u> two numbers in standard notation according to to the format,

$$\boxed{(a \times 10^{n})(b \times 10^{m}) = ab \times 10^{n+m}}$$

<u>Example 1:</u> $2 \times 10^{6} \times 3 \times 10^{2} = (2 \times 3) \times 10^{6+2}$

$= 6 \times 10^{8}$

<u>Example 2:</u> $4.7 \times 10^{6} \times 6.2 \times 10^{-3}$

$= 29 \times 10^{3}$ (product justified to 2 S.F.)

$= 2.9 \times 10^{4}$ (express answer in S.N.)

4. <u>Divide</u> two numbers in standard notation according to the format,

$$\boxed{(a \times 10^{n}) \div (b \times 10^{m}) = (a \div b) \times 10^{n-m}}$$

<u>Example 1:</u> $(7 \times 10^{6}) \div (2 \times 10^{-2}) = (7 \div 2) \times 10^{6-(-2)}$

$= 3.5 \times 10^{8}$

Example 2: $2.4 \times 10^5 \div 6.9 \times 10^9$

$= 0.35 \times 10^{-4}$ (quotient justified to 2 S.F.)

$= 3.5 \times 10^{-5}$ (express answer in S.N.)

Evaluating Complex Expressions Using Numbers in Standard Notation

(1) Do operations in established order of precedence (cf lesson 421.10-1).

(2) Retain one more D.P. or S.F. than justified in intermediate calculations (to avoid introducing unnecessary 'rounding-off error').

(3) Round off final answer to correct number of digits justified.

Example 1: $2.2 \times 10^2 \div (8.1 \times 10^4) + 1.7 \times 10^{-6} \times 4.6 \times 10^3$

$= 0.272 \times 10^{-2} + 7.82 \times 10^{-3}$ (÷, x precede +; retain 3 S.F. temporarily)

$= 2.72 \times 10^{-3} + 7.82 \times 10^{-3}$ (convert to same power)

$= 10.54 \times 10^{-3}$ (last digit not significant)

$= 1.05 \times 10^{-2}$ (answer in S.N.)

Example 2: Recall that division bar acts as a bracket, requiring evaluation of numerator and denominator prior to division, as follows:

$$\frac{4.7 \times 10^6 + 2.1 \times 10^7}{6.8 \times 10^{11} \times 1.4 \times 10^{-6}}$$

$$= \frac{.47 \times 10^7 + 2.1 \times 10^7}{6.8 \times 1.4 \times 10^{11+(-6)}}$$ (convert to same powers in numerator)

$$= \frac{2.57 \times 10^7}{9.52 \times 10^5}$$ (retain extra digit temporarily)

$= 0.27 \times 10^2$ (answer justified to 2 S.F.)

$= 2.7 \times 10^1$ (answer in S.N.)

ASSIGNMENT

1. Evaluate:

(a) $10^3 \times 10^4 =$ (b) $10^3 \div 10^2 =$

(c) $10^9 \times 10^{-3} =$ (d) $10^9 \div 10^{-3} =$

(e) $10^{-4} \times 10^{-4} =$ (f) $10^{11} \div 10^{20} =$

(g) $10^4 \times 10^{-8} =$ (h) $10^4 \div 10^8 =$

2. Change to a simpler form:

(a) $\frac{1}{10^2} =$ (b) $\frac{1}{10^6 \times 10^3} =$

(c) $\frac{1}{10^{-2}} =$ (d) $\frac{1}{10^{-9} \times 10^9} =$

(e) $-\frac{1}{10^7} =$ (f) $\frac{1}{10^{-13}} =$

(g) $\frac{10^9 \times 10^7}{10^6} =$ (h) $\frac{10^{-17} \times 10^{19}}{10^{20} \times 10^{-5}} =$

(i) $\frac{10^{-11} \times 10^{12}}{10^{-8}} =$ (j) $\frac{10^{21} \times 10^{-19}}{10^3 \times 10^4 \times 10^6} =$

(k) $\frac{10^3}{10^{-12} \times 10^2} =$ (l) $\frac{-10^2 \times 10^3 \times 10^{17}}{10^4 \times 10^{17}} =$

3. Rewrite the following in decimal form:

a) 10^2 b) 10^{-3}

c) 10^5 d) 10^{-6}

e) 10^6 f) 10^{-4}

4. Convert the following to standard notation:

(a) 165 000
(b) .00693
(c) 37.5
(d) .025
(e) 2934
(f) .00101
(g) 10000
(h) .00020
(i) -249
(j) .97
(k) 176×10^{-3}
(l) $.0027 \times 10^{3}$
(m) 957×10^{2}
(n) $.0175 \times 10^{-12}$
(o) $.024 \times 10^{9}$
(p) $.032 \times 10^{14}$

5. Calculate the following:

(a) $9.3 \times 10^{2} + 1.5 \times 10^{3} =$
(b) $4.6 \times 10^{12} + 9.9 \times 10^{11} =$
(c) $9.4 \times 10^{12} - 1.2 \times 10^{14} =$
(d) $7.5 \times 10^{2} - 5.0 \times 10^{3} =$
(e) $4.5 \times 10^{12} - 4.5 \times 10^{9} =$

6. Express answers in scientific notation:

(a) $3.7 \times 10^{2} \times 2.5 \times 10^{3} =$
(b) $2.5 \times 10^{9} \div 3.6 \times 10^{3} =$

(c) $$\frac{9.7 \times 10^{12} \times 3.3 \times 10^{10}}{9.5 \times 10^{15}} =$$

(d) $$\frac{3.2 \times 10^{13} \times 2.2 \times 10^{-12}}{1.3 \times 10^{10} \times 9.9 \times 10^{2}} =$$

(e) $$\frac{2.8 \times 10^{-12} \times 1.1 \times 10^{11}}{8.0 \times 10^{3} \times 7.0 \times 10^{-8}} =$$

7. Express answers in scientific notation.

(a) $\dfrac{7.5 \times 10^{2} + 5.0 \times 10^{3} \times 2.0 \times 10^{-1}}{2.5 \times 10^{2} \times 3.0 \times 10} =$

(b) $\dfrac{(8.6 \times 10^{-14} + 9.9 \times 10^{-13}) \times 2.0 \times 10^{12}}{4.6 \times 10^{3} \times 5.0} =$

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Mathematics - Course 421

ALGEBRA FUNDAMENTALS

I Introduction

Basic operations in algebra are the same as they are in arithmetic, except that letters are used to stand for numbers. This gives the advantage that one can manipulate numbers without knowing their values. As will be seen in lesson 421.20-2, this advantage is useful in setting up and solving proportions, manipulating formulas, and solving problems in one unknown.

II Evaluation of Algebraic Expressions by Substitution

To evaluate an algebraic expression by substitution, substitute the given numerical values for the variables (letters), and then simplify using "BEDMAS" for correct order of operations (cf lesson 421.10-1, section V).

Example 1:

Evaluate $a + 3b$ if $a = 5$ and $b = -2$.

Solution:

$$\begin{aligned} a + 3b &= 5 + 3(-2) && \text{(substitute)} \\ &= 5 + (-6) && \text{(x precedes +)} \\ &= -1 \end{aligned}$$

Example 2:

Evaluate $(x + y) \div (x)(y)$ if $x = 7$, $y = -4$.

Solution:

$$\begin{aligned} (x + y) \div (x)(y) &= (7 + (-4)] \div (7)\ (-4) && \text{(substitute)} \\ &= (3) \div 7(-4) && \text{(brackets first)} \\ &= (\tfrac{3}{7})(-4) && \text{(÷, x as they occur)} \\ &= (\tfrac{3}{7})(\tfrac{-4}{1}) \\ &= \frac{-12}{7} \\ &= -1\tfrac{5}{7} \end{aligned}$$

III Powers

a) Notation

Recall that a^n stands for n factors of a:

$$a^n = \underbrace{a \cdot a \cdot a \cdot \ldots\ a}_{\text{n factors of a}}$$

eg, $x^5 = x \cdot x \cdot x \cdot x \cdot x$

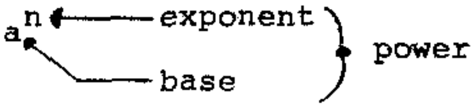
Note the use of the dot to indicate multiplication, in order to avoid confusion of the times sign "x" with the letter "x". Sometimes brackets are used to indicate multiplication.

eg, $3x(-y)$ means 3 times x times $-y$, but
$3x-y$ means 3 times x, subtract y.

Most often, however, when variables are multiplying each other, the sign is omitted altogether.

eg, $-3xy$ means -3 times x times y.

A *power* consists of a *base* and an *exponent:*



** NB *Exponentiation* (raising a base to an exponent) takes precedence over multiplication and division.

eg, $xy^2 = xyy$ (y must be squared before multiplying by x)

But this natural order of precedence can be overruled with the use of brackets:

eg, $xy^2 = xyy$, but $(xy)^2 = (xy)(xy)$

eg, $-2x^2 = -2xx$, but $(-2x)^2 = (-2x)(-2x)$

eg, $-10^2 = -(10)(10)$, but $(-10)^2 = (-10)(-10)$

b) Power Laws

Nine basic laws governing operations with exponents follow. A brief rationale and one or more examples are included with each law.

Law 1:

$$\boxed{x^n \cdot x^m = x^{m + n}}$$

Rationale:

(n factors of x)(m factos of x) = (m + n) factors of x

Example:

$x^5 \cdot x^7 = x^{12}$

Law 2:

$$\boxed{\left.\begin{array}{l} x^n \div x^m \\ \text{or } \dfrac{x^n}{x^m} \end{array}\right\} = x^{n - m}}$$

Rationale:

a) If $n > m$, cancelling m common factors of x leaves $n - m$ factors of x in the numerator.

b) If $n < m$, cancelling n common factors of x leaves $m - n$ factors of x in the donominator.

ie, $\dfrac{1}{x^{m - n}} = x^{-(m - n)} = x^{n - m}$ (cf law 4)

Examples:

$$\left.\begin{array}{l} x^7 \div x^5 = x^{7-5} = x^2 \\ \\ x^5 \div x^7 = x^{5-7} = x^{-2} \end{array}\right\} \begin{array}{l}\text{amounts to} \\ \text{cancelling 5} \\ \text{factors of x} \\ \text{in either case}\end{array}$$

** NB <u>In laws 1 and 2, the bases of the powers must be identical</u>

ie, 2^5 x $2^7 = 2^{12}$, but 2^5 x 3^7 <u>cannot</u> be simplified as a power

Similarly, $\frac{2^7}{2^5} = 2^2$, but $\frac{2^7}{3^5}$ <u>cannot</u> be simplified as a power

Law 3:

$$\boxed{(x^n)^m = x^{mn}}$$

Rationale:

m factors of (n factors of x) = mn factors of x

Example:

$$(x^7)^5 = x^{35}$$

Law 4:

$$\boxed{\begin{array}{l} x^{-m} = \frac{1}{x^m} \\ \\ \text{or} \quad \frac{1}{x^{-m}} = x^m \end{array}}$$

Rationale:

Negative exponents are defined this way to make the other laws consistent.

eg, $\frac{a^3}{a^5} = \frac{\cancel{a}\cdot\cancel{a}\cdot\cancel{a}}{\cancel{a}\cdot\cancel{a}\cdot\cancel{a}\cdot a\cdot a} = \frac{1}{a^2}$

But law 2 gives $\frac{a^3}{a^5} = a^{3-5} = a^{-2}$

These are consistent only if $\frac{1}{a^2} = a^{-2}$

Examples:

$$x^{-5} = \frac{1}{x^5}$$

$$\frac{1}{x^{-5}} = x^5$$

Thus powers may be shifted from numerator to denominator, and vice versa, merely by changing the sign of their exponents.

Law 5:

$$\boxed{x^0 = 1}$$

Rationale:

$\frac{x^n}{x^n} = x^{n-n} = x^0$ by law 2

But $\frac{x^n}{x^n} = 1$ by cancelling numerator and denominator

$\therefore$ $x^0 = 1$ to make answers consistent

Eamples:

$$10^0 = 1$$

$$(-13)^0 = 1$$

$$(xy)^0 = 1$$

Law 6:

$$\boxed{(xy)^m = x^m y^m}$$

Rationale:

m factors of xy = (m factors of x)(m factors of y) just by reordering the x's and y's.

Examples:

$$(xy)^5 = x^5 y^5$$

$$(-p)^5 = (-1 \text{ x } p)^5 = (-1)^5 \, p^5 = -p^5$$

$$(x^2 y)^5 = (x^2)^5 \, y^5 = x^{10} y^5$$

$$(2y^3)^8 = 2^8 (y^3)^8 = 256 \, y^{24}$$

Law 7:

$$\boxed{\begin{array}{l} (x \div y)^m = x^m \div y^m \\ \text{or} \quad (\frac{x}{y})^m = \frac{x^m}{y^m} \end{array}}$$

Rationale:

$$\text{m factors of } \frac{x}{y} = \frac{\text{m factors of x}}{\text{m factors of y}}$$

by rule for multiplication of fractions.

Examples:

$$(\frac{a}{b})^7 = \frac{a^7}{b^7}$$

$$\left(-\frac{2x}{3}\right)^3 = \left(\frac{-2x}{3}\right)^3 = \frac{(-2x)^3}{3^3} = \frac{(-2)^3x^3}{3^3} = \frac{-8x^3}{27}$$

or $$\left(-\frac{2x}{3}\right)^3 = \left((-1)\frac{2x}{3}\right)^3 = (-1)^3\left(\frac{2x}{3}\right)^3 = (-1)\frac{(2x)^3}{3^3} = -\frac{8x^3}{27}$$

Law 8:

$$\boxed{x^{\frac{1}{n}} = \sqrt[n]{x}}$$

Rationale:

By law 3, $(a^{\frac{1}{n}})^n = a^{\frac{n}{n}} = a^1 = a$

Thus n factors of $a^{\frac{1}{n}}$ equals a.

But, by definition, $\sqrt[n]{a}$ is that number, n factors of which equals a

$\therefore \quad a^{\frac{1}{n}} = \sqrt[n]{a}$

Examples:

$$8^{\frac{1}{3}} = \sqrt[3]{8} = 2$$

$$(-125)^{\frac{1}{3}} = \sqrt[3]{-125} = -5$$

$$(27x^6)^{\frac{1}{3}} = 27^{\frac{1}{3}}(x^6)^{\frac{1}{3}} = \sqrt[3]{27}\ x^2 = 3x^2$$

Law 9:

$$\boxed{x^{\frac{m}{n}} = \sqrt[n]{x^m} = (\sqrt[n]{x})^m}$$

Rationale:

$$x^{\frac{m}{n}} = (x^{m})^{\frac{1}{n}} \qquad \text{(law 3)}$$

$$= \sqrt[n]{x^{m}} \qquad \text{(law 8)}$$

$$\text{But } x^{\frac{m}{n}} = (x^{\frac{1}{n}})^{m} \qquad \text{(law 3)}$$

$$= (\sqrt[n]{x})^{m} \qquad \text{(law 8)}$$

<u>Examples:</u>

$$8^{\frac{2}{3}} = (\sqrt[3]{8})^{2} = (2)^{2} = 4$$

$$\text{or } \sqrt[3]{8^{2}} = \sqrt[3]{64} = 4$$

$$(-32x^{10})^{.6} = (-32x^{10})^{\frac{3}{5}} = (-32)^{\frac{3}{5}}(x^{10})^{\frac{3}{5}} = (\sqrt[5]{-32})^{3}x^{6}$$

$$= (-2)^{3}x^{6} = -8x^{6}$$

c) <u>Additional Examples of Use of Power Laws</u>

<u>Example 1:</u>

$$(3xy^{2})^{3} \div (6x^{6}y^{4})$$

$$= \frac{(3xy^{2})^{3}}{6x^{6}y^{4}}$$

$$= \frac{3^{3}x^{3}(y^{2})^{3}}{6x^{6}y^{4}} \qquad \text{(law 6 on numerator)}$$

$$= \frac{27x^{3}y^{6}}{6x^{6}y^{4}} \qquad \text{(complete exponentiation, law 3)}$$

$$= \frac{9}{2} x^{3-6} \ y^{6-4}$$

(apply law 2 to x's, y's separately)

$$= 4.5 \ x^{-3} y^{2}$$

Example 2:

$$\frac{(-x)^{2}(-x^{2})}{(-x)^{-2}(-x^{-2})}$$

$$= \frac{(-x)^{2}\cancel{(-1)}x^{2}}{(-x)^{-2}\cancel{(-1)}x^{-2}}$$

(show $-x^2$ as $(-1)x^2$ to separate numerical coefficient (-1) from base x)

$$= (-x)^{2-(-2)} \ x^{2-(-2)}$$

(law 2 for each base)

$$= (-x)^{4}x^{4}$$

$$= x^{4}x^{4}$$

(even no. negative factors yields positive result)

$$= x^{8}$$

(law 1)

Example 3:

$$(-\ \frac{1}{32}\ )^{-.4}$$

$$= (\frac{1}{-32})^{-2/5}$$

$$= \frac{1^{-2/5}}{(-32)^{-2/5}}$$

(law 7)

$$= \frac{1}{(-32)^{-2/5}}$$

($1^{x} = 1$ for any x value)

$$= (-32)^{2/5}$$

(law 4)

$= (\sqrt[5]{-32})^2$ (law 9)

$= (-2)^2$ ($\sqrt[5]{-32} = -2$ since $(-2)^5 = -32$)

$= 4$

IV The Four Basic Operations with Algebraic Terms

a) Definitions:

An *algebraic term* is a group of numbers and/or letters associated by multiplication or division only, and separated from other terms by addition or subtraction,

eg, $3x^2$, $-5xy$, 16, $xypg$ are terms

Like terms are terms having identical letter combinations, including exponents,

eg, x, $3x$, $-17x$ and

xy^2, $5xy^2$, $-4xy^2$ are groups of like terms,

but $5xy$ and $- 4xy^2$ are not like terms since

the exponent on y differs.

The *numerical coefficient* of a term is the number which multiplies the letter combination,

eg, $\underset{\uparrow}{3}xy$, $\underset{\uparrow}{\pi}q^2$, $\underset{\uparrow}{-15}tsw$

numerical coefficients

b) Addition and Subtraction of Terms

Like terms ONLY are added/subtracted by adding/subtracting their numerical coefficients. The process of adding/subtracting like terms to simplify an algebraic expression is called *collecting terms*.

Example 1:

$$3x^2 + 5x^2 = (3 + 5)x^2$$
$$= 8x^2$$

(Note that letter combination does not change)

Example 2:

$$-15yp^2 + 9yp^2 = (-15 + 9)yp^2$$
$$= -6yp^2$$

Example 3:

$$5qr - 3qr = (5 - 3)qr$$
$$= 2qr$$

Example 4:

$$-15x^2y - 2x^2y = (-15 - 2)x^2y$$
$$= -17x^2y$$

Example 5:

Simplify $-15x^2 + 4xy - y^2 + 2x^2 - 3y^2$

Solution:

$$-15x^2 + 4xy - y^2 + 2x^2 - 3y^2$$
$$= (-15 + 2)x^2 + 4xy + (-1 + (-3))y^2$$ (collect like terms)
$$= -13x^2 + 4xy - 4y^2$$

c) Multiplication and Division of Terms:

Terms are multiplied/divided by multiplying/dividing first the numerical coefficients, then each group of *like powers* (same bases) successively.

Example 1:

$$(5x^2y)(1.3xy^3)$$
$$= (5 \times 1.3)(x^2\ x)(y\ y^3)$$ (group like powers)
$$= 6.5\ x^3y^4$$

Example 2:

$$(-4pq^2)(-3qr^3)$$
$$= (-4 \times (-3))(p)(q^2\ q)(r^3)$$
$$= 12pq^3r^3$$

Example 3:

$$(15x^6) \div (3x^2)$$

$$= (\frac{15}{3})(\frac{x^6}{x^2})$$

$$= 5x^4$$

Example 4:

$$\frac{120p^2q}{-15q^3r^2} = \frac{120}{-15}(\frac{p^2}{1})(\frac{q}{q^3})(\frac{1}{r^2})$$

$$= -8p^2q^{-2}r^{-2}$$

(Imagine factors of $r^0 = 1$ in the numerator, and $p^0 = 1$ in the denominator if this helps.)

V Multiplication and Division of Polynomials

Definitions:

Monomials, *binomials* and *polynomials* are algebraic expressions having one, two and several *terms*, respectively.

a) Multiplying Binomials by Monomials

Multiply each term of the binomial by the monomial.

Example 1:

$$\overset{\swarrow\,\text{terms}}{\underset{\uparrow \atop \text{Monomial}}{a}\underbrace{(b + c)}_{\nwarrow\text{Binomial}}} = ab + ac$$

Example 2:

$$5x(2x - y)$$

$$= 5x(2x + (-y))$$ (Optional step: express binomial as sum of 2 terms.)

$$= 5x(2x) + 5x(-y)$$

$$= 10x^2 + (-5xy)$$

$$= 10x^2 - 5xy$$

b) Multiplying Two Binomials

Multiply each term of second binomial by each term of the first binomial.

Example 1:

$$(a + b)(c + d) = ac + ad + bc + bd$$

Example 2:

$$(2x + y)(x - 5y)$$

$$= 2x(x) + 2x(-5y) + y(x) + y(-5y)$$

$$= 2x^2 - 10xy + xy - 5y^2$$

$$= 2x^2 - 9xy - 5y^2$$ (collect terms in xy)

c) Dividing Binomials by Monomials

Divide each term of the binomial by the monomial.

Example 1:

$$\frac{12x^2 + 4xy}{2x}$$

$$= \frac{12x^2}{2x} + \frac{4xy}{2x}$$ (problem reduces to dividing terms)

$$= 6x + 2y$$

Example 2:

$$\left.\begin{aligned} & -\frac{10x^2 - 4y^2}{2xy} \\ & = -\left(\frac{10x^2}{2xy} - \frac{4y^2}{2xy}\right) \end{aligned}\right\}$$ Note that minus sign in front of quotient applies to entire expression, hence the brackets

$$= -\left(\frac{5x}{y} - \frac{2y}{x}\right)$$

$$= -\frac{5x}{y} + \frac{2y}{x} \quad \underline{\text{or}} \; -5xy^{-1} + 2x^{-1}y$$

d) Generalizations to Polynomials

To multiply two polynomials, multiply each term of the first by each and every term of the second polynomial.

To divide a polynomial by a monomial, divide each term of the polynomial by the monomial.

VI Simplification of Algebraic Expressions

The order of operations, "BEDMAS" (see 421.10-1, V), holds for simplifying algebraic expressions just as for arithmetic expressions. The following examples illustrate the preceding rules for operations on powers, terms, and polynomials.

Example:

Simplify the following:

(a) $x^2 \div x + x^3 \div x^2 + x$

(b) $aba + aab$

(c) $4x - 7y - (3x - 4y) + x + 3y$

(d) $abc - abc\ (-2)(-\frac{1}{2})(-3)$

(e) $\frac{-6ab - 12a^2}{-3a} - \frac{3b^2 - 6ab}{-3b}$

Solutions:

(a) $x^2 \div x + x^3 \div x^2 + x$

$= x + x + x$ (÷ precedes +)

$= 3x$ (collect terms)

(b) $aba + aab$

$= aab + aab$ (order of a's, b's does not affect value of product)

$= a^2b + a^2b$

$= 2a^2b$ (collect terms)

(c) $4x - 7y - (3x - 4y) + x + 3y$

$= 4x - 7y - 3x + 4y + x + 3y$ (remove brackets preceded by minus sign by changing sign of <u>all</u> enclosed terms)

$= (4 - 3 + 1)x + (-7 + 4 + 3)y$ (collect like terms)

$= 2x$

(d) $abc - abc\ (-2)\ (-\frac{1}{2})(-3)$

$= abc - abc\ (-3)$ (3 negative factors (odd no.) give negative product)

$= abc + 3abc$ (to subtract, add the opposite)

$= 4abc$

(e) $\dfrac{-6ab - 12a^2}{-3a} - \dfrac{3b^2 - 6ab}{-3b}$

$= \dfrac{-6ab}{-3a} + \dfrac{-12a^2}{-3a} - (\dfrac{3b^2}{-3b} + \dfrac{-6ab}{-3b})$ (Express binomials as sum of 2 terms)

$= 2b + 4a - (-b + 2a)$

$= 2b + 4a + b - 2a$ (remove brackets)

$= 2a + 3b$ (collect terms

<u>NB</u> Brackets preceded by a "+" may be inserted or removed without altering enclosed terms, but brackets preceded by a "-" may be inserted or removed only by altering signs of <u>all</u> terms enclosed.

Assignment

1. If a = 12, b = 2 and c = -3, evaluate the following:

   (a) $-a + \frac{5b}{6} + \frac{cb}{a}$

   (b) $a + 2a - 3c^2$

   (c) $6b^2 - a - b^2 + c$

2. Simplify

   (a) a^4a^6 (b) $\frac{1}{2}a(\frac{1}{4}a^7)(\frac{3}{4}a^2)$

   (c) $b^3b^4b^5$ (d) $3 \times 3^2 \times 3^4$

   (e) $m^7 \cdot m^4 \div m^5$ (f) $a^6 \div a^{-5} \cdot a^8$

   (g) $\frac{a^7}{a^5} \cdot \frac{a}{a^4}$ (h) $\frac{b^6b^4}{b^3}$

   (i) $(a^7)^2$ (j) $(-3a^2)^3$

   (k) $(\frac{1}{2} x^4)^5$ (l) $\sqrt[3]{a} \quad \sqrt{a}$

   (m) $\sqrt{\sqrt[3]{a}}$ (n) $(-3xy^{\frac{1}{2}})^2$

   (o) $x^6x^{-2}x^{-4}$ (p) $\left(\frac{\sqrt{x}}{y^2}\right)^2$

3. Evaluate:

   (a) $3 \cdot \sqrt{3} \cdot \sqrt{3^3}$ (b) $\left(\frac{1}{4}\right)^{2{\cdot}5}$ (c) $(16)^{-0{\cdot}25}$

   (d) $\left(\frac{2^2}{3^3}\right)^{-1}$ (e) $(-3)^{-3}$ (f) $36^{1/2}$

   (g) $\left(-\frac{1}{32}\right)^{-1/5}$ (h) $(-8)^{5/3}$ (i) $\left(-\frac{27}{64}\right)^{-2/3}$

4. Write each expression without negative or zero exponents and simplify.

(a) $\frac{3a^0 - b^0}{a^0 + (3b)^0}$

(b) $(16\ x^{16})^{1/2}$

(c) $-(-3a^{0.4}\ b^{0.6})^5$

(d) $\frac{3x^{-3}\ y^2}{6^{-1}\ Z^{-2}}$

5. The mass of an electron is 0.00055 a.m.u. and 1 a.m.u. is 1.66×10^{-24} g. Calculate the mass of the electron in grams.

6. It requires 3.1×10^{10} fissions per second to produce 1 watt of energy. How many fissions per second are required to produce 200 Megawatts.

7. Simplify:

(a) $2a + 3a + 6a$

(b) $5x^2 + 2x + 3 + 5xy + 4x + 2 + x^2$

(c) $5x + 12y + 20x + 8y$

(d) $2c + 8a + 6c + 2b + 3b + 4c$

(e) $3j + k - 4j + 11k - 7k - j$

(f) $a + a + a + a - 5a + 11a$

(g) $x + 3xy^2 + 2x + y - 3x + y^2x$

(h) $x^2y^3 + 3x^2y^3 + 1 - x$

(i) $x + y + z - 2y + 3z - 6x$

8. Simplify:

(a) $\frac{-6y^2}{3y}$ (b) $\frac{7x}{21xy}$

(c) $\frac{15ab}{3a}$ (d) $-\frac{30pg}{15pg}$

(e) $\frac{-2x^2}{-2x}$ (f) $\frac{-27ab^2c}{9b}$

(g) $\frac{6y}{3x}$ (h) $\frac{x^2yz}{2yz}$

(i) $(6x^2y)(-4y^3p)$ (j) $(-11pq)(-2ps^2t)$

9. Simplify:

(a) $(x + 4y)(x - 8y)$ (b) $(3x + 2)(5x + 4)$

(c) $(3a - 2c)(4a - 5c)$ (d) $(x^2 - y)(y + x^2)$

(e) $\frac{6x^2 - 2x}{-2x} - \frac{9x - 3}{-3}$ (f) $\frac{4x + 10y}{2}$

(g) $\frac{-10a^2 - 5}{-5} - \frac{-3a^2 - 6}{-3}$ (h) $\frac{8x^2 + 10x}{2x}$

(i) $\frac{3x^2 - 15x}{3x} - \frac{12x - 18}{-6}$ (j) $\frac{14x^2 + 21x}{7x} - \frac{3x^2 + 9x}{3}$

10. Simplify:

(a) $(8mn)(4mx)$

(b) $(9abc)(-4bcd)$

(c) $-3y(6m - 5t)$

(d) $5(4h - 6k)$

(e) $-3(x + y) + 10(2x - 3y) + 5(2y - 3x)$

(f) $2x(x + y) - (x^2 + xy) + (x - x)$

(g) $8c + 3k - (5c + 2k)$

10. Cont'd

(h) $2b - c + (8c - 4b) + b$

(i) $3a - 5x - (4a + x) - 2a$

(j) $ab + ab^2 \div b$

(k) $xy + x^2y^2 \div xy$

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Mathematics - Course 421

GRAPHS

I Uses of Graphs

Graphs are used to

(1) display the relationship between 2 or more variables
(2) summarize data pictorially for easy assimilation.

II Rectangular Co-ordinate System

A *rectangular co-ordinate system* is set up by drawing two mutually perpendicular lines *(axes)* which intersect at the *origin, 0.* The vertical axis is usually called the *y-axis;* its upward branch is labelled "*y*" to indicate that *y* increases vertically upwards. The horizontal axis is usually called the *x-axis;* its rightward branch is labelled "*x*" to indicate that *x* increases horizontally rightward. The axes divide the *xy-plane* into four *quadrants*, as in Figure 1.

A uniform scale of length units is marked on each axis, starting from 0. The position of a point in the plane is specified by its distance from the y-axis (the *x co-ordinate* or *abscissa*) and its distance from the x-axis (the *y co-ordinate* or *ordinate).* For example the point P(2, 3), with x co-ordinate 2 and y co-ordinate 3, is located in the plane at the intersection of perpendiculars erected to the x-axis, 2 units from 0, and to the y-axis, 3 units from 0 (see Figure 2).

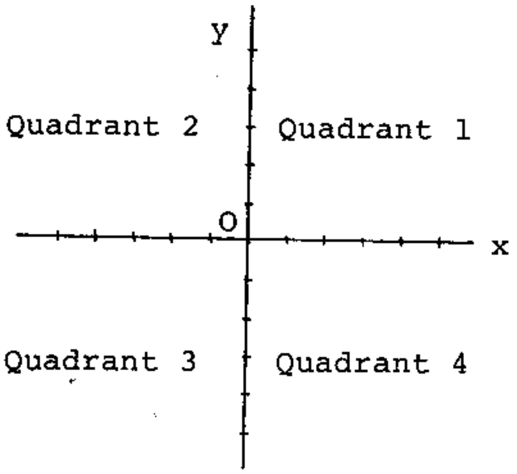


Figure 1

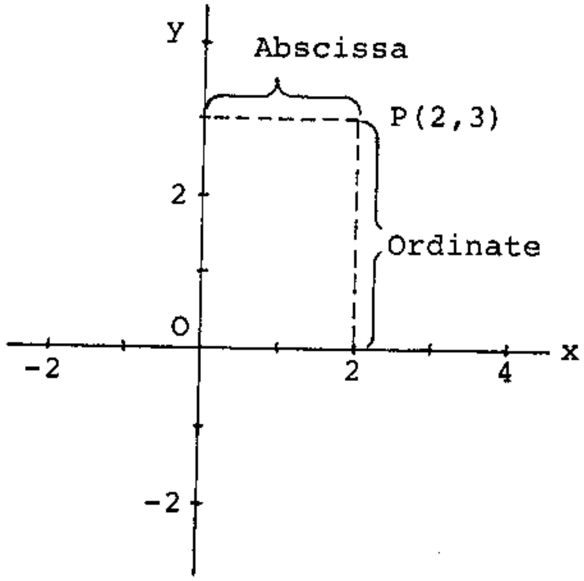


Figure 2

August 1978

The tedious process of constructing perpendiculars is usually eliminated by the use of *squared paper*.

III Data Graphs

Steps to Plotting a Data Graph

Step 1:

Select a piece of graph paper of suitable dimensions and size of grid squares to display data.

Step 2:

Select the *independent variable* (the one deliberately varied in an experiment) to be displayed horizontally and the *dependent variable* (the one which responds to changes in the independent variable) to be displayed vertically. For example, temperature would normally be plotted vertically on a temperature - time graph. Note, however, that choice of variable to be displayed vertically is often a matter of personal judgement or convenience - eg, graph of voltage vs current, where either variable could be independent.

Step 3:

Choose display ranges and scales to spread data over about two-thirds or more of available space along either axis. Draw axis and mark on scales.

Step 4:

Label axes with respective quantities and units thereof.

Step 5:

Plot data. Make data points visible by some method such as circling them.

Step 6:

Indicate the pattern or trend of the data by

(a) joining successive data points by straight line segments to produce a *LINE GRAPH*, if the data does not obey a simple relationship, or

(b) drawing a smooth averaging *CURVE* through the data ("*curve*" here includes the case of the straight line), if the data does obey a simple law.

Step 7:

Place a suitable title on the graph.

Example 1: Hospital Patient's Temperature Chart

The following table indicates a patient's temperature readings taken at 6-hour intervals May 1 to 3 inclusive. Plot a temperature-time graph for the patient.

Day	May 1				May 2				May 3			
Time (hr)	0000	0600	1200	1800	0000	0600	1200	1800	0000	0600	1200	1800
Temp (°C)	37.6	37.3	37.1	36.9	36.9	36.9	37.1	38.9	38.1	37.2	36.9	36.9

Note that all the above temperature readings lie between 36.9° and 38.9°C. The above data have been plotted in Figure 3, using an unsuitable temperature display range of 0° to 40°C, and again in Figure 4 using a temperature display range of 36.8° to 39.0°C. Figure 4 is obviously much easier to read and interpret than Figure 3. This contrast between Figures 3 and 4 illustrates the importance of choosing a suitable scale and display range (step 3 above).

Note that a *line graph* has been produced (step 6 above), since there is no simple law relating a patient's temperature with time.

Figure 3: HOSPITAL PATIENT'S TEMPERATURE CHART

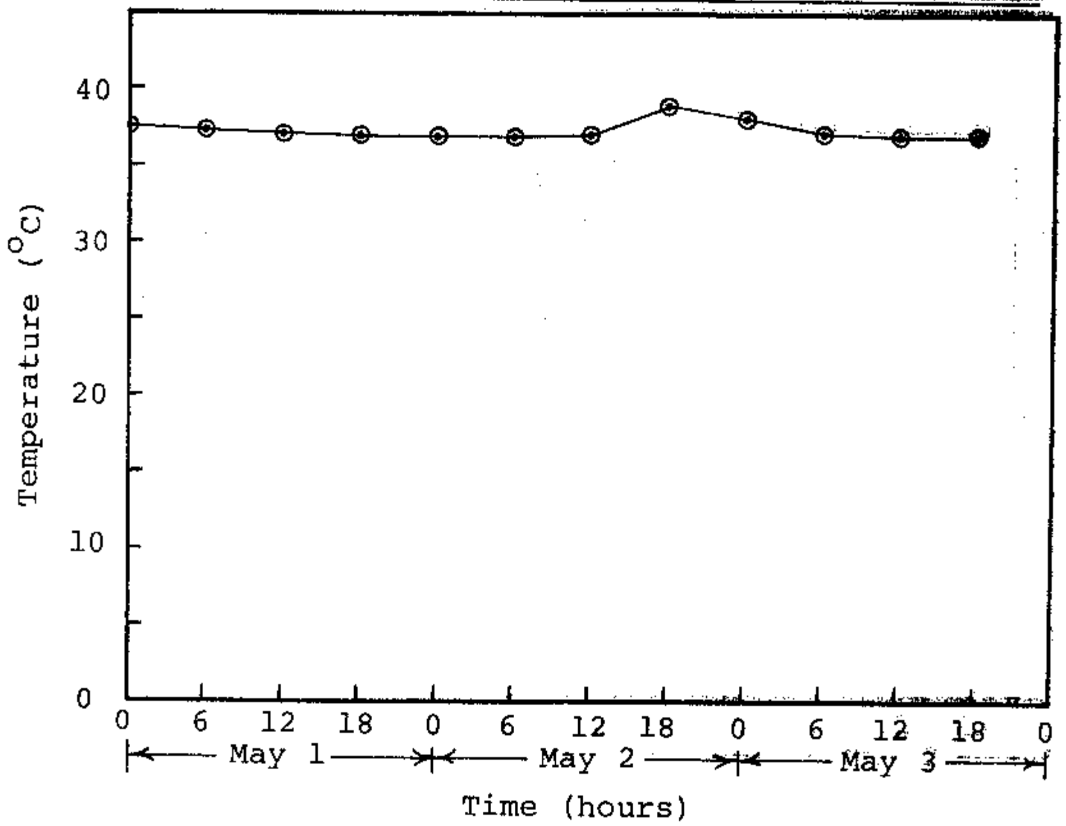
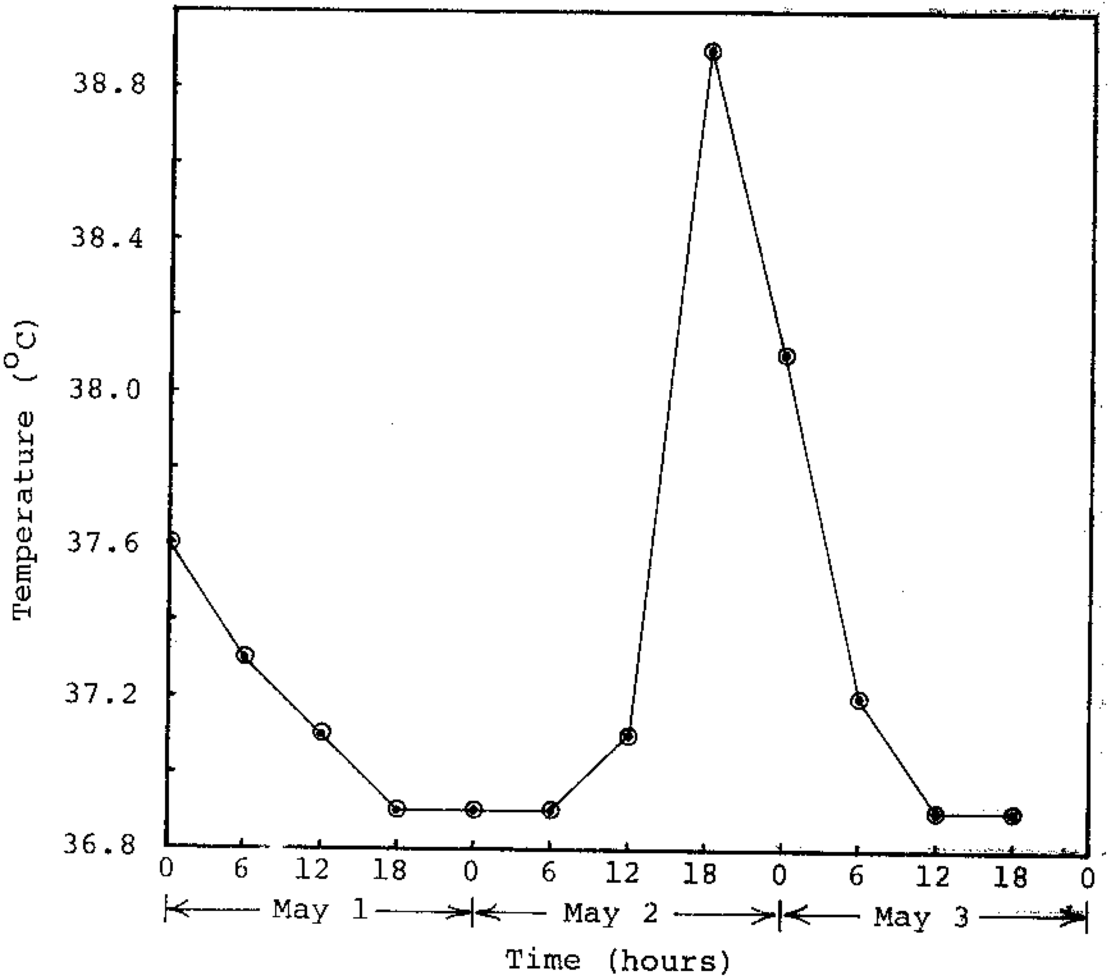


Figure 4: HOSPITAL PATIENT'S TEMPERATURE CHART



Example 2: Average Weight vs Age of Teenage Boys

The following table gives the weights of boys of various ages. Draw a graph to illustrate this variation. From the graph, find the average weight of (a) a 12½ year old boy (b) a 16 year old boy.

Age (years)	10	11	12	13	14	15
Weight (kg)	34.7	36.3	38.6	41.7	45.8	51.7

A step-by-step solution is given for this example:

Step 1:

Graph paper with one millimeter squares is suitable for this application.

Step 2:

Weight will be plotted vertically and age horizontally. (Weight is responding to age, not age to weight.)

Step 3:

y-axis: weights from 34 to 60 kg, at scale 2 kg = 1cm

x axis: ages 10 to 16 years at scale 1 year = 2 cm.

Steps 4 and 5:

See Figure 5 for axis labels, data plot.

Step 6:

Since there is an obvious relationship between average weight and age of boys 10 to 15 years old, a smooth curve is drawn through the data.

Step 7:

See Figure 5 for title.

The curve drawn in Figure 5 represents the relationship between average weight and age of boys of all ages from 10 to 15 years, inclusive. Thus the average weight of a 12½ year old boy, from Figure 5, is 40.1 kg. This is an example of *INTERPOLATION* - estimating values of variables between given data points.

By assuming that the trend of the relationship continues to age 16 years, one can estimate the average weight of a 16 year old boy. As seen from Figure 5, this weight is 60.0 kg. This is an example of *EXTRAPOLATION* - estimating values of variables outside of the range of the given data.

Note that interpolation gives more reliable estimates than extrapolation because the former is guided by given data on both sides of the estimate, whereas the latter is guided by data on one side only of the estimate, and the assumption that the trend of the data continues as far as the estimated value.

Example 3: Load - Effort Relationship for a Machine

The following table contains experimentally determined values of the effort required to move various loads, using a certain machine. Draw a graph to show the load-effort relationship.

Load (kg)	30	40	60	70	80
Effort (kg)	2.13	2.6	3.8	4.3	5.1

The required graph is shown in Figure 6. Note that the curve best fitting the data in this example is a straight line. The curve itself represents an estimate of the true relationship between load and effort. The various data points lie slightly above or below the curve simply because of the uncertainty inherent in the experimental measurements.

When the curve best fitting data is deemed to be a straight line, as in this example, the relationship between the variables is said to be *linear* (noun "line"; adjective "linear").

Figure 5: AVERAGE WEIGHT vs. AGE FOR TEENAGE BOYS

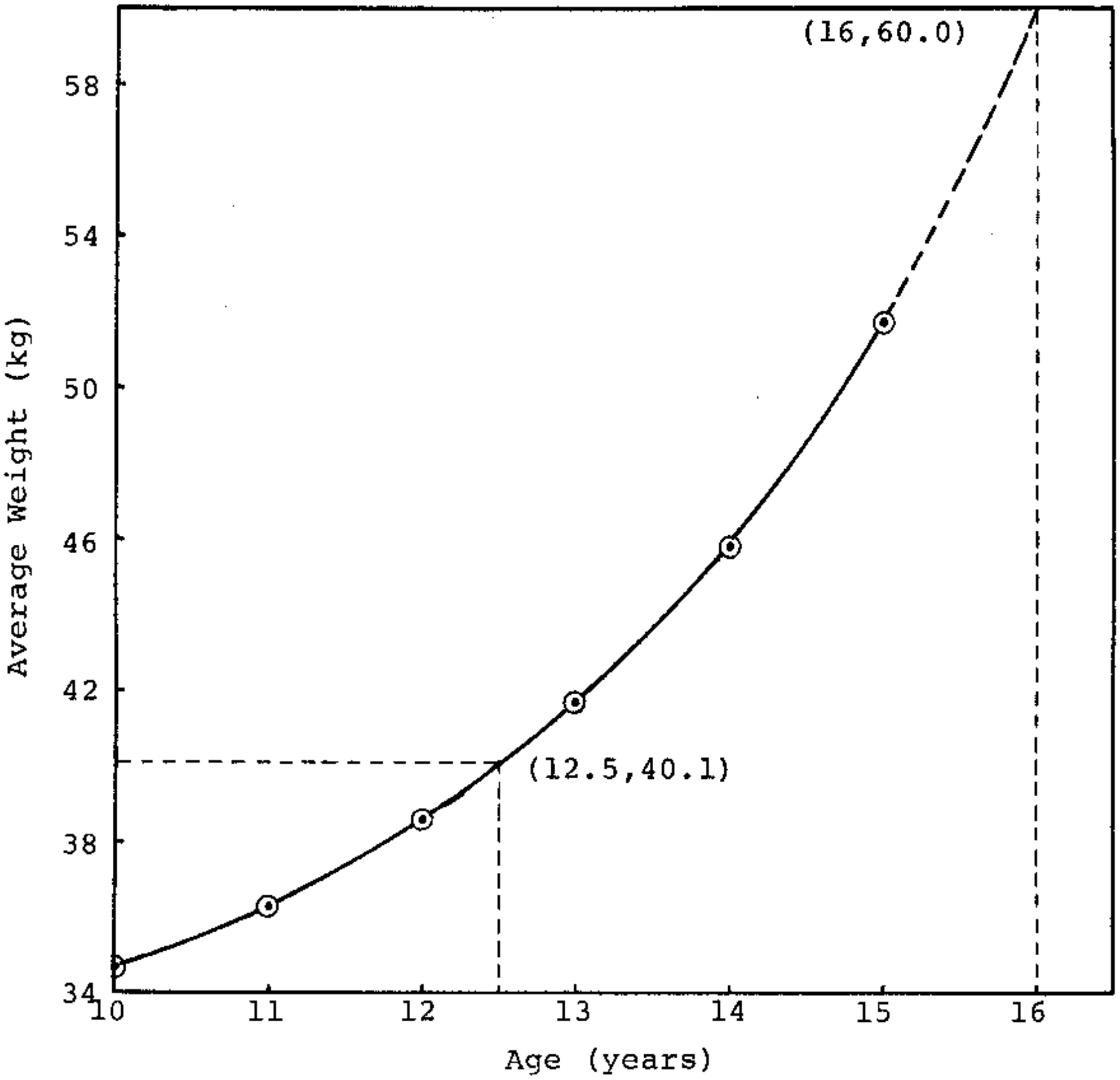
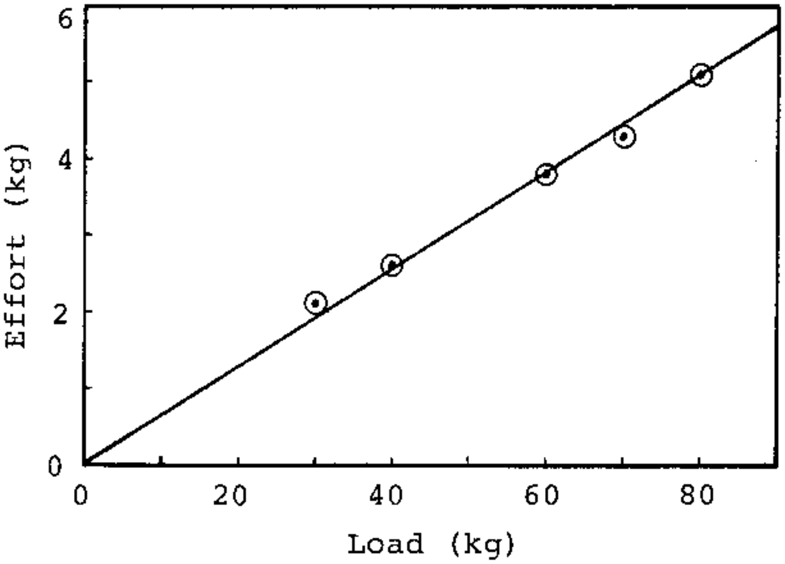


Figure 6: LOAD-EFFORT RELATIONSHIP FOR A MACHINE



ASSIGNMENT

1. Plot the following points:

   (a) P(3,4)
   (b) Q(-2,4)
   (c) R(-5,-4)
   (d) S(4,-2)
   (e) T(15,10)
   (f) U(-3,8)

2. The following table shows the temperature at two-hourly intervals for one day. Plot a graph to illustrate this variation in temperature.

Time	2am	4	6	8	10	Noon	2pm	4	6	8	10	12
Temp (°C)	9	8	9	12	14	18	23	26	22	20	16	14

3. The following table gives the current, I, in a circuit, for various values of the resistance, R, when the voltage remains constant.

R(ohms)	2	4	8	12	16	20	40	60
I(amperes)	60	30	15	10	7.5	6	3	2

   Plot a graph showing how the current varies with the resistance and estimate

   (a) The current when R = 10 ohms and
   (b) The resistance required to give a current of 50 amperes.

4. The pressure, P, at different depths, h, in a liquid is found to be as follows:

h(cm)	0	10	20	30	35
P(kPa)	103	261	419	577	656

   Plot the graph and from the graph estimate:

   (a) The pressure at a depth of 50 cm.
   (b) The depth at which the pressure is 300 kPa.

L. Haacke

Mathematics - Course 421

GRAPHING FUNCTIONS

I Introduction to Functions

Definition:

One variable is a *function* of another variable if a unique value of the first variable corresponds to each value of the other, ie, if the two variables are related by some formula (loosely speaking).

Notation:

The notation $f(x)$, $A(r)$, $P(T)$, etc is used to denote f as a function of x, A as a function of r, P as a function of T, etc.

Example 1:

The area A of a circle is a function of its radius r according to the formula,

$A(r) = \pi r^2$ (read "A at r equals πr^2")

ie, a definite value of A corresponds to each value of r

eg, $A(1) = \pi(1)^2 = 3.14$

$A(5) = \pi(5)^2 = 78.5$

$A(0.1) = \pi(0.1)^2 = 0.0314$

etc.

Example 2:

$f(x) = x^3 - 5x$ (read "f at x equals $x^3 - 5x$")

Here f is a function of x since the formula gives a unique value of f for each value of x

eg, $f(0) = 0^3 - 5(0) = 0$

$f(1) = 1^3 - 5(1) = -4$

$f(-2) = (-2)^3 - 5(-2) = 2$

etc.

Functions of Several Variables:

If G is a function of n variables, x_1, x_2, ..., X_n) one writes

$$G(x_1, x_2, \ldots, X_n)$$

Example 3:

Cylinder volume V is a function of both height h and radius r, according to the formula,

$$V(r, h) = \pi r^2 h$$

ie, each pair of r and h gives a unique volume

eg, $V(1, 1) = \pi(1)^2\ (1) = 3.14$

$V(2, 5) = \pi(2)^2\ (5) = 62.8$

etc.

Dependent and Independent Variables:

The *independent variable* is the one to which values are assigned arbitrarily, and the *dependent variable* is the one given by the formula.

eg,

dependent variables

$$\overset{\downarrow}{A}(\underset{\uparrow}{r});\qquad \overset{\downarrow}{f}(\underset{\uparrow}{x});\qquad \overset{\downarrow}{G}(\underset{\uparrow}{x_1}, \underset{\uparrow}{x_2}, \ldots, \underset{\uparrow}{X_n})$$

independent variables

II Graphing Functions

Usually the independent variable is plotted along the x-axis (horizontally) and the dependent variable along the y-axis (vertically) - cf 421.40-1, part III.

The steps to graphing a function are similar to those outlined in § 221.40-1, part III for data graphs, with the following notable differences:

(1) The table of values must be calculated, using the function relationship.

(2) The plotted points are <u>always</u> joined by a smooth curve (except for discontinuous functions, which are beyond the scope of this text).

(3) The curve is labelled with the equation which it represents.

Example 1:

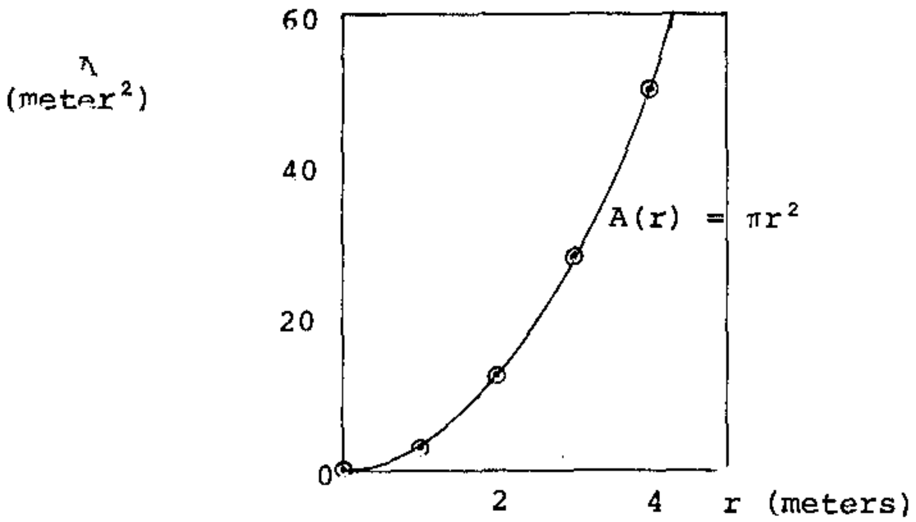
Plot a graph showing circle area A as a function of radius r in meters, $0 \le r \le 4$.

Solution:

Use $A(r) = \pi r^2$ to generate a table of values.

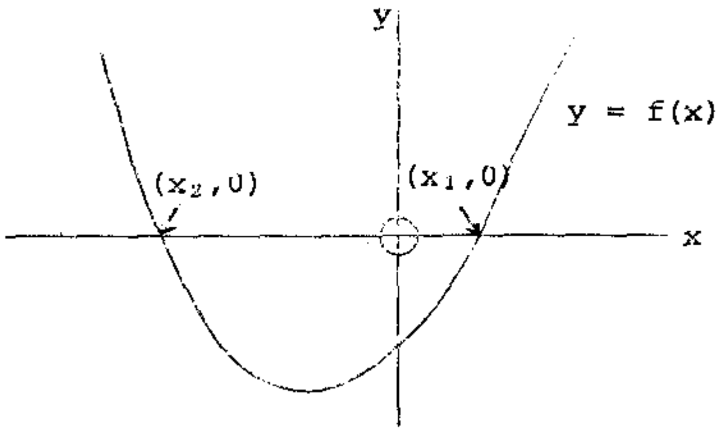
r meters	0	1	2	3	4
A(r) meters^2	0	3.1	12.6	28.3	50.3

Graph of Circle Area vs Radius



Roots of an Equation:

The *roots* of any equation of the form $f(x) = 0$ are the x values which satisfy this equation (make it true). Clearly, the x-coordinates of the x-intercepts of the curve $y = f(x)$ are the roots of $f(x) = 0$, as illustrated below:



x_1, x_2 are the roots of $f(x) = 0$

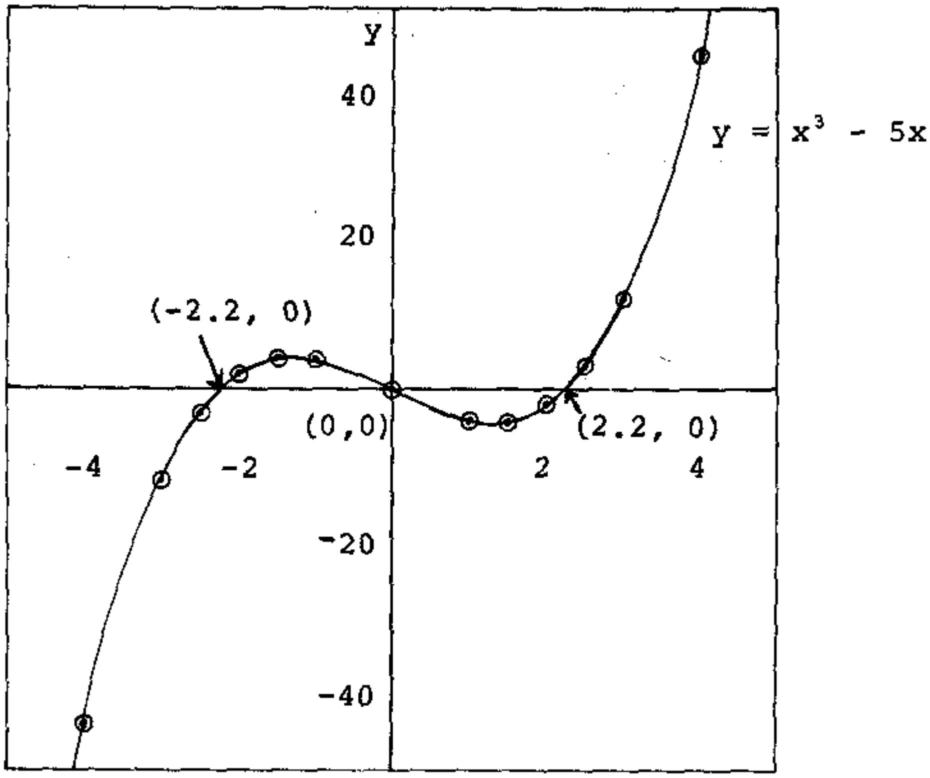
Example 2:

Graph the function $f(x) = x^3 - 5x$ and find the roots of $x^3 - 5x = 0$ from the graph.

Solution:

Let $y = f(x)$, and use $y = x^3 - 5x$ to generate a table of values

x	0	±1	±1.5	±2	±2.5	±3	±4
y	0	∓4	∓4.1	∓2	±3.1	±12	±44



Roots of $x^3 - 5x = 0$ are $x = \pm 2.2$ and $x = 0$

ASSIGNMENT

1. Express each of the following statements in functional notation, and give the exact formula for the notation:

   (a) The circumference C of a circle is a function of its radius r.

   (b) The distance d travelled in time t at a uniform speed v is a function of t and v.

   (c) The total area A of the surface of a right circular cylinder is a function of its height h and radius r of its base.

2. Given $f(x) = 2x - 3$, find $f(6)$, $f(0)$, $f(-2)$.

3. Given $H(x) = x(x - a)(x - 1)$ find $H(0)$, $H(1)$, $H(a)$.

4. Find the length d of a diagonal of a square as a function of the perimeter p of the square.

5. Graph the following functions $f(x)$ and find the roots of $f(x) = 0$ from the graphs:

   (a) $4 - x^2$

   (b) $x^2 + 2x + 2$

   (c) $2 + 9x - x^3$

   (d) $x^2 - x - 6$

   (e) $x^3 - 3x - 1$

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Mathematics - Course 321

LOGARITHMS AND EXPONENTIALS

I INTRODUCTION

(a) Exponential Functions

DEFINITION:

An *exponential function* is a function of the form

$f(x) = a^x$, where "a" is a real positive constant.

The distinction between the exponential function, a^x, and the more familiar power function, x^a, should be clear from the following example in which a = 2:

Example 1:

Plot on the same graph the functions $y = 2^x$ and $y = x^2$ over the domain $-4 \leqslant x \leqslant 4$.

x	-4	-3	-2	-1	0	1	2	3	4
2^x	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8	16
x^2	16	9	4	1	0	1	4	9	16

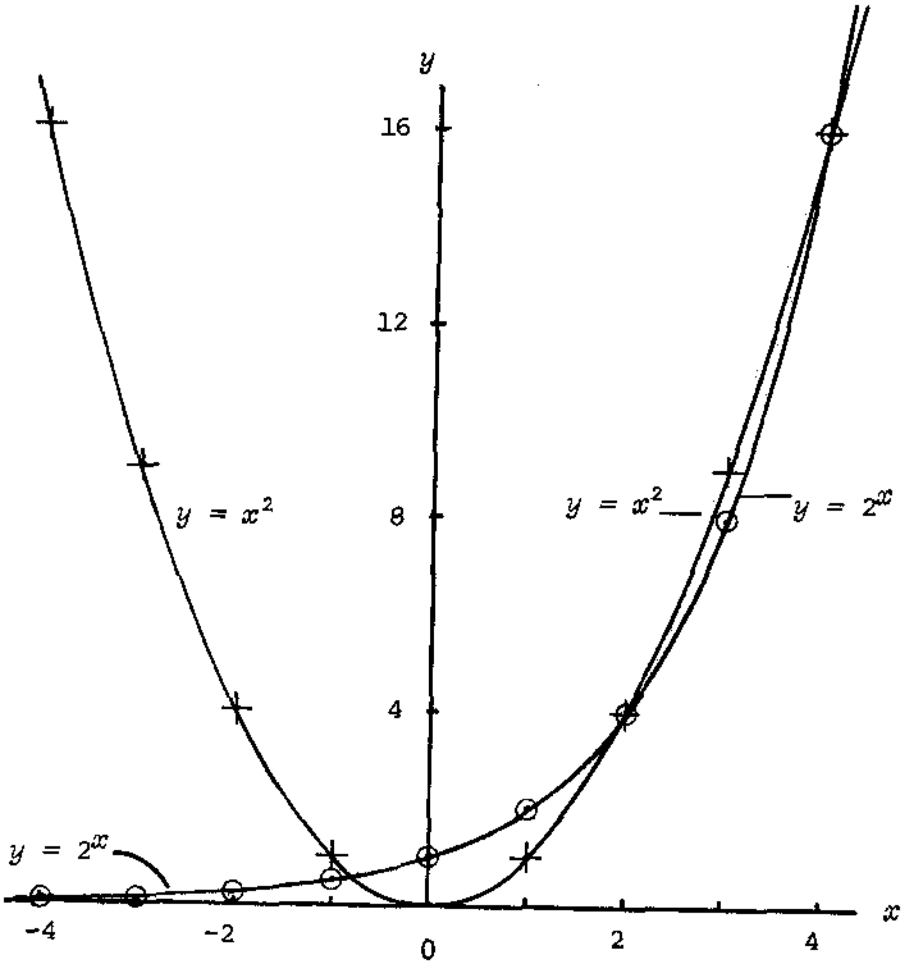


Figure 1

Note that the curve $y = 2^x$ is *asymptotic* to the negative x-axis, ie, the curve approaches ever more closely to the negative x-axis as the magnitude of x grows, but never actually reaches the axis for any finite x-value. The curve $y = 2^{-x}$ is the mirror image in the y-axis of $y = 2^x$, and is asymptotic to the positive x-axis. (Check this.)

Thus, in general, if $a > 1$, the exponential functions a^x and a^{-x} have the characteristic shapes illustrated in Figure 2.

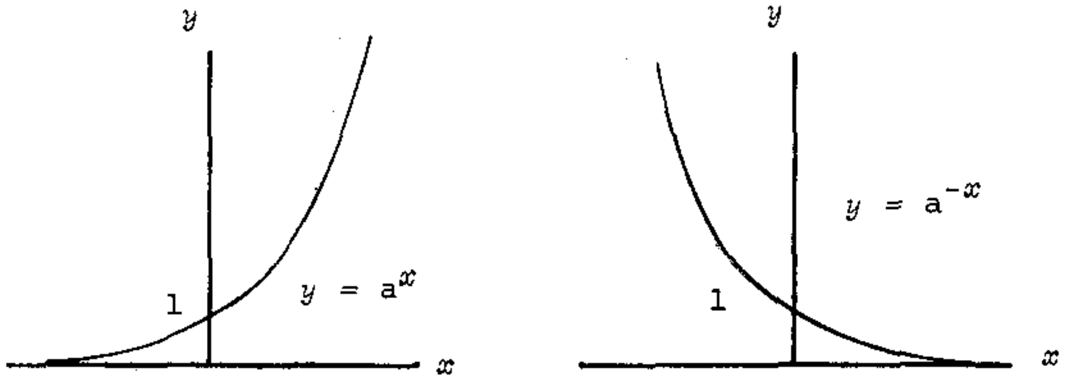


Figure 2

(b) Logarithmic Functions

DEFINITION:

The *logarithm of x to the base "a"* designated "$\log_a x$", is the exponent to which "a" must be raised to produce x.

That is: $\log_a x = y \Longleftrightarrow a^y = x$

eg, $\log_3 9 = 2$, since $3^2 = 9$

eg, $\log_2 64 = 6$, since $2^6 = 64$

In general, the curve $y = \log_a x$ has the characteristic shape shown in Figure 3.

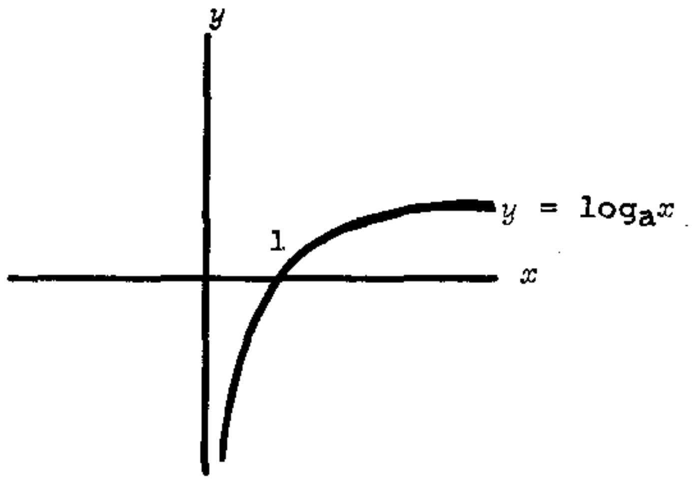


Figure 3

The bases 10 and e are so commonly used as to justify log x and ℓnx functions on scientific calculators. These are the so-called:

(1) *Common Logarithms,* to base 10, and

(2) *Natural Logarithms,* to base "e".

(e = 2.718281828...)

The rationale behind the special provisions for common logarithms is our use of the decimal system (base 10), while the rationale for natural logarithms is the fact that exponential functions (base "e") are abundant in technical applications.

eg: $V(t) = V_0 e^{-t/RC}$, where
Vo = initial voltage at t=o
V(t) = voltage at time t
t = time in seconds
R = resistance
C = capacitance

The above definition for $\log_a x$ is restated here specifically for common and natural logarithms:

DEFINITION:

The *common logarithm of x,* designated "$\log_{10} x$" (or simply "log x"), is the exponent to which 10 must be raised to produce x.

eg, $\log 1000 = 3$, since $10^3 = 1000$

eg, $\log \sqrt{10} = \frac{1}{2}$, since $10^{\frac{1}{2}} = \sqrt{10}$

DEFINITION:

The *natural logarithm of x,* designated "$\log_e x$" (or simply "ℓnx"), is the exponent to which "e" must be raised to produce x.

eg, $\ln e^5 = 5$, since $e^5 = e^5$

eg, $\ln \sqrt[3]{e} = \frac{1}{3}$, since $e^{1/3} = \sqrt[3]{e}$

II USE OF LOGARITHMS IN COMPUTATION OF COMPLEX ARITHMETIC EXPRESSIONS

Logarithms are used to reduce the operations of multiplication, division, and exponentiation to addition, subtraction, and multiplication, respectively, according to the following laws:

LAW 1: $\log XY = \log X + \log Y$

LAW 2: $\log \frac{X}{Y} = \log X - \log Y$

LAW 3: $\log X^n = n \log X$

With the introduction of the scientific calculator, the computation of complex arithmetic expressions has been greatly simplified. In fact, the use of logarithms in their evaluation has been rendered virtually obsolete. However, the trainee should become fully familiar with the laws governing the use of logarithms as an aid in solving some types of problems which will be introduced later in this lesson. To this end, several examples are now presented which illustrate the use of logarithms in the evaluation of complex arithmetic and algebraic expressions.

Example 1:

Evaluate $\sqrt[5]{(0.007294)^3}$

Solution:

(a) Modern calculator technique (use of Y^x)

$$\begin{aligned}\sqrt[5]{(0.007294)^3} &= (0.007294)^{3/5}\\ &= (0.007294)^{0.6}\\ &= 0.05221\end{aligned}$$

(b) Use of logarithms (obsolete method)

$$\begin{aligned}\text{Let } x &= \sqrt[5]{(0.007294)^3}\\ \text{then } \log x &= \log (0.007294)^{3/5}\\ &= \frac{3}{5} \log (0.007294)\\ &= \frac{3}{5} (-2.1370)\\ &= -1.2822\\ \therefore \log x &= -1.2822\end{aligned}$$

How does one now find the value of x?

Recall that log x, by definition, is the exponent to which 10 is raised to produce x. Thus,

$$x = 10^{-1.2822}$$

(This process of exponentiating to find x is also called taking the antilogarithm, to base 10, of -1.2822.)

<u>Example 2</u>

Evaluate $$\frac{(7.236)^{1/3} \times (4.36)^2}{(0.00287)^4}$$

<u>Solution:</u>

$$\text{Let} \quad x = \frac{(7.236)^{1/3} \times (4.36)^2}{(0.00287)^4}$$

$$\begin{aligned}
\text{Then log} \quad x &= \log \left[\frac{(7.236)^{1/3} \times (4.36)^2}{(0.00287)^4}\right] \\
&= \log \left[(7.236)^{1/3} \times (4.36)^2\right] - \log(0.00287)^4 \\
&= \log \quad (7.236)^{1/3} + \log(4.36)^2 - \log(0.00287)^4 \\
&= \frac{1}{3} \log 7.236 + 2 \log 4.36 - 4 \log(0.00287) \\
&= \frac{1}{3} (0.8595) + 2(0.6395) - 4(-2.5421) \\
&= 11.7339
\end{aligned}$$

$$\begin{aligned}
\therefore \quad \log x &= 11.7339 \\
x &= 10^{11.7339} \\
&= 5.42 \times 10^{11}
\end{aligned}$$

Example 3:

Express $\sqrt[3]{\dfrac{X^2\ \sqrt{Y}}{Z^5}}$ in terms of log X, log Y and log Z.

Solution:

$$\log \sqrt[3]{\frac{X^2\ \sqrt{Y}}{Z^5}} = \log\left(\frac{X^2\ \sqrt{Y}}{Z^5}\right)^{1/3} \qquad \text{(power law: } x^{1/n} = \sqrt[n]{x}\text{)}$$

$$= \frac{1}{3}\log\left(\frac{X^2\ \sqrt{Y}}{Z^5}\right) \qquad \text{(Law 3)}$$

$$= \frac{1}{3}\left[\log\ (X^2\ \sqrt{Y}) - \log\ Z^5\right] \qquad \text{(Law 2)}$$

$$= \frac{1}{3}\left[\log\ X^2 + \log\ \sqrt{Y} - \log\ Z^5\right] \qquad \text{(Law 1)}$$

$$= \frac{1}{3}\left[2\ \log\ X + \frac{1}{2}\ \log\ Y - 5\ \log\ Z\right] \qquad \text{(Law 3)}$$

III CONNECTION BETWEEN EXPONENTIALS AND LOGARITHMS

Taking the logarithm of x to base "a" and raising "a" to the exponent x are opposite operations in the same sense that multiplication and division are opposite operations, ie, the one operation 'undoes' the other.

For example, any one of the following sequences of operations on x will give x back again as the final result:

(1) first multiply by 2, then divide result by 2, ie, $(2x) \div 2 = x$.

(2) first divide by 2, then multiply result by 2, ie, $(x \div 2)\ (2) = x$.

(3) first take logarithm to base 2, then raise 2 to the result, ie $2^{\log_2 x} = x$.

(4) first raise 2 to exponent x, then take logarithm of result to base 2; ie, $\log_2 2^x = x$.

The above explanation can also be presented in tabular form.

Table Illustrating the Effect of Applying Opposite Operations Consecutively

Start With	x	x	x	x
First Operation	add k	multiply by k	exponentiate with base a	take log to base a
Interim Result	$x + k$	kx	a^x	$\log_a x$
Second (Opposite) Operation	subtract k	divide by k	take log to base a	exponentiate with base a (ANTILOG)
Final Result	$(x+k)-k=x$	$(kx)\div k=x$	$\log_a a^x=x$	$a^{\log_a x}=x$
Example	$(x+2)-2=x$	$(2x)\div 2=x$	$\log_2 2^x=x$	$2^{\log_2 x}=x$

The connection between logarithms and exponentials can be further summarized as follows:

$$\boxed{\log_a a^x = x = a^{\log_a x}}$$

The corresponding statements for common and natural logarithms are:

$\log 10^x = x = 10^{\log x}$ (common logs)

$\ln e^x = x = e^{\ln x}$ (natural logs)

At this point the trainee should be able to evaluate simple expressions involving logarithms, without recourse to aids, by applying the foregoing definitions.

Example 4:

Evaluate without recourse to aids: $5^{\log_5 x}$

Solution:

By definition, $\log_5 x$ represents the number to which 5 must be raised to produce x. Therefore, in the above expression, 5 is being raised to that number which will produce x.

ie: $5^{\log_5 x} = x$

Example 5:

Evaluate without recourse to aids: $e^{\ln x}$

Solution:

By definition, $\ln x$ represents the number to which e must be raised to produce x. Therefore, in the above expression, e is being raised to that number which will produce x.

ie: $e^{\ln x} = x$

IV SOLVING EXPONENTIAL EQUATIONS

Example 6:

Solve for x correct to 2 decimal places: $e^{-0.6x} = 5$

Solution:

$\ln e^{-0.6x} = \ln 5$ (take natural log both sides)

ie, $-0.6\,x = \ln 5$

$$\therefore\ x = \frac{\ln 5}{-0.6}$$

$$= \frac{1.6094}{-0.6}$$

$$= -2.6823$$

$\therefore$ correct to two decimal places, $x = -2.68$

Example 7:

Solve for x correct to 2 decimal places: $3^x = 5$

Solution:

Method (i)

$$\log 3^x = \log 5$$

$$x \log 3 = \log 5$$

$$x = \frac{\log 5}{\log 3}$$

$$= \frac{0.6990}{0.4771}$$

$$= 1.4649$$

$\therefore$ $x = 1.46$, correct to 2 decimal places

Method (ii)

$\ln 3^x = \ln 5$ (take natural log of both sides)

$x \ln 3 = \ln 5$ (law 3)

$$x = \frac{\ln 5}{\ln 3}$$

$$= \frac{1.6094}{1.0986}$$

$$= 1.4649$$

$\therefore$ $x = 1.46$, correct to 2 decimal places.

This example has been evaluated using both common and natural logarithms to demonstrate that, regardless of which base is used, the answer will be the same.

Example 8:

The activity of a radioactive source after t seconds is given by:

$$A(t) = A_0 e^{-\lambda t}$$

where A_0 = original activity at t = 0, and λ is the decay constant in s^{-1} (per second)

(a) If A_0 = 9.5 Ci, A(t) = 7.2 Ci, and t = 2 hr, calculate λ.

(b) Using that value of λ, calculate the half life of the radionuclide (ie, the time for the activity to decrease by a factor of 2).

Solution:

Method (i)

(a) $7.2 = 9.5\ e^{-7.2 \times 10^3 \lambda}$, (t = 2 hr = 7.2×10^3 sec)

$\log 7.2 = \log 9.5 - 7.2 \times 10^3\ \lambda \log e$ (common log of both sides)

$$\therefore \quad \lambda = \frac{\log 7.2 - \log 9.5}{-\ 7.2 \times 10^3 \log e}$$

$$= \frac{0.8573 - 0.9777}{-7.2 \times 10^3 \times 0.4343}$$

$$= 3.85 \times 10^{-5}$$

$$\therefore \quad \lambda = 3.85 \times 10^{-5}\ s^{-1}$$

Method (ii)

$$7.2 = 9.5e^{-7.2 \times 10^3\lambda}$$

$$\ln 7.2 = \ln 9.5 - 7.2 \times 10^3\lambda \ln e \qquad \text{(nat. log of both sides)}$$

$$\therefore \quad \lambda = \frac{\ln 7.2 - \ln 9.5}{-7.2 \times 10^3 \ln e}$$

$$= \frac{1.9741 - 2.2513}{-7.2 \times 10^3 \times 1}$$

$$= 3.85 \times 10^{-5}$$

$$\therefore \quad \lambda = 3.85 \times 10^{-5}\ s^{-1}$$

NOTE: Whether you use common logs or natural logs, the answer is the same.

(b) Let T be the half-life of the radionuclide

After 1 half-life, $A(T) = .5\ A_0$

ie: $$.5\ A_0 = A_0e^{-3.85 \times 10^{-5}\ T}$$

$$\therefore \quad .5 = e^{-3.85 \times 10^{-5}\ T}$$

$$\therefore \ln\ .5 = -3.85 \times 10^{-5}\ T \ln e$$

$$\therefore \quad T = \frac{\ln .5}{-3.85 \times 10^{-5} \times \ln e}$$

$$= \frac{-0.6931}{-3.85 \times 10^{-5} \times 1}$$

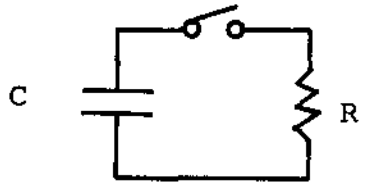
$$= 1.8 \times 10^4\ s$$

$$\therefore \quad T = 1.8 \times 10^4\ s = 5\ h$$

ASSIGNMENT

1. If, at $t = o$, the switch is closed in the circuit illustrated below, the voltage V across the capacitor after t seconds is given by the formula,

$$V(t) = V_o e^{-t/RC}$$



where V_o volts is the original voltage across the capacitor at time $t = o$,

R ohms is the resistance in the circuit, and

C farads is the capacitance of the capacitor.

Find (i) R

(ii) the discharge current, $I(t)$ (recall Ohms Law: $I = \frac{V}{R}$)

if (a) $V_o = 12$ V, $V(t) = 2$ V, $t = 6$ s, $C = 2$ F

(b) $V_o = 1$ V, $V(t) = 0.1$ V, $t = 10^{-3}$s, $C = 200$ μF.

2. A radioactive source decays from 10 Ci to 4.5 Ci in 3.0 hours. Calculate

(a) the decay constant λ in s^{-1}

(b) the half-life in hours

of the source. ($A(t) = A_o e^{-\lambda t}$)

3. Evaluate without recourse to aids:

(a) $\ln \sqrt{e}$

(b) $10^{\log 0.6}$

(c) $e^{2\ln 8}$

(d) $3^{\log_3 4}$

(e) $\log_5 5^{-0.2}$

(f) $2 \log 10^{-5}$

(g) $\log_2 1024$

(h) $2 \log_5 625$

4. A 0.5 μF capacitor, resistor and switch are placed in series in a circuit. The capacitor is charged to a voltage of 12 V when the switch is closed. If the voltage decays to 0.1 V after 2 ms, what is the resistance value in the circuit?

$(V(t) = V_o e^{-t/RC})$

5. In a certain quantity of a radioactive substance, there are 10^{20} radioactive nuclei, each of which will eventually decay by a single disintegration to a stable daughter. If $\lambda = 3.0 \times 10^{-5}\ s^{-1}$, find the time required for the number of radioactive nuclei to decrease to 10^{15}.

$(N(t) = N_o e^{-\lambda t})$

6. Find x, correct to 2 significant figures:

(a) $e^{-1.17x} = 37$

(b) $(1.73)^x = 0.0046$

(c) $3^x = 17$

(d) $e^{0.003x} = 146.2$

(e) $\frac{2^{3x}}{7} = 1.3$

(f) $e^{-0.3x} = 25$

*(g) $\log_3 x = 2.7$

*(h) $\log_7 x = 4.8$

*(i) $\log_9 x = 2.1$

*(j) $\log_4 x = 5.3$

*(k) $\log_{17} x = 16.8$

*(l) $\log_6 x = 7.5$

*If your calculator does not have a Y^X function key, derive an expression for your answer.

7. Express in terms of log X, log Y, and log Z:

(a) $\log \sqrt[4]{X^3 \quad \sqrt[3]{Y^2} \ Z^5}$

(b) $\log \left[\dfrac{X^{1/3} \ \sqrt{Z^5}}{Y^3} \right]^3$

(c) $\log \left[\dfrac{X^5 \ \sqrt[4]{Y^7} \ Z^9}{\sqrt{X^3} \quad Y^3 \ Z^{3/2}} \right]^{1/3}$

8. Find x (correct to 3 significant figures):

   (a) $e^{7.2} = x$

   (b) $e^{-3.5} = x$

   (c) $e^{0.4} = x - 23.82$

   (d) $10^{1.7} = x$

   (e) $10^{-4.3} = x$

   (f) $10^{0.83} = x + 6$

9. Find x, correct to 2 significant figures:

   (a) $7^{3.5} = x$

   (b) $3^{-0.7} = x$

   (c) $1.4^{0.65} = x$

   (d) $6^{4.5} = x$

L. Haacke
W. Western

Mathematics - Course 321

USE OF LOGARITHMIC SCALED GRAPH PAPER

We have discussed the use of graphs for many purposes in previous courses. In all the cases considered, the graphs have been plotted on squared paper on which all the divisions are equal. These divisions may be 1/4" long or 1/6", they may be 1/10" or 1 millimeter, but in all cases they are all equal divisions. The scales used on such graph paper are known as LINEAR scales in the same way as the scale on a foot rule is a linear scale. The scale on a foot rule may be subdivided into inches and further subdivided into tenths, eighths or sixteenths of an inch, but all the subdivisions are equal in length.

When linear scales are used on graph paper, they form a grid of squares all equal in area. This is why such graph paper is frequently referred to as "squared" paper. This type of graph paper is known as LINEAR graph paper or, in order to indicate that linear scales are being used along both x- and y-axis, the term LINEAR-LINEAR graph paper may be used.

Linear scales and graph paper have many uses and can be useful tools for the solution of mathematical, scientific, or engineering problems. There are some instances, however, where the use of linear scales is limited and where LOGARITHMIC scales have a distinct advantage. This lesson will describe logarithmic scales and the circumstances under which they can be usefully employed.

Logarithmic Scales

On a logarithmic scale the divisions, instead of being equally spaced, are made proportional to logarithms of numbers rather than to the numbers themselves. An excellent example of a logarithmic scale is that to be found on the scales C and D on a slide rule which are used for multiplication and division.

Figure 1 shows a 5-inch length of line divided linearly into 10 equal parts. The equal parts are numbered from 1 to 10, but could equally well have been 0.1 to 1.

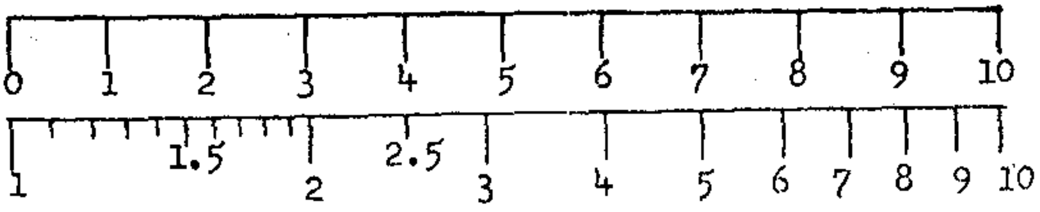


Figure 1

The scale goes from 0 to 10. Below the linear scale is shown the same length of line divided logarithmically. This logarithmic scale goes from 1 to 10 or 0.1 to 1 or 10 to 100.

Note that the logarithm of 1 (on logarithmic scale) is zero (on the linear scale). Also, log 2 (on log scale) is 0.3010 (on linear scale), log 4 (on log scale) is 0.6021 (on linear scale) and log 10 (on log scale) is 1.0 (on linear scale).

A logarithmic scale going from 0.01 to 0.1, or 0.1 to 1.0, or 1.0 to 10, etc, is said to cover or span one DECADE. A logarithmic scale can span several such decades, eg, it could go from 0.01 to 100. Such a scale would be made of 4 decades, each like the one in Figure 1, and this scale is shown in Figure 2.

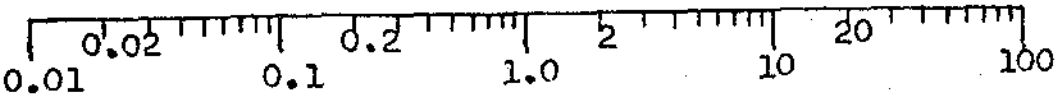


Figure 2

It can be seen from Figure 2 that each decade of the scale is subdivided in exactly the same manner. The scale in Figure 2 spans 4 decades or 4 CYCLES.

Uses of Logarithmic Scales

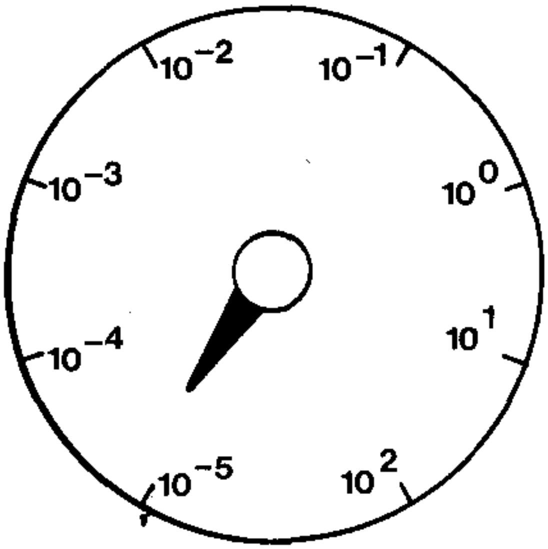
If the linear scale in Figure 1 is examined, it is clear that the distance between 0 and 1 is only 1/10 of the total length of the scale. If this distance is further subdivided into 10 equal parts, each part would be 1/1000 of the full scale value of 10, ie, each part is 0.1. Such a scale then could be used to measure to 0.1, since these subdivisions could be read with fair accuracy. However, it would not be possible to subdivide each 0.1 any further, because the subdivisions would be too small. Therefore, with a linear scale, fractional value of a measured quantity cannot be measured with any accuracy.

The same length of scale can, however, be spanned with as many decades of a logarithmic scale as is desirable. For example, the same length of scale as in Figure 1 is spanned by 4 decades in Figure 2. If the scale in Figure 2 went from 0.001 to 10, it would be easy to measure a 0.001 or 0.002 on this scale, ie, 0.01% of the full scale reading. If more decades were used, the measurement could be even smaller than this. It must be remembered, however, that the distance between 1 and 10 now only occupies the top decade and that there is, therefore, a loss of accuracy with the larger values. We can say that:

> The advantage with a logarithmic scale is that it expands the low end of the scale.

The disadvantage with a logarithmic scale is that it contracts the high end of the scale with consequent loss of accuracy.

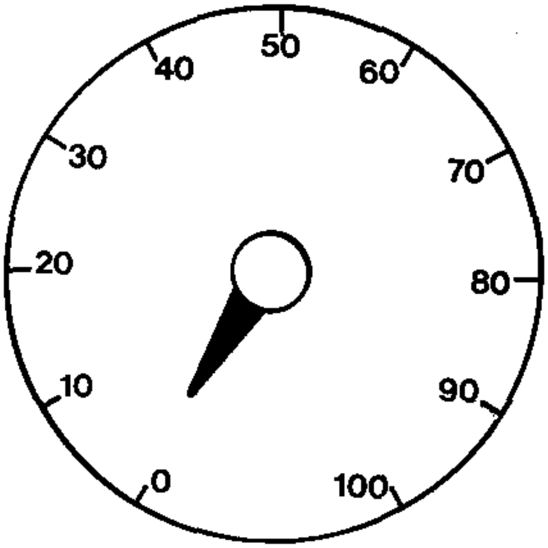
A logarithmic scale would, therefore, be used where a large range of values are to be measured. For example, reactor neutron power may vary from full power (100%) down to zero. During normal operation of a reactor, a linear scale from 0 to 100% neutron power would be adequate. However, when the reactor is started up, reactor power may only be 0.001% or less of full power, but it is important that these low power values be measured. A guage with a scale as shown in Figure 3, is in fact used on start up from 0.001% to approximately 10% full power. Above 10%, the linear scale becomes more accurate.



% Full Power

(Log Scale)

Figure 3



% Full Power

(Linear Scale)

Figure 4

Note that low values of power, such as 0.001% and 0.01% are easily read and can be determined much more accurately than on a linear scale of the same size. However, values of power from 10% and up could not be measured as accurately as on the linear scale, ie, 92% full power could be much more accurately determined on the linear scale.

The only method of obtaining the same accuracy over the whole range of values is to use a linear scale, the range of which can be varied with some suitable range switch. In effect, this replaces one scale with a number of scales, each covering, say, 1 decade of the logarithmic scale.

Figure 5 shows another example of the use of a logarithmic scale. The radiation field in a room may normally vary from 0.1 mr/hr to 10 mr/hr, but it may well increase up to 100 mr/hr,

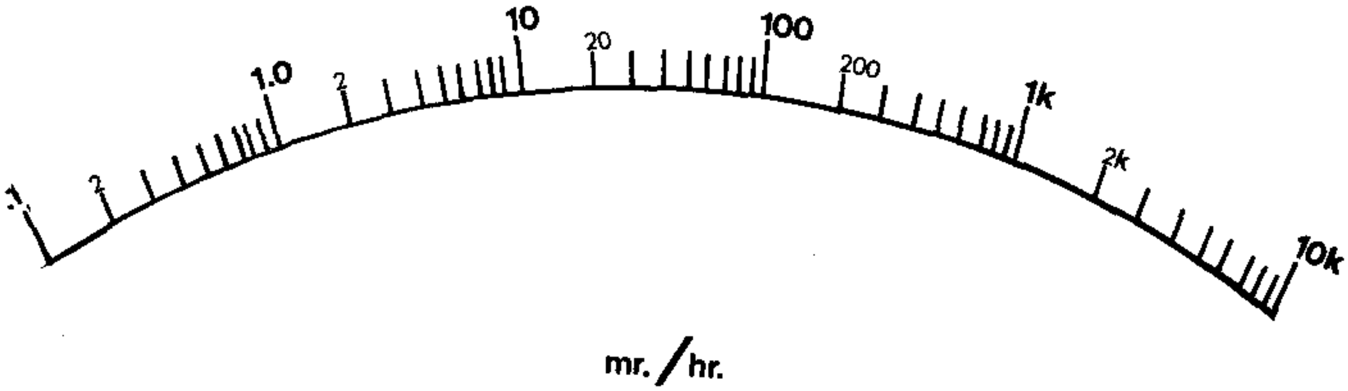


Figure 5

1000 mr/hr, or even higher. The only method of covering such a range on one scale is to use a 5-decade logarithmic scale as shown. Normal fields are clearly read and high fields can also be measured to the accuracy required. It would not be possible to say, with any certainty, whether the field was 8400 or 8500 mr/hr but such accuracy would not be required.

Logarithmic Graph Paper

Logarithmic graph paper is graph paper which is ruled with logarithmic divisions or scales instead of linear scale with the divisions all equal. There are as many different types of logarithmic graph paper as there are uses for such graph paper but they all fall into one of two main groups:

1. SEMILOGARITHMIC or LOG-LINEAR graph paper, in which the paper is ruled with a logarithmic scale in one direction (say, along the y-axis) and with equal divisions, or a linear scale in the perpendicular direction. Examples of such graph paper are shown in Figures 6 and 7.

2. LOG-LOG graph paper, in which logarithmic spacing is used in both directions. Log-log graph paper has been used in Figure 8.

Logarithmic graph paper is further classified by the number of decades covered by the logarithmic scale. The number of decades covered is known as the number of CYCLES. Thus, 6-cycle semilog graph paper will have a 6-decade logarithmic scale in one direction and a linear scale in the other direction. A 4 x 6 cycle log-log graph paper spans 4 decades one way and 6 decades in a perpendicular direction.

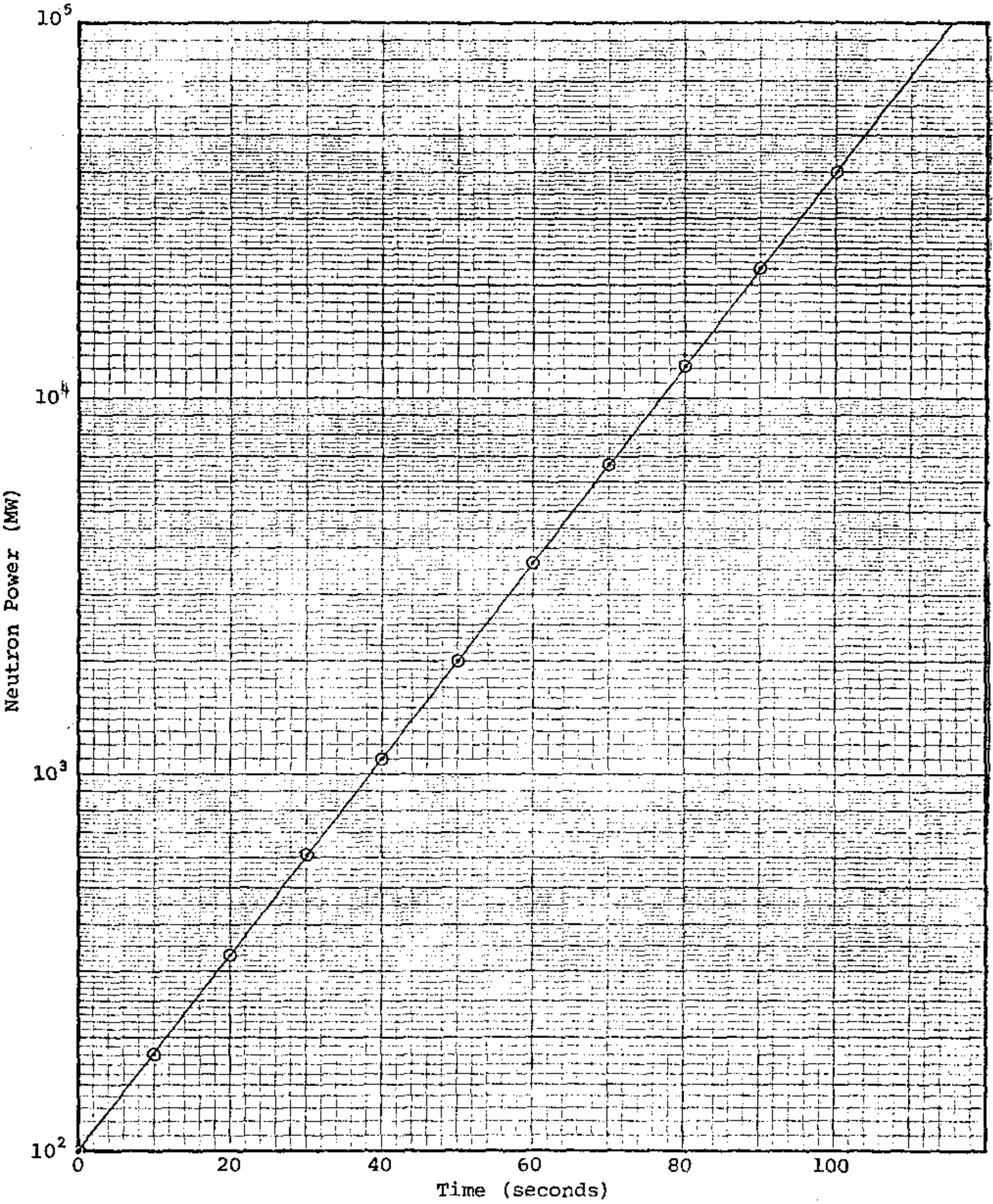


Figure 6

Reactor Neutron Power Versus Time

Gamma Dose Rate vs Penetration Depth in NPD Concrete Shield

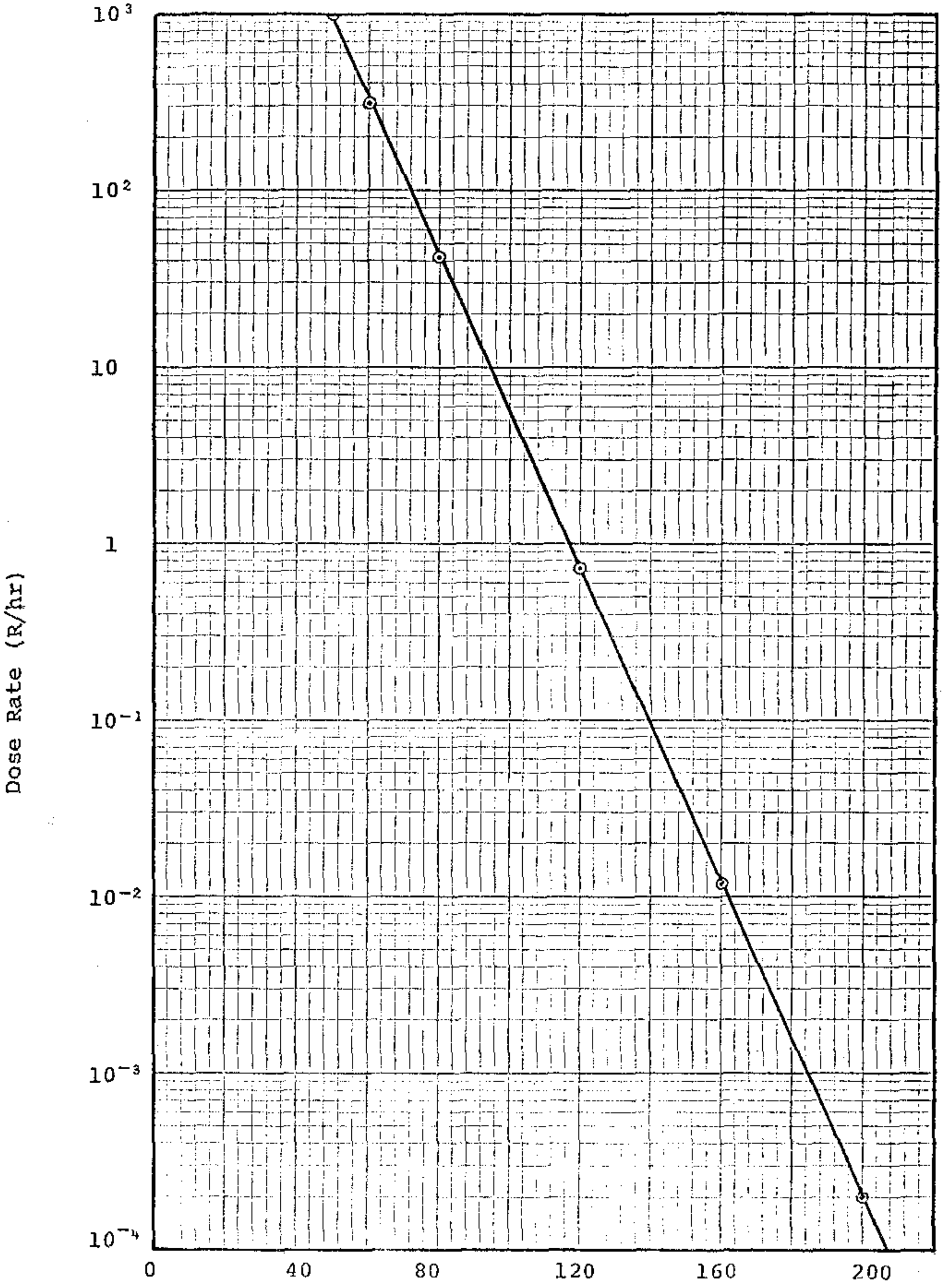


Figure 7

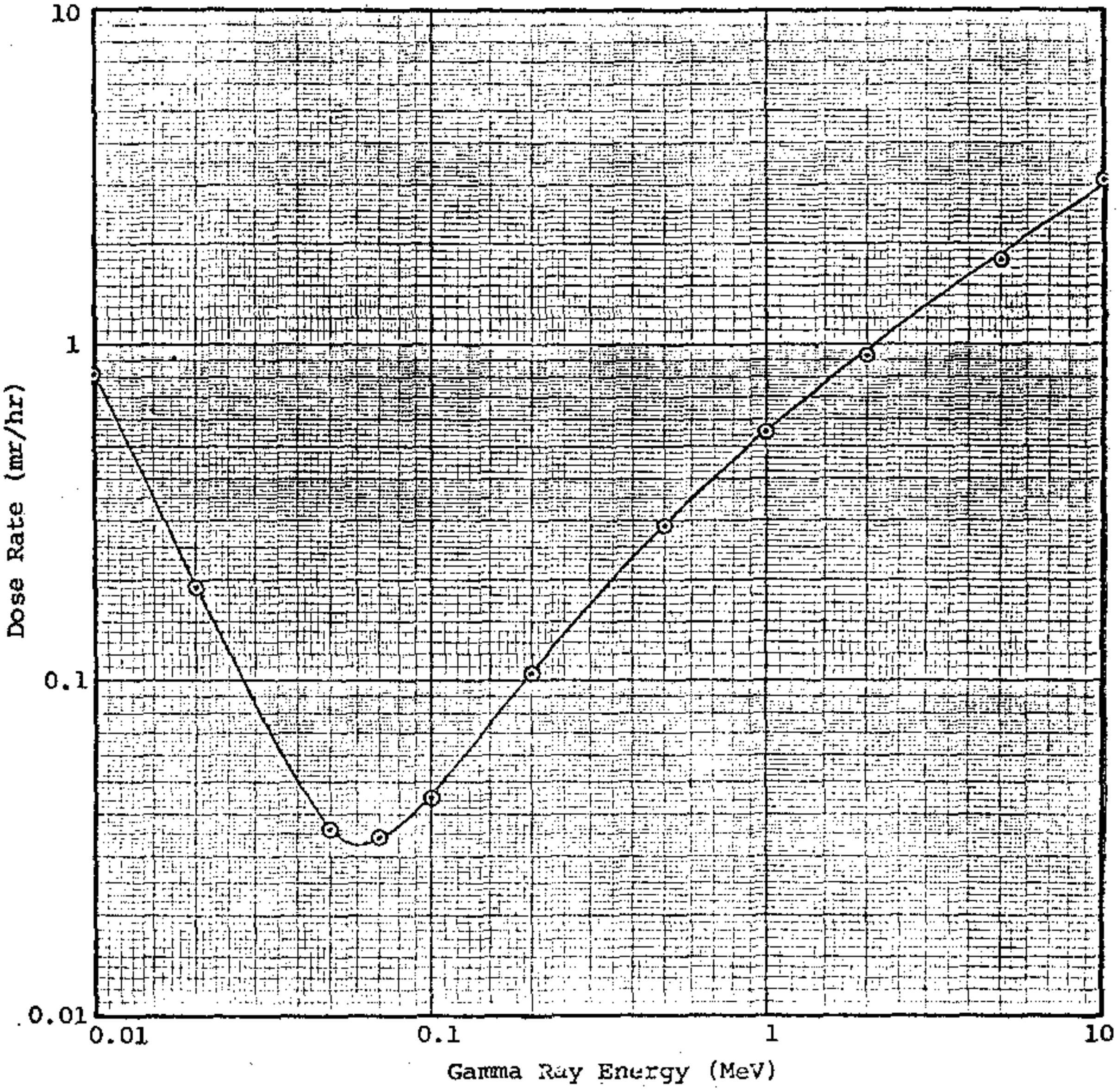


Figure 8

Dose Rate at One Meter from Gamma Source Versus Gamma Ray Energy

Uses of Logarithmic Graph Paper

The selection of graph paper for a particular purpose will be illustrated by the following examples:

Example 1:

The neutron power of a reactor, after a sudden reactivity increase, changes with time according to the equation:

$$P = 100\ e^{0.06t} \text{ Megawatts}$$

Plot the graph of the power against the time for 100 sec and determine from the graph the reactor power after 60 sec.

The calculated values of neutron power are as follows:

Time t(sec)	0	10	20	30	40	50	60	70	80	90	100
Power P(Mw)	100	182	332	605	1100	2010	3670	6650	12200	22100	40000

From the table it may be seen that a linear scale is required for the time and a 3-cycle log scale for the power. The graph is shown in Figure 6, page 5.

Power after 100 sec = 40,000 Megawatts.

Note that on semilog graph paper an exponential graph is a straight line.

Example 2:

The following gamma radiation dose rate measurements were taken at various distances through the NPD concrete shield:

Distance into shield from inner face (cm)	50	60	80	120	160	200
Dose Rate (R/hr)	1×10^{3}	3.1×10^{2}	42	0.72	1.2×10^{-2}	2×10^{-4}

The distance scale must again be a linear one but the dose rate has to be a logarithmic scale covering 7 decades. The graph is shown in Figure 7, page 6.

Since the graph is again a straight line, it can be concluded that the gamma dose rate decreases exponentially through the shield.

If the acceptable radiation dose rate outside the shield is 1 mr/hr or 1×10^{-3} R/hr, a shield thickness of 184 cms, or just over 6 ft, would have been sufficient.

Example 3:

The dose rate, at a distance of 1 meter from a source of 1 millicurie, varies with the energy of the gamma rays emitted by the source. The following table shows the dose rate for various energy gamma rays. Plot the curve of dose rate against gamma energy and estimate the energy when the dose rate is a minimum.

Gamma Energy (MeV)	0.01	0.02	0.05	0.07	0.1	0.2	0.5	1.0	2.0	5.0	10.0
Dose Rate (mr/hr) at 1 meter from source	0.82	0.19	0.036	0.034	0.045	0.105	0.29	0.55	0.93	1.8	3.1

Both quantities span 3 decades and so we require 3 x 3 cycle log-log graph paper. The graph is shown in Figure 8, page 7.

From the graph, the dose rate is a minimum when energy = 0.062 MeV.

Example 4:

The radiation dose received in one hour from a small gamma source varies inversely with the square of the distance from the source. Consider a gamma source which causes an exposure of 400 millirems per hour at a distance of one foot. At other distances, the dose rates can be found by using the inverse square law. A few calculated values follow:

Distance (ft)	1	2	4	10	20	100
Dose Rate Millirems/hr	400	100	25	4	1	0.04

Plotting this graph on log-log paper has two advantages:

1. A wide range of values can be covered.
2. The curve becomes a straight line. (See Figure 9, page 10)

If the student will try to plot a graph of the above information on a linear-linear graph sheet, he will immediately see the difficulties involved.

Dose Rate from Gamma Source vs Distance From Source

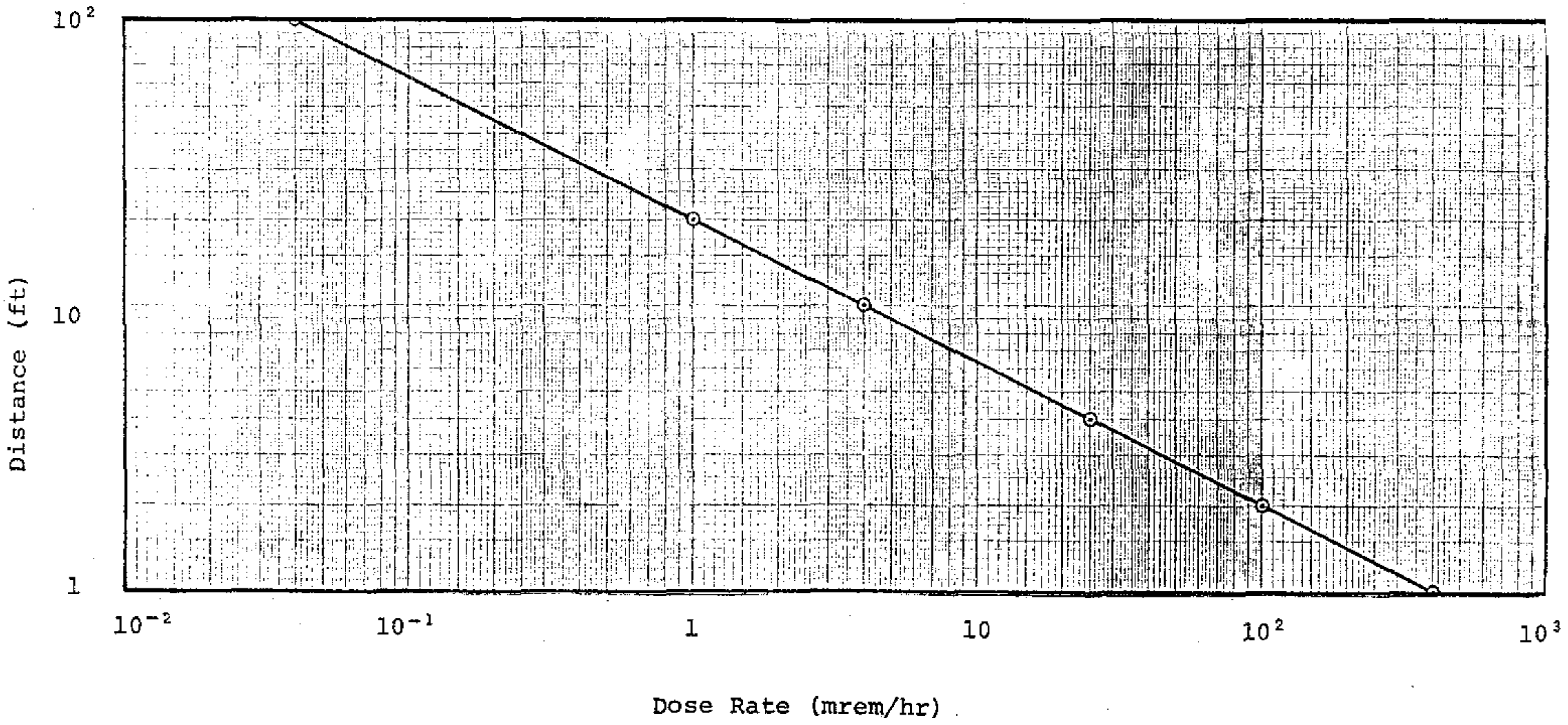


Figure 9

ASSIGNMENT

1. (a) What is the basic difference between the divisions on a logarithmic scale and a linear scale?

   (b) What is a decade on a logarithmic scale?

2. State the advantage and disadvantage of a logarithmic scale over a linear scale.

3. Under what circumstances would a logarithmic scale be used?

4. The following table shows the decrease in neutron power in a reactor following a trip.

Neutron Power (% Full Power)	100	2.2	1.0	0.3	0.058	0.013	0.0028	0.0013	0.001
Time (Minutes)	0	0.5	1.0	2	4	6	8	10	12

Plot the graph of neutron power against time and determine from the graph the time required for the neutron power to decrease to 0.1% of full power.

5. The total weight of heavy water in the air in the boiler room of a nuclear electric station required to produce a certain tritium concentration is given in the following table.

Tritium Concentration (M.P.C.)	100	500	1000	5000	10000	50000
Weight D_2O (lb)	1.62	8	16.2	80	162	800

Show graphically how the tritium concentration varies with the weight of heavy water in the room. From the graph determine the tritium concentration when there are 25 pounds of D_2O in the air in the room

6. The thermal power in a reactor following a reactor trip varies with time as shown in the following table.

Time (seconds)	0	0.5	1	5	10	100	1000	10,000
Thermal Power (% full power)	100	92	67	12.2	7.5	3.9	2.2	1.25

Plot the graph of thermal power against time and, from the graph, determine how long it takes for the power to drop to 6% of full power.

W. McKee

Mathematics - Course 221

BASIC RELIABILITY CONCEPTS

The material in this lesson is intended to provide the basic probability and reliability concepts required in the reliability evaluation of nuclear power station systems. The emphasis is on the analysis of safety systems, eg, ECC, shutdown systems, containment.

I. BASIC PROBABILITY

The word *probability* is often used very loosely, and it is important that it is recognized as a technical word implying "a measure of chance".

Probability is expressed over a scale of 0 to 1 as shown in Figure 1.

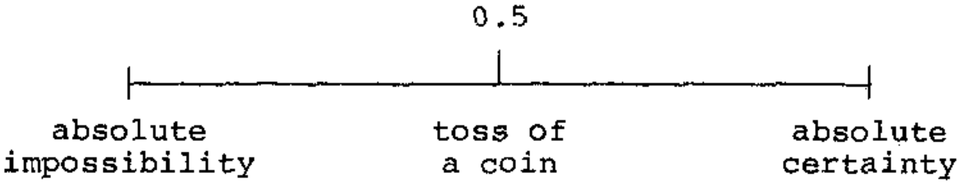


Figure 1

Probability Scale

Example 1

Roll a die. What is the probability that a two appears? There are only six possible outcomes to the experiment and only one of them gives a two.

The probability of a two appearing is 1/6. Symbolically:

$$P(2) = 1/6$$

The proability of a two not appearing $P(\bar{2})$is 5/6. Symbolically:

$$P(\bar{2}) = 5/6$$

If p is the proability that an event occurs and q is the probability that the event does not occur, then:

$$p + q = 1$$

In engineering applications, component success or failure probabilities cannot usually be determined by their geometries as in the case of a coin, a die, a roulette wheel, a deck of cards, etc. A *frequency interpretation of probability* must be used.

If n is the number of times an experiment is repeated and f is the number of occurrences of a particular event E, then the probability of E's occurring,

$$P(E) = \lim_{n\to\infty} \left(\frac{f}{n}\right)$$

Independent Events

Consider two events: if the outcome of one cannot be affected by the outcome of the other, they are said to be independent.

If there are two independent events, Event A and Event B, the probability of both Event A and Event B happening equals the product of the probabilities of each happening. The combined event is designated Event AB.

$$P(AB) = P(A) \times P(B)$$

Generalizing to n independent events:

$$P(A_1A_2\ldots A_n) = P(A_1)P(A_2)\ldots P(A_n)$$

This relation is of utmost importance in reliability work. For example, consider an electronic system which is composed of 5 components: the probability that component No. 1 survives $P(A_1)$ is some value p_1; for component No. 2 $P(A_2) = p_2$ and so on. The system will survive (ie, maintain the ability to perform its task) only if all its components survive. The probability of this event is:

$$P(\text{system survives}) = p_1 \cdot p_2 \cdot p_3 \cdot p_4 \cdot p_5$$

In words, the probability that the system survives is the product of the survival probabilities of its components.

Where do the p_i's come from? They are based on empirically (based on practical experience rather than theory) determined failure rates. For systems in the design stage, one uses historical data collected from tests under simulated operating conditions or from items used in similar duty. For operating equipment, this data can be refined using actual experience; in this way, the effects of any "local" or individual conditions can be included.

II. BASIC RELIABILITY

Reliability (R) is the probability, at any given instant, that a component or system will be available to perform its intended function. *Unreliability* (Q) is just the opposite of reliability, ie, the probability of being unavailable at any given instant. Both are dimensionless quantities and represent the fraction of total time spent in either condition. R + Q must always equal 1; ie, if a component is out of service 2% of the time, it means the component must be in-service 98% of the time. Hence, $R = 0.98$ and $Q = 0.02$ and $R + Q = 1$.

In safety system analysis, we generally speak of unreliability (Q). This is purely for arithmetical convenience; ie, it is easier to write an unreliability of 10^{-5} than a reliability of 0.99999.

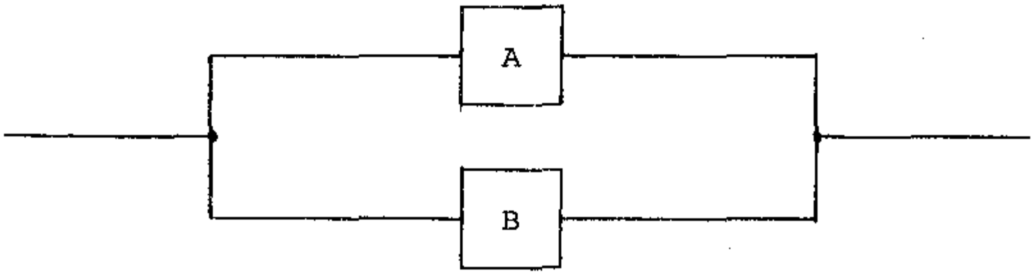
Redundancy

In some instances requirements are beyond the inherent reliability of the equipment. To meet the requirements, one can employ redundant components. The justification for redundancy is simply that multiple random failures are less likely than single ones.

There are two types of redundancy - active and standby. In active redundancy, all components operate simultaneously, while in standby redundancy, the components operate in solo and require a switching operation to change from an operating one to a standby one.

Example of Active Redundancy

The simplest form has only two components, eg, two 100% control valves. If one or both of them survive (operate as required), the system is said to be 'successful'.



'Success Modes'

Both operate as required (each allowing 50% flow)

$$P(A) \times P(B)$$

A operates allowing 100% flow, B failed

$$P(A) \times [1-P(B)]$$

A failed, B operates allowing 100% flow

$$[1-P(A)] \times P(B)$$

The probability of the system success (required operation) is equal to the sum of all success modes:

$$P = P(A) \times P(B) + P(A) [1-P(B)] + [1-P(A)] \times P(B)$$

$$= P(A) + P(B) - P(A) \times P(B)$$

If $P(A) = P(B)$, it is easily shown that:

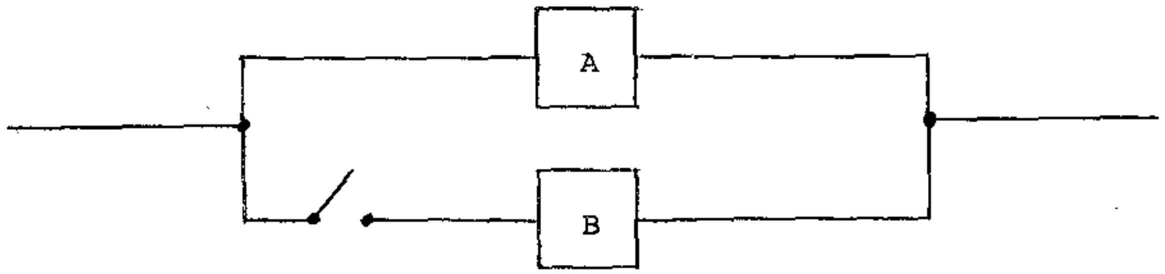
$$P = 2P(A) - P^2(A)$$

$$= 1 - Q^2(A) \qquad \text{when } P(A) + Q(A) = 1$$

The probability of at least one component success is equal to 1 minus the probability that both components fail.

Example of Standby Redundancy

A similar approach can be taken with standby redundancy (eg, 2 x 100% pumps, when one fails the other is switched on).



Two approaches are common: one which assumes failure free switching and the other which considers that switching may fail.

III. SAFETY SYSTEMS UNRELIABILITY

Safety systems are standby, 'guardian angel' systems - they normally operate only when process equipment fails. For example, the reactor *protective system* trips off the reactor only when the *regulating system* fails; otherwise the protective system does nothing. Similarly, the *containment system* operates to confine the spread of radioactivity within plant boundaries only in the event that both regulating and protective systems fail simultaneously.

For safety systems, the unreliability is numerically equivalent to the *unavailability*.

Definition

The unavailability Q of a component or system is the fraction of time during which it would not function as required.

Thus:
$$Q = \lambda t,$$

where λ is the *failure rate* in failures per year, and
t is the *average fault duration* in years.

Failure rates can be calculated on the basis of operating experience.

Example 2

Calculate the failure rate of a component, given that 6 component failures occurred during 4 years' operation of 12 such components.

Solution

$$\lambda = \frac{\text{No. of component failures}}{\text{No. component-years of operation}}$$

$$= \frac{6}{4 \times 12}$$

$$= 0.125 \text{ failures/year.}$$

Since safety systems are passive until hazardous circumstances arise, and since it is unwise to wait for such circumstances to arise before finding out whether the safety systems are still operative, the systems are *tested* periodically. For reliability evaluation purposes, the system is assumed to be in a failed state for one-half the *test period* each time it fails. This is obviously the long-term average fault duration, although the actual fault duration on any particular occasion can be anything from 0 (ie, component fails just as test occurs) to one test period (ie, component failed at conclusion of previous test).

The fault duration, $t = \frac{T}{2} + r,$

where T is the test period in years, and r is the repair time in years.

Normally $r \ll T$, and is neglected in reliability calculations. Accordingly, the usual formula for unreliability of safety systems is:

$$\boxed{Q = \lambda \frac{T}{2}}$$

Example 3

Pickering Pressure Relief Valves are tested at a rate of 1 per month. Since there are 12 valves the test interval for each is one year, and hence:

$$\begin{aligned} t &= \frac{T}{2} + r \\ &= \frac{1}{2}\text{ year} + \text{few days} \\ &\cong \frac{1}{2}\text{ year} \end{aligned}$$

Example 4

During six years of operation, a power reactor experienced the following independent faults:

- two faults in the regulating system which rapidly increased the power to such an extent that the reactor was shut down by the protective system,
- three faults which would have prevented operation of the protective system if it had been called on to act, were detected by routine daily testing of the protective system.

Assuming the faults were repaired within minutes of being discovered, calculate the annual risk of a run-away accident in this reactor (ie, the average annual frequency of such accidents).

Solution

The annual risk A.R. of a run-away accident equals the regulation system failure rate λ_R times the fraction of time the protective system is unavailable, Q_p,

$$\text{ie,}\quad \text{A.R.} = \lambda_R Q_p$$

$$= \lambda_R \lambda_p \frac{Tp}{2} \qquad (\because\ Q_p = \lambda_p \frac{Tp}{2})$$

$$= \left(\frac{2\ \text{failures}}{6\ \text{years}}\right) \left(\frac{3\ \text{faults}}{6\ \text{years}}\right) \left(\frac{\frac{1}{365}\ \text{years}}{2}\right)$$

$$= 2 \times 10^{-4}\ \text{accidents/year.}$$

ASSIGNMENT

1. In 12 years of operation of 30 pressure detection instrument lines in the containment system, 5 failures were detected. The instrumentation is tested semi-annually. What is the unreliability of a pressure detection line?

2. In 12 years of operation of 6 dump valves, 3 failures were found. The dump valves are tested twice weekly. Determine the valve unreliability.

3. Assume that the expected frequency of a complete unsafe failure of the NPD regulating system is once every 2 years. What is the annual risk of power excursions if the failure rate of the protective system is:

   (a) Complete system failure occurs once each year and the system remains in the failed state for 1 day.

   (b) Complete system failure occurs 6 times each year and failures are detected and corrected at the beginning of each shift.

4. Two pumps P_1 and P_2 operate in series. P_1 raises line pressure to meet P_2's intake requirements. The system will fail if either pump fails. If P_1 and P_2 have unreliabilities of 1.2×10^{-2} and 5×10^{-3}, respectively, calculate system unreliability.

5. Two identical pumps, each with unavailability of 2×10^{-2} are operated in a 2 x 100% arrangement. Calculate the unavailability of the system.

6. Weekly testing of a system of 15 switches has revealed 50 switch failures in 10 years' operation. Calculate the unreliability of a switch.

7. How often should a system of 12 dousing valves be tested in order to meet an unreliability target of 1.0×10^{-2}, if 15 valve failures have occurred during the past 5 years?

L.C. Haacke

Mathematics - Course 221

THE STRAIGHT LINE

I Slope of a Straight Line

The *slope* of a straight line in the xy-plane is a measure of how steeply the line rises or falls relative to the x-axis.

More precisely, the slope of a line is the increase in y per unit increase in x,

OR *the rate of change of y with respect to x.*

In Figure 1, for line segment P_1P_2,

$\Delta y = y_2 - y_1$ is called the *rise*

$\Delta x = x_2 - x_1$ is called the *run*, and

θ is called the *angle of inclination* of the line.

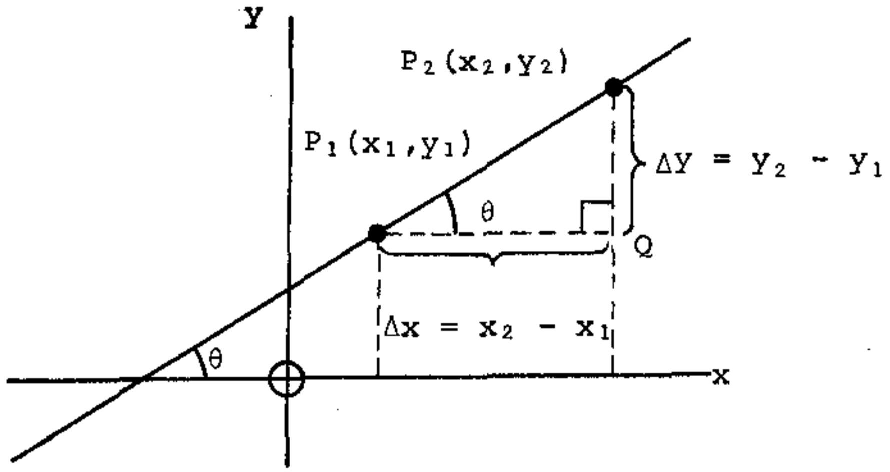


Figure 1

The numerical value of the slope, usually designated "m", is given by

$$\boxed{\text{slope } m = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1}}$$

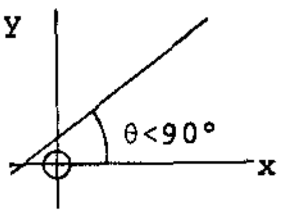
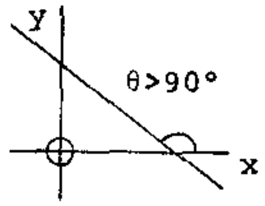
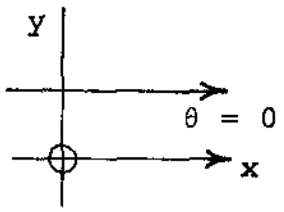
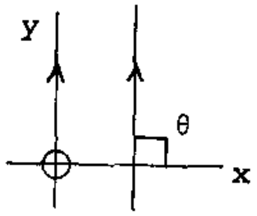
By trigonometry applied to right triangle P_1P_2Q of Figure 1,

$$\boxed{\tan \theta = \frac{\Delta y}{\Delta x} = m}$$

ie, the slope of a line is numerically equal to the tangent of the line's angle of inclination.

Note that the angle of inclination is defined as the smallest angle measured counterclockwise from the positive x-axis to the line, and therefore is always less than 180°.

The following table summarizes the correlation between the slope and orientation of a line in the plane:

Line Orientation	Typical Sketch	Slope Value
Rising to the right		$m > 0$
Falling to the right		$m < 0$
Parallel to x-axis		$m = 0$ $(\Delta y = 0)$
Perpendicular to x-axis		m undefined $(\Delta x = 0)$

Example 1

Find the (a) slope (b) angle of inclination of the line which passes through (-2,4) and (3,-5)

Solution

(a) Slope $= \frac{y_2 - y_1}{x_2 - x_1}$

$= \frac{-5 - 4}{3-(-2)}$

$= \frac{-9}{5}$ or -1.8

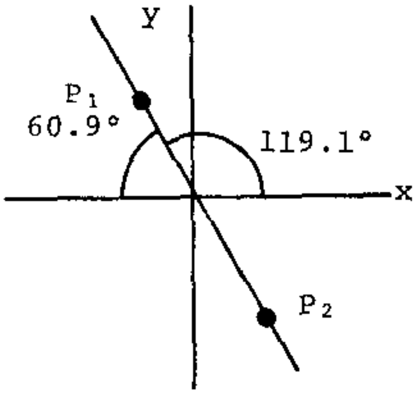
NOTE: In the previous solution, $P_1(x_1, y_1) = (-2, 4)$ and $P_2(x_2, y_2) = (3, -5)$. However, the choice for P_1 and P_2 could have been reversed without affecting the answer. (Check this.)

(b) $\tan \theta = -1.8$

$\Longrightarrow$ associated acute angle $= \tan^{-1}1.8$ (cf lesson 321.20-3)
$= 60.9°$

$\therefore$ angle of inclination, $= 180-60.9°$

$= \underline{\underline{119.1°}}$



Example 2

Given that the slope of a line is 1.5, find the change in

(a) x corresponding to an increase of 3 in y.

(b) y corresponding to a decrease of 4 in x.

Solution

Let $P(x,y)$ and $Q(x + \Delta x, y + \Delta y)$ be any two points on the line (see Figure 2).

Then slope of $PQ = \frac{\Delta y}{\Delta x} = 1.5$

(a) $\Delta y = 3 \Rightarrow \frac{3}{\Delta x} = 1.5$

ie, $\Delta x = \frac{3}{1.5}$

$= 2$

$\therefore$ x increases by 2 if y increases by 3 (between any two points on the line.)

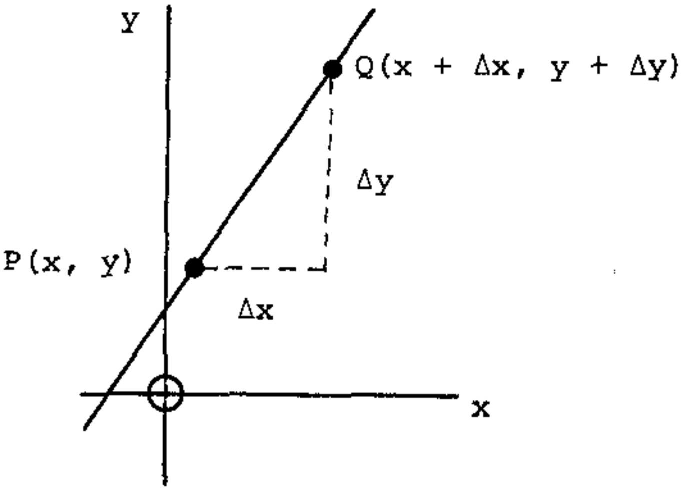


Figure 2

(b) $\Delta x = -4$ (x *in*creases by -4 if x *de*creases by 4).

Then $\frac{\Delta y}{-4} = 1.5$

$\therefore$ $\Delta y = (-4)(1.5)$
$= -6$

$\therefore$ y decreases by 6 if x decreases by 4.

II Parallel and Perpendicular Lines

(a) Parallel lines have equal slopes,

ie, line $L_1 \parallel$ line $L_2 \Leftrightarrow m_1 = m_2$

(b) The slopes of perpendicular lines are negative reciprocals,

ie, line $L_1 \perp$ line $L_2 \Leftrightarrow m_1 = -\frac{1}{m_2}$

Example 3

Find the slope of the family of lines (a) parallel (b) perpendicular to a line L with slope $m = \frac{2}{5}$.

Solution

(a) Slope of family of lines parallel to L $= m$

$$= \frac{2}{5}$$

(b) Slope of family of lines perpendicular to L $= -\frac{1}{m}$

$$= -\frac{1}{\frac{2}{5}}$$

$$= -\frac{5}{2}$$

III Equation of a Line

The *equation of a line* is the relationship which is satisfied by the coordinates of all points on the line, and by no others.

(a) Two-Point Form

Required: to find the equation of the line which passes through points $P_1(x_1,y_1)$ and $P_2(x_2,y_2)$.

Solution: Let $P(x,y)$ be any point (other than P_1 or P_2) on the line (see Figure 3).

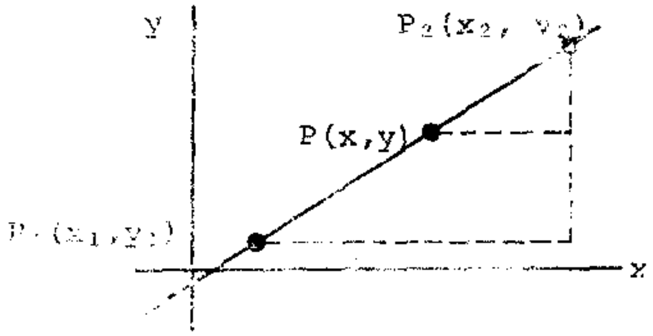


Figure 3

Then slope P_1P = slope P_1P_2 (all line segments have same slope)

ie, $\frac{y-y_1}{x-x_1} = \frac{y_2-y_1}{x_2-x_1}$

$$\therefore \quad \boxed{y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)} \quad \text{Two-point form.}$$

Example 4

Find the equation of the line passing through points (-2,4) and (3,-5).

Solution: Using two-point form,

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

ie, $y - 4 = \frac{-5 - 4}{3 - (-2)}(x - (-2))$

$$= \frac{-9}{5}(x + 2)$$

ie, $5y - 20 = -9x - 18$

ie, $\underline{\underline{9x + 5y - 2 = 0}}$

Note:

(i) The answer has been expressed in the so-called *general form* of the straight line equation, $Ax + By + C = 0$.

(ii) Points $P_1(x_1,y_1)$ and $P_2(x_2,y_2)$ can be interchanged in the above solution without affecting the answer. (Check this.)

(b) Slope-Point Form

Required: to find the equation of the line having slope m and passing through $P_1(x_1, y_1)$.

Solution: Let $P(x,y)$ be any point on the line (see Figure 4).

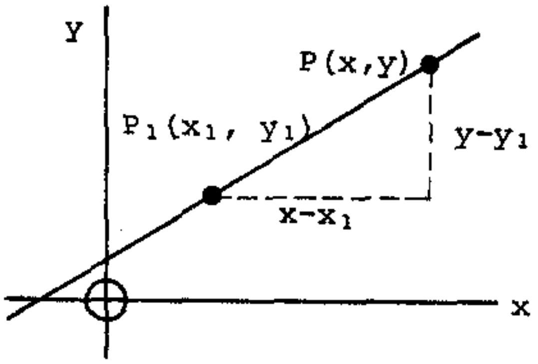


Figure 4

Then slope $P_1P = m$

ie, $\frac{y - y_1}{x - x_1} = m$

$\therefore$ $\boxed{y - y_1 = m(x - x_1)}$ Slope-Point Form.

Example 5

Find the equation of a line with slope -2 and passing through (-3,5).

Solution: Using slope-point form,

$y - y_1 = m(x - x_1)$

ie, $y - 5 = -2(x - (-3))$ (substitute (-3,5) for (x_1,y_1))

ie, $y - 5 = -2x - 6$

ie, $\underline{\underline{2x + y + 1 = 0}}$

(c) Slope-Intercept Form

Required: to find the equation of the line with slope m and y-intercept b.

Solution: Let P(x,y) be any point on the line (see Figure 5).

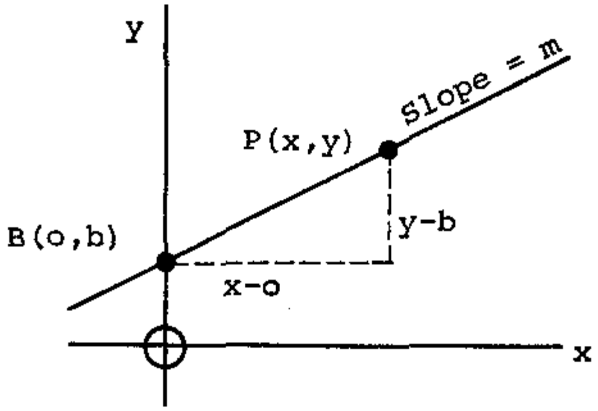


Figure 5

Then slope BP = m

ie, $\frac{y - b}{x - o} = m$

ie, $y - b = xm$

ie, $\boxed{y = mx + b}$ Slope-Intercept Form

Example 6

Find the equation of the line having slope $\frac{2}{3}$ and y-intercept -3.

Solution: Using slope-intercept form,

$y = mx + b$

ie, $y = \frac{2}{3}x + (-3)$

ie, $3y = 2x - 9$ (mult. both sides by 3)

ie, $\underline{\underline{2x - 3y - 9 = 0}}$

Example 7

Find the (a) slope (b) y-intercept (c) x-intercept of the line $5x - 2y + 10 = 0$.

Solution: The simplest way to find the slope and y-intercept is to express the equation in slope-intercept form by solving for y:

$$5x - 2y + 10 = 0$$

$$\therefore \quad -2y = -5x - 10$$

$$\therefore \quad y = \underset{\substack{\uparrow \\ m}}{\frac{5}{2}}x + \underset{\substack{\uparrow \\ b}}{5} \qquad (y = mx + b)$$

(a) slope $m = \frac{5}{2}$, and

(b) y-intercept $b = 5$

(c) At the x-intercept, $y = o$. Thus the x-coordinate is found by substituting $y = o$ in the equation, and solving for x:

$$5x - 2(0) + 10 = 0$$

$$\therefore \quad x = -2$$

$$\therefore \quad \text{x-intercept} = -2$$

Example 8

Find the equation of the line L_2 passing through the point $(-4,1)$, and perpendicular to line L_1 $3x - y - 2 = 0$.

Solution: Equation of L_1 in "$y = mx + b$" form is $y = 3x - 2$

$$\therefore m_1 = 3$$

$$\therefore m_2 = -\frac{1}{m_1}$$

$$= -\frac{1}{3}$$

$\therefore$ Equation of L_2 is $y - y_1 = m(x - x_1)$ (slope-point form)

$$y - 1 = -\frac{1}{3}(x - (-4)) \qquad ((x_1,y_1) = (-4,1))$$

$$\text{ie,} \quad 3y - 3 = -(x + 4)$$

$$= -x - 4$$

$$\therefore \quad \underline{\underline{x + 3y + 1 = 0}}$$

IV Graphing Lines

Recall that all equations of the form

$Ax + By + C = 0$ (general form) or

$y = mx + b$ (slope-intercept form),

represent straight lines in the xy-plane. The (x,y) co-ordinates of every point on a line (and no others) satisfy the equation of the line.

Steps to Graphing a Line

1. Solve the equation for y (or x).
2. Make a table of values containing at least three points. (The third point serves as an internal check: if all three points do not line up on graph, at least one point is in error.)
3. Plot points.
4. Draw and label line.

Example 9

Graph the line $2x - 5y + 6 = 0$

Step 1: $y = \frac{2x + 6}{5}$

Step 2:

x	-8	0	2
y	-2	$\frac{6}{5}$	2

Step 3, 4:

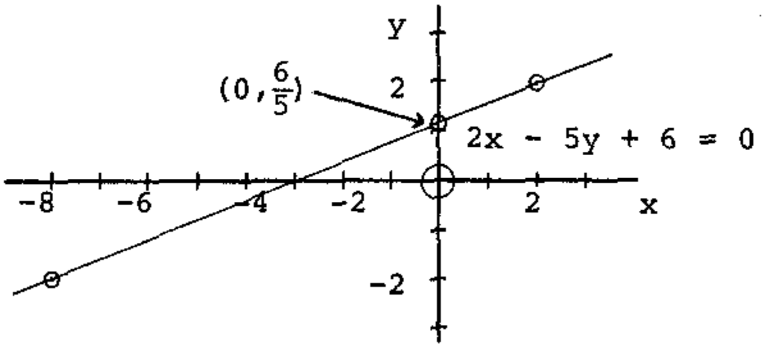


Figure 6

ASSIGNMENT

1. Find (i) the slope, (ii) the angle of inclination, and (iii) the equation the line passing through the points,

   (a) (0,0) and (3,4)

   (b) (0,2) and (3,0)

   (c) (2,-2) and (-2,2)

   (d) (5,2) and (0,2)

   (e) (-3,1) and (-3,4)

2. Show that the following three points lie on the same straight line:

   P(-5,-3), Q(-1,-1), R(5,2)

3. Graph the following lines and find their slopes and intercepts:

   (a) $x + y = 4$

   (b) $5x - 4y - 20 = 0$

   (c) $5y - 6 = 0$

   (d) $15x + 4 = 0$

4. State the slope of the family of lines (a) parallel (b) perpendicular to each of the lines in question 3.

5. Find the equation of the line passing through the given point with the given slope.

   (a) (4,3), $m = 1/3$

   (b) (-4,-1), $m = -5$

   (c) (-7,-5), $m = 0$

6. Find the equation of the line passing through the given point with the given angle of inclination.

   (a) (3,3), $\theta = 45°$

   (b) (-1,4), $\theta = 30°$

   (c) (2,-5), $\theta = 135°$

7. Find the slope and y-intercept of each of the following lines:

   (a) $2x - 5y + 6 = 0$

   (b) $8x + 3y - 7 = 0$

8. For each line in question #7, state the change in

   (a) x corresponding to an increase of 3 in y.

   (b) y corresponding to a decrease of 5 in x.

9. Find the equations of the following lines:

   (a) passing through (-1,4) and (-1,-2)

   (b) passing through (-2,-5) with slope $\frac{5}{3}$

   (c) with y-intercept $-4\frac{1}{2}$ and slope $-\frac{2}{3}$

   (d) passing through (0,0) and parallel to $4x + y - 2 = 0$

   (e) with y-intercept 6 and perpendicular to $x - 5y + 3 = 0$

   (f) passing through (6,0) with angle of inclination 45°.

L.C. Haacke

Mathematics - Course 221

THE DERIVATIVE

I LINEAR FUNCTIONS

Recall that *linear functions* are functions of the form

$$f(x) = mx + b,$$

where "m" is the slope, and "b" is y-intercept of the line $y = f(x)$.

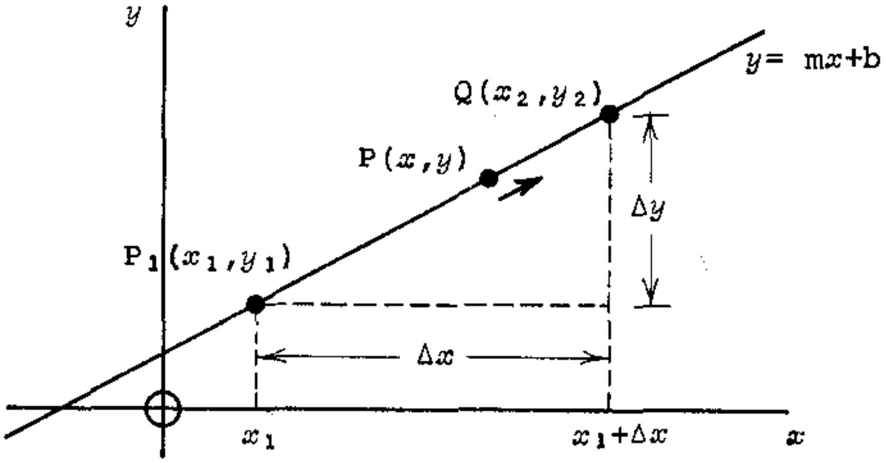


Figure 1

For example, as the point $P(x,y)$ moves up the line from P_1 to Q in Figure 1, x increases by Δx and y increases by Δy, and y increases m times as fast as x, where

$$m = \frac{\Delta y}{\Delta x}$$

ie, for a line with slope 2, y increases twice as fast as x as point $P(x,y)$ moves along the line.

In other words, the slope of a line gives the rate of change of y with respect to x along the line.

In Figure 1, as P moves from P_1 to Q, x and y are both continually changing. Therefore the rate of change of y with respect to x (the slope) must have meaning not only over the whole segment from P_1 to Q, but at every point along the line. The slope of the line at a specific point P_1 may be called the *'instantaneous' rate of change of y with respect to x at P_1.*

Note that "instantaneous" is placed in inverted commas since $x = x_1$ represents an instant only in a figurative sense.

The slope of the line at point P_1 is found by *taking the limit* of the slope of segment P_1Q as Q moves to P_1 along the line,

ie, symbolically,

$$\boxed{\begin{aligned}\text{slope of line at } P_1 &= \lim_{Q\to P_1} \text{slope segment } P_1Q \\ &= \lim_{\Delta x\to 0} \frac{\Delta y}{\Delta x}\end{aligned}}$$

Note: Read "$\lim\limits_{Q\to P_1}$" as "limit as Q tends to P_1 of..."

and "$\lim\limits_{\Delta x\to 0}$" as "limit as Δx tends to zero of..."

Example 1

Find the 'instantaneous' rate of change of $f(x) = 2x + 1$ with respect to x at $x = 3$.

Solution

The problem may be restated as follows: "Find the slope of the line $y = 2x + 1$ at the point $P_1(3,7)$".

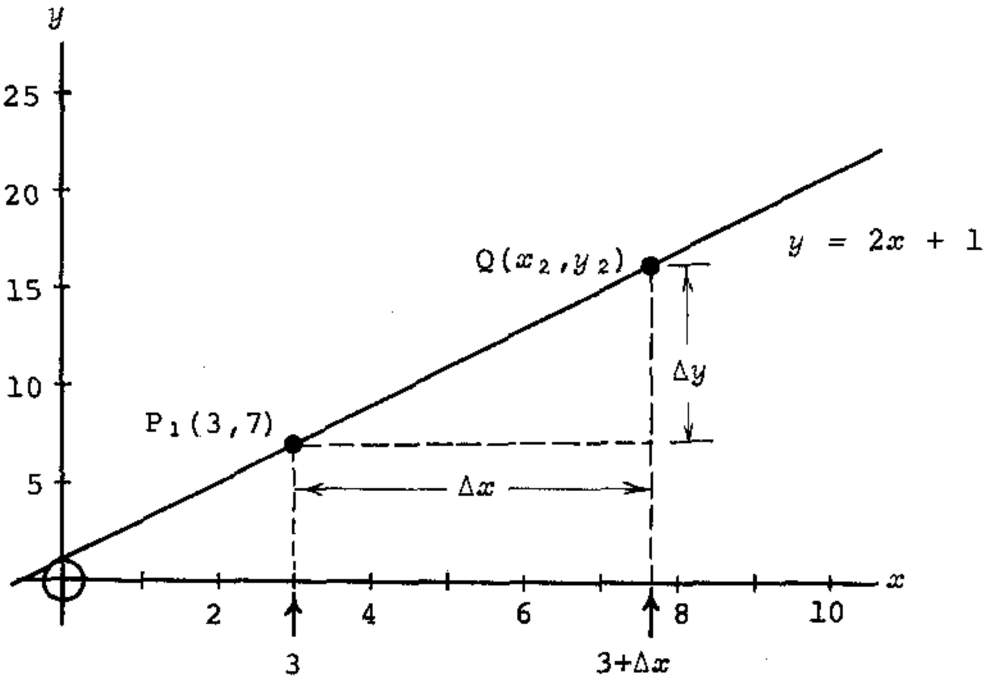


Figure 2

The following table has been constructed with reference to Figure 2, showing the slopes of segments P_1Q for various positions of Q as Q moves towards P_1 along the line:

Δx	Coord's of Q		Slope $P_1Q = \frac{y_2-7}{x_2-3}$
	x_2	y_2	
10	13	27	$\frac{27-7}{13-3} = 2$
5	8	17	$\frac{17-7}{8-3} = 2$
1	4	9	$\frac{9-7}{4-3} = 2$
.1	3.1	7.2	$\frac{7.2-7}{3.1-3} = 2$
.01	3.01	7.02	$\frac{7.02-7}{3.01-3} = 2$
10^{-6}	$3 + 10^{-6}$	$7 + 2 \times 10^{-6}$	$\frac{7+2\times10^{-6}-7}{3+10^{-6}-3} = 2$

The pattern of these results indicates that, no matter how close Q gets to P_1 the slope of P_1Q equals 2, and that the slope of $y = 2x + 1$ AT $P_1(3,7)$ is therefore probably equal to 2.

This can be proved algebraically as follows:

Slope of line at $P_1(3,7) = \lim_{Q \to P_1}$ slope of segment P_1Q,

where Q has coordinates $x_2 = 3 + \Delta x$

$$\begin{aligned} \text{and } y_2 &= f(x_2) \\ &= f(3 + \Delta x) \\ &= 2(3 + \Delta x) + 1 \\ &= 6 + 2\Delta x + 1 \\ &= 7 + 2\Delta x \end{aligned}$$

$$\begin{aligned} \therefore \quad \text{slope of line at } P_1(3,7) &= \lim_{\Delta x \to 0} \frac{y_2 - y_1}{x_2 - x_1} \\ &= \lim_{\Delta x \to 0} \frac{(7+2\Delta x)-7}{3+\Delta x-3} \\ &= \lim_{\Delta x \to 0} \frac{2\cancel{\Delta x}}{\cancel{\Delta x}} \\ &= \lim_{\Delta x \to 0} 2 \\ &= 2 \quad (\text{"2" is a constant, independent of } \Delta x) \end{aligned}$$

Note that it would be improper to substitute "0" for "Δx" before the second-last line above, since this would lead to the *indeterminate form*, "0÷0".

Exercise:

Do an analysis similar to the above to prove that the 'instantaneous' rate of change of $f(x) = 5x - 2$ at (1,3) equals 5.

Example 2

Prove that the 'instantaneous' rate of change of the linear function

$$f(x) = mx + b$$

with respect to x, at point $P_1(x_1,y_1)$, equals "m".

Solution

The problem is equivalent to proving that the slope of the line $y = mx + b$ at the point $P_1(x_1,y_1)$ equals "m".

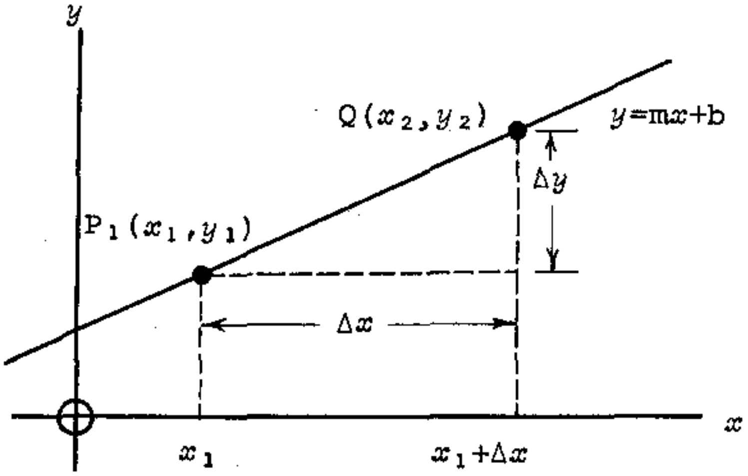


Figure 3

$$\text{Slope of } y = mx + b \text{ at } P_1 = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \text{ , (see Figure 3)}$$

$$\begin{aligned} \text{where } \Delta x &= x_2 - x_1 \\ &= (x_1 + \Delta x) - x_1 \\ &= \Delta x \end{aligned}$$

$$\begin{aligned} \text{and } \quad \Delta y &= y_2 - y_1 \\ &= f(x_2) - f(x_1) \\ &= (mx_2 + b) - (mx_1 + b) \\ &= m(x_2 - x_1) \\ &= m\Delta x \end{aligned}$$

$$\begin{aligned} \therefore \text{ slope at } P_1 &= \lim_{\Delta x \to 0} \frac{m\cancel{\Delta x}}{\cancel{\Delta x}} \\ &= \lim_{\Delta x \to 0} m \\ &= m \end{aligned}$$

CONCLUSION: THE 'INSTANTANEOUS' RATE OF CHANGE OF A LINEAR FUNCTION EQUALS THE AVERAGE RATE OF CHANGE OF THE SAME FUNCTION, AND BOTH ARE EQUIVALENT TO THE SLOPE OF THE LINE REPRESENTED BY THE FUNCTION.

Notation: "$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$" is abbreviated "$\frac{dy}{dx}$"

read "dee y by dee x", and is called the *derivative of y with respect to x.*

Definition:

The derivative of a function f(x) with respect to x is the 'instantaneous' rate of change of the function with respect to x.

Thus the words "'instantaneous' rate of change" are interchangeable with "derivative" in the foregoing.

II GENERALIZATION TO INCLUDE NONLINEAR FUNCTIONS

Definition:

The *derivative ('instantaneous' rate of change) of a function f(x)* at the point $P_1(x_1,y_1)$ is the limit as Δx tends to zero, of the average rate of change of f(x) with respect to x over the interval $x = x_1$ to $x = x_1 + \Delta x$.

Symbolically,

$$\boxed{f'(x_1) = \lim_{\Delta x \to 0} \frac{f(x_1+\Delta x) - f(x_1)}{\Delta x}}$$

The notation "$f'(x_1)$", read "f-primed at x_1", stands for

"the derivative of function f(x), evaluated at $x = x_1$"

OR "the instantaneous rate of change of f(x) with respect to x at $x = x_1$".

Hereafter "rate of change of" will be abbreviated "R/C" and "with respect to" will be abbreviated "wrt".

Graphical Significance of Definition of Derivative

Definitions:

A *secant* to a curve $y = f(x)$ is a straight line cutting the curve at two points.

A *tangent* to a curve $y = f(x)$ is a straight line touching the curve at one point only.

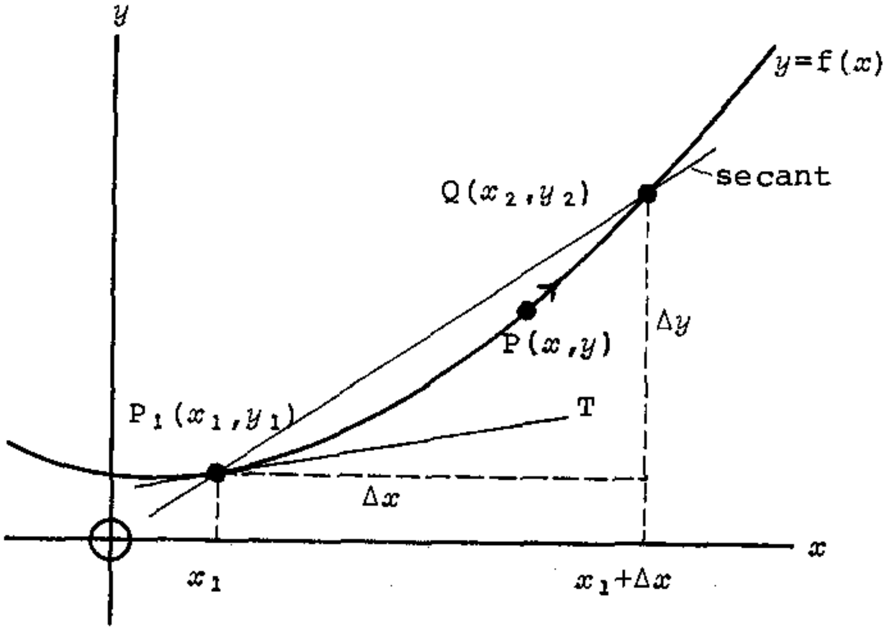


Figure 4

With reference to Figure 4, as point $P(x,y)$ moves up the curve from $P_1(x_1,y_1)$ to $Q(x_2,y_2)$ x changes by Δx, from x_1 to $x_1 + \Delta x$, and y changes by Δy, from $f(x_1)$ to $f(x_1 + \Delta x)$

$$\therefore \text{ average R/C } f(x) \text{ wrt } x = \text{slope of secant } P_1Q$$

$$= \frac{\Delta y}{\Delta x}$$

$$= \frac{f(x_1+\Delta x) - f(x_1)}{\Delta x}$$

Now imagine point Q moving down the curve towards P_1. As Q moves towards P_1, the secant P_1Q rotates clockwise and the interval Δx shortens, until, in the limiting position Q coincides with P_1, $\Delta x = 0$, and secant P_1Q coincides with tangent P_1T. Furthermore, the average R/C $f(x)$ wrt x (secant slope) becomes the 'instantaneous' R/C $f(x)$ wrt x (tangent slope).

It should be obvious that the tangent slope at P_1 equals $f'(x_1)$, the derivative at P_1, since the tangent takes the same direction as the curve at P_1. Thus the R/C y wrt x along the tangent line is the same as along the curve at the point of tangency. In fact, when one speaks of the "slope of a curve" one is understood to mean the "slope of the tangent to the curve".

To summarize, the following are equivalent:

(1) 'instantaneous' R/C f(x) wrt x at $x = x_1$

(2) the derivative of f(x) evaluated at $x = x_1$:

$$f'(x_1) = \lim_{\Delta x \to o} \frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x}$$

(3) the instantaneous R/C y wrt x at $x = x_1$, where $y = f(x)$:

$$\frac{dy}{dx} = \lim_{\Delta x \to o} \frac{\Delta y}{\Delta x} \qquad (\Delta x = x_2 - x_1)$$

(4) $\lim_{Q \to P_1}$ (slope of secant P_1Q)

(5) tangent slope at $P_1(x_1, y_1)$

(6) slope of curve $y = f(x)$ at $x = x_1$

Example 3

Find the 'instantaneous' R/C $f(x) = x^2$ wrt x at $x = 2$.

Solution

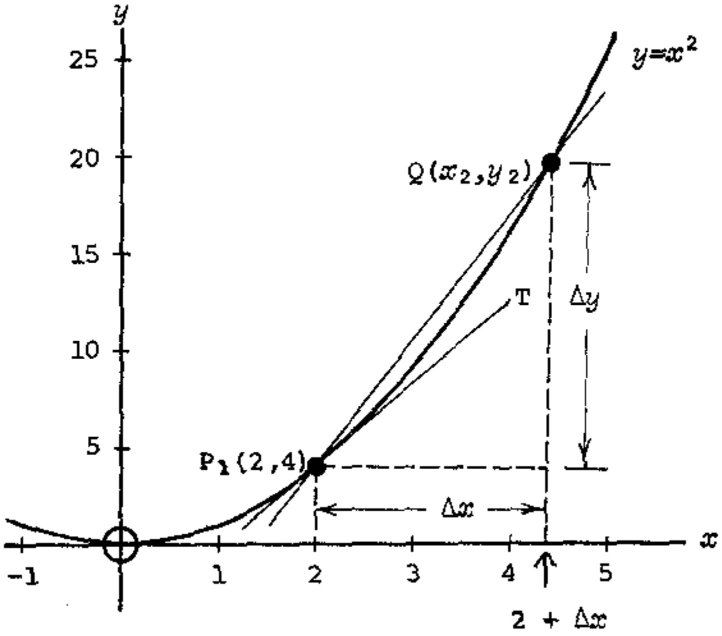


Figure 5

The following table has been constructed with reference to Figure 5, showing the slopes of secant P_1Q for various positions of Q as Q moves towards P_1 along the curve:

Δx	Coord's of Q		Slope $P_1Q = \frac{y_2-4}{x_2-2}$	
	x_2	y_2		
5	7	49	$\frac{49-4}{7-2}$	$= 9$
1	3	9	$\frac{9-4}{3-2}$	$= 5$
0.1	2.1	4.41	$\frac{4.41-4}{2.1-2}$	$= 4.1$
0.01	2.01	4.0401	$\frac{4.0401-4}{2.01-2}$	$= 4.01$
10^{-6}	$2 + 10^{-6}$	$4+4\text{x}10^{-6}+10^{-12}$	$\frac{4+4\text{x}10^{-6}+10^{-12}}{2+10^{-6}-2}$	$= 4 + 10^{-6}$

The pattern of these results indicates that the slope of secant P_1Q approaches ever more closely to 4 as Q approaches P_1 along the curve, ie, that the tangent slope of P_1 is likely equal to 4.

This will now be <u>proved</u> algebraically:

$$\begin{aligned}
\text{Tangent slope at } P_1(2,4) &= f'(2) \\
&= \lim_{\Delta x \to o} \frac{f(2+\Delta x)-f(2)}{\Delta x} \\
&= \lim_{\Delta x \to o} \frac{(2+\Delta x)^2-2^2}{\Delta x} \\
&= \lim_{\Delta x \to o} \frac{4+4\Delta x+(\Delta x)^2-4}{\Delta x} \\
&= \lim_{\Delta x \to o} (4+\Delta x) \\
&= 4
\end{aligned}$$

Exercise:

Do an analysis similar to the foregoing to show that the 'instantaneous' R/C $f(x) = 2x^2 + 5$ wrt x at $x = 3$ equals 12.

Example 4 - Power Functions

Definition:

A *power function* is a function of the form $f(x) = x^n$, n a constant.

The derivative of $f(x) = x^n$ at point $P_1(x_1,y_1)$ is

$$f'(x_1) = \lim_{\Delta x \to o} \frac{f(x_1+\Delta x)-f(x_1)}{\Delta x}$$

$$= \lim_{\Delta x \to o} \frac{(x_1+\Delta x)^n - x_1{}^n}{\Delta x}$$

It can be shown with the use of the binomial expansion formula, which is beyond the scope of this course, that this limit equals $nx_1{}^{n-1}$, ie,

$$f'(x_1) = nx_1{}^{n-1}$$

Since x_1 can take any value, the subscript on x_1 can be dropped, and the general result for a power function is:

$$\boxed{f(x) = x^n \Longrightarrow f'(x) = nx^{n-1}}$$

NOTE that $f'(x)$ is the *derivative function*, ie, $f'(x)=nx^{n-1}$, is a formula for calculating the 'instantaneous' R/C $f(x) = x^n$ wrt x at any point $P(x,y)$.

Example 5

Use the result of Example 4 to obtain the 'instantaneous' R/C $f(x) = x^2$ wrt x at $x = 2$ (cf Example 3).

Solution

$$f(x) = x^2 \Rightarrow f'(x) = 2x^{2-1}$$
$$= 2x$$

$$\therefore \quad f'(2) = 2(2)$$
$$= 4$$

$\therefore$ 'instantaneous' R/C $f(x) = x^2$, at $x = 2$, equals 4.

Example 6

Find the slope of the tangent to $y = x^3$ at $x = -2.5$.

Solution

$$f(x) = x^3 \Rightarrow f'(x) = 3x^2$$

$$\therefore \quad f'(-2.5) = 3(-2.5)^2$$
$$= 18.75$$

$\therefore$ slope of tangent to $y = x^3$, at $x = -2.5$, equals 18.75.

NOTE that alternative notations for writing down the result for power functions are:

$$\boxed{y = x^n \Rightarrow \frac{dy}{dx} = nx^{n-1}}$$

or, simply,

$$\boxed{\frac{d}{dx} x^n = nx^{n-1}}$$

In the latter notation "$\frac{d}{dx}$", read "dee by dee x of...", is regarded as an operator, which operates on the function x^n to produce its rate of change, nx^{n-1}.

III STANDARD DIFFERENTIATION FORMULAS

Definition:

To *differentiate* a function is to find its derivative.

The process of differentiating is called *differentiation.*

Trainees are expected to be able to apply the following formulas:

(1) $\frac{d}{dx} x^n = nx^{n-1}$ (power rule)

(2) $\frac{d}{dx} cf(x) = c \frac{d}{dx} f(x)$, where "c" is a constant

(3) $\frac{d}{dx} c = 0$, where "c" is a constant

(4) $\frac{d}{dx} (f(x) \pm g(x)) = \frac{d}{dx} f(x) \pm \frac{d}{dx} g(x)$

The power rule was developed in the preceding section. Formula (2) may be stated epigrammatically as follows: "The derivative of a constant times a function equals the constant times the derivative".

Proof of Formula 2:

Let $g(x) = cf(x)$

Then,
$$g'(x) = \lim_{\Delta x \to o} \frac{g(x+\Delta x)-g(x)}{\Delta x}$$

$$= \lim_{\Delta x \to o} \frac{cf(x+\Delta x)-cf(x)}{\Delta x}$$

$$= c \lim_{\Delta x \to o} \frac{f(x+\Delta x)-f(x)}{\Delta x}$$

$$= cf'(x)$$

$$\therefore \frac{d}{dx} cf(x) = c \frac{d}{dx} f(x)$$

Example 7

$$\frac{d}{dx}\,7x^5 = 7\,\frac{d}{dx}\,x^5$$

$$= 7(5x^4)$$

$$= 35x^4$$

Proof of Formula 3:

Let $f(x) = c$.

Then, $$f'(x) = \lim_{\Delta x \to o} \frac{f(x+\Delta x)-f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to o} \frac{c - c}{\Delta x}$$

$$= \lim_{\Delta x \to o} \frac{0}{\Delta x}$$

$$= 0$$

Aside:

Note that if "0" were actually substituted for "Δx" in the second-last line above, the result would be the indeterminate form "0÷0"; however, the process of taking the limit as $\Delta x \to o$ is not that of simply substituting "0" for "Δx", but rather that of ascertaining the value of an expression as "Δx" tends to "0". (A more advanced or rigorous treatment would include a formal discussion of *limit theory*; this text glosses over many subtleties of the subject.) Note that $0 \div \Delta x = 0$ for any finite value of Δx, no matter how small.

Note that the graph of $y = f(x) = c$ is a straight line, parallel to the x-axis, with slope equal to zero (see Figure 6), consistent with a zero derivative value.

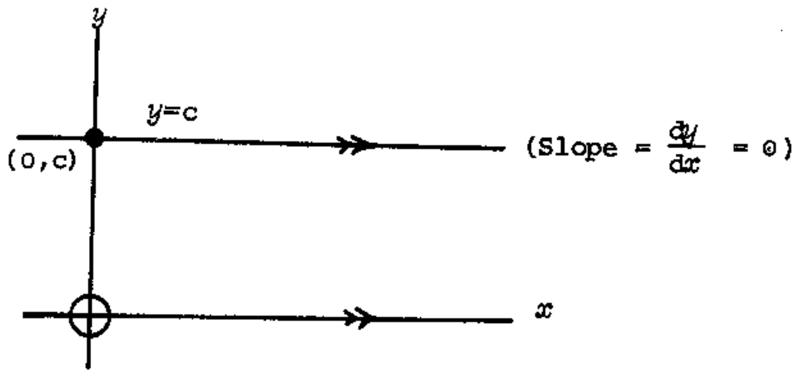


Figure 6

Example 8

(a) $\frac{d}{dx} 8 = 0$

(b) $\frac{d}{dx} (-13) = 0$

(c) $\frac{d}{dx} \pi = 0$

Proof of Formula 4:

Let $h(x) = f(x) + g(x)$

Then,

$$h'(x) = \lim_{\Delta x \to 0} \frac{h(x+\Delta x)-h(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{[f(x+\Delta x)+g(x+\Delta x)]-[f(x)+g(x)]}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{[f(x+\Delta x)-f(x)]+[g(x+\Delta x)-g(x)]}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \left(\frac{f(x+\Delta x)-f(x)}{\Delta x} + \frac{g(x+\Delta x)-g(x)}{\Delta x} \right)$$

$$= \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x} + \lim_{\Delta x \to 0} \frac{g(x+\Delta x)-g(x)}{\Delta x}$$

$$= f'(x) + g'(x)$$

ie, $\frac{d}{dx}\ [f(x)+g(x)] = \frac{d}{dx}\ f(x) + \frac{d}{dx}\ g(x)$

The proof is similar that

$$\frac{d}{dx}\ [f(x)-g(x)] = \frac{d}{dx}\ f(x) - \frac{d}{dx}\ g(x)$$

Example 9

(a)
$$\begin{aligned}\frac{d}{dx}\ [x^3+x^7] &= \frac{d}{dx}\ x^3 + \frac{d}{dx}\ x^7 && \text{(law (4))}\\ &= 3x^2 + 7x^6 && \text{(law (1))}\end{aligned}$$

(b)
$$\begin{aligned}\frac{d}{dx}\ [6x^2-2x^3] &= \frac{d}{dx}\ 6x^2 - \frac{d}{dx}\ 2x^3 && \text{(law (4))}\\ &= 6\ \frac{d}{dx}\ x^2 - 2\ \frac{d}{dx}\ x^3 && \text{(law (2))}\\ &= 6(2x) - 2(3x^2) && \text{(law (1))}\\ &= 12x - 6\ x^2\end{aligned}$$

(c)
$$\begin{aligned}\frac{d}{dx}\ [15x^2+10] &= \frac{d}{dx}\ 15x^2 + \frac{d}{dx}\ 10 && \text{(law (4))}\\ &= 15\ \frac{d}{dx}\ x^2 + 0 && \text{(laws (2), (3))}\\ &= 15(2x) && \text{(law (1))}\\ &= 30\ x\end{aligned}$$

(d)
$$\begin{aligned}\frac{d}{dx}\ 2\sqrt{x} &= \frac{d}{dx}\ 2x^{\frac{1}{2}} && (\sqrt[n]{x} = x^{1/n})\\ &= 2\ \frac{d}{dx}\ x^{\frac{1}{2}} && \text{(law (2))}\\ &= \cancel{2}\left(\frac{1}{\cancel{2}}\ x^{\frac{1}{2}-1}\right) && \text{(law (1))}\\ &= x^{-\frac{1}{2}} \text{ or } \frac{1}{x^{\frac{1}{2}}} \text{ or } \frac{1}{\sqrt{x}}\end{aligned}$$

Example 10

Find the tangent slope to the curve $y = \sqrt{x}\ (x^2+5)$ at $x = 1$.

Solution

Since the rule for differentiating a product of two functions of x ($\sqrt{x}$ and (x^2+5)) has not been given, the product must first be evaluated:

$$\begin{aligned} y &= \sqrt{x}\ (x^2+5) \\ &= x^{\frac{1}{2}}\ (x^2+5) \\ &= x^{\frac{5}{2}} + 5x^{\frac{1}{2}} \end{aligned}$$

Then
$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}\left(x^{\frac{5}{2}} + 5x^{\frac{1}{2}}\right) \\ &= \frac{d}{dx} x^{\frac{5}{2}} + \frac{d}{dx} 5x^{\frac{1}{2}} && \text{(law (4))} \\ &= \frac{5}{2} x^{\frac{3}{2}} + 5 \frac{d}{dx} x^{\frac{1}{2}} \\ &= \frac{5}{2} x^{\frac{3}{2}} + \frac{5}{2} x^{-\frac{1}{2}} \\ &= \frac{5}{2} \sqrt{x^3} + \frac{5}{2\sqrt{x}} \end{aligned}$$

$\therefore$ at $x = 1$, tangent slope
$$\begin{aligned} &= \frac{5}{2} \sqrt{1^3} + \frac{5}{2\sqrt{1}} \\ &= \frac{5}{2} + \frac{5}{2} \\ &= 5 \end{aligned}$$

ASSIGNMENT

1. Find the tangent slope at $(x, f(x))$ for each of the following functions:

   (i) by evaluating $\lim\limits_{\Delta x \to o} \dfrac{f(x+\Delta x)-f(x)}{\Delta x}$

   (ii) by applying the differentiation formulas.

   Include graphs of the functions, and evaluate the tangent slope at $x = 2$ in each case.

   (a) $f(x) = 5x^2 - 2x + 1$

   (b) $f(x) = \frac{2}{x}$

2. Find $\frac{dy}{dx}$:

   (a) $y = 2x^4 - 4x^3 + 15$

   (b) $y = \frac{x^2}{a^2} + \frac{a^2}{x^2}$ where "a" is a constant

   (c) $y = \frac{3}{\sqrt{x}}$

3. Find $f'(x)$:

   (a) $f(x) = x^2 - 6x + 3$

   (b) $f(x) = x^3\ (2x^2-1)$

   (c) $f(x) = ax^2 + bx + c$

   (d) $f(x) = \sqrt[3]{x^2} - 3\sqrt[3]{x} - 5$

4. Find

   (a) the 'instantaneous' R/C $y = 2x^3 - 3x^2 - x + 5$ at $x = 2$.

   (b) the slope of the tangent to $y = \frac{x+1}{\sqrt{x}}$ at $x = \frac{1}{4}$

   (c) the values of x at which the derivatives of x^3 and $x^2 + x$ wrt x are equal. (See Appendix 3 for methods of solving quadratics.)

L.C. Haacke

Mathematics - Course 221

SIMPLE APPLICATIONS OF DERIVATIVES

I Equations of Tangent and Normal to a Curve

This exercise is included to consolidate the trainee's concept of derivative as tangent slope, and to review the procedure for finding the equation of a straight line.

DEFINITION: The *normal* to the curve y = f(x) at a point P(x,y) on the curve is the straight line passing through P, which is perpendicular to the tangent at P.

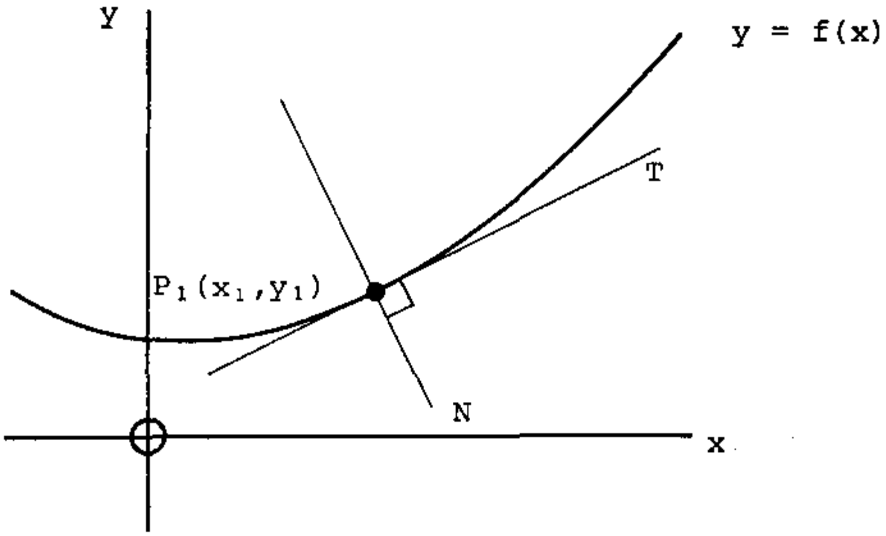


Figure 1

In Figure 1, P_1T is the tangent, and P_1N is the normal to the curve y = f(x), at the point $P_1(x_1,y_1)$.

The slope of tangent $P_1T = f^1(x_1)$.

$\therefore$ Equation of tangent P_1T, is

$$\boxed{y - y_1 = f^1(x_1)(x - x_1)} \quad \text{(slope - point form)}$$

Since the slopes of perpendicular lines are negative reciprocals (cf 221.20-1),

$\therefore$ equation of normal P_1N is

$$\boxed{y - y_1 = -\frac{1}{f^1(x_1)}(x - x_1)}$$

Example 1

Find the equations of the tangent and normal to the curve $y = 4x - x^3$ at $x = 2$. Sketch the graph of $y = 4x - x^3$, showing tangent and normal at $x = 2$.

Solution

First find the y co-ordinate at $x = 2$, using curve equation $y = 4x - x^3$:

$$\begin{aligned} y &= 4(2) - (2)^3 \\ &= 0 \end{aligned}$$

$\therefore$ Curve, tangent and normal intersect at $(2,0)$.

$\because$ $\frac{dy}{dx} = 4 - 3x^2$

$\therefore$ at $(2,0)$, tangent slope $= 4 - 3(2)^2$

$$= -8$$

$\therefore$ tangent equation is $y - y_1 = m(x - x_1)$

$$\begin{aligned} \text{ie,} \quad y - 0 &= -8(x - 2) \\ &= -8x + 16 \\ \text{ie,} \quad \underline{\underline{8x + y - 16 = 0}} \end{aligned}$$

$$\text{Slope of normal} = -\frac{1}{\text{tangent slope}}$$

$$= -\frac{1}{-8}$$

$$= \frac{1}{8}$$

$\therefore$ Equation of normal is $y - y_1 = m(x - x_1)$

$$\text{ie,}\quad y - 0 = \frac{1}{8}(x - 2)$$

$$\text{ie,}\quad 8y = x - 2$$

$$\text{ie,}\quad \underline{\underline{x - 8y - 2 = 0}}$$

The curve, tangent and normal are shown in Figure 2.

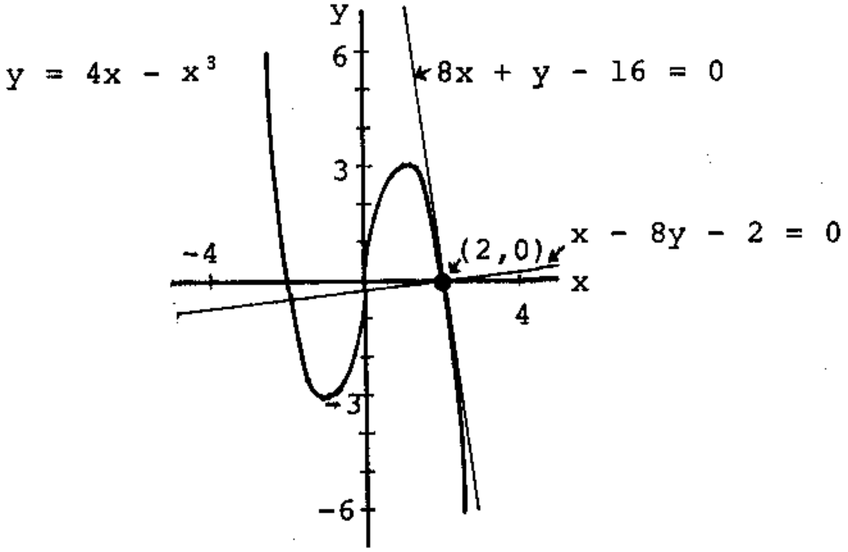


Figure 2

II Displacement, Velocity and Acceleration

The application of derivatives to such familiar concepts as velocity and acceleration should reinforce the trainee's intuitive grasp of the significance of a derivative as a rate of change.

The present discussion of displacement, velocity and acceleration will be limited to the case of motion in one dimension only.

DEFINITION: The *displacement* (designated "s") of a particle, restricted to move along an axis, is given by its co-ordinate relative to the origin on the axis.

eg, displacements of particles #1, #2, respectively, in Figure 3 are -3 and +5.

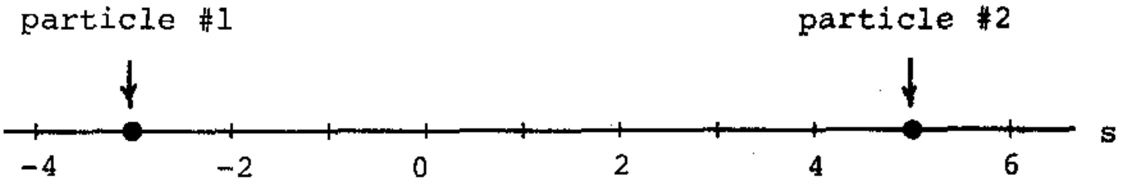


Figure 3

DEFINITION: *Velocity* (designated "v") is the rate of change of displacement with respect to time.

DEFINITION: *Acceleration* (designated "a") is the rate of change of velocity with respect to time.

Suppose a particle moving along the displacement axis passes points A and B, separated by a distance Δs, at times t_1 and $t_1 + \Delta t$, respectively (see Figure 4).

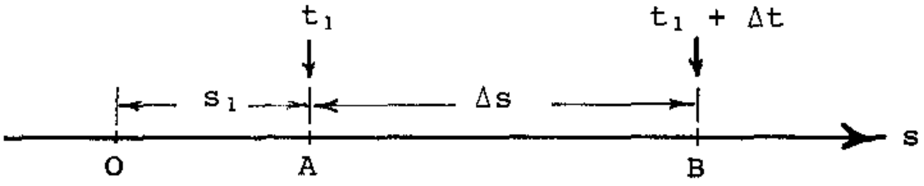


Figure 4

The particle's average velocity between A and B is

$$\bar{v}_{AB} = \frac{\Delta s}{\Delta t}$$

Its instantaneous velocity AT point A is

$$v_A = \lim_{B \to A} \bar{v}_{AB}$$

$$= \lim_{\Delta t \to o} \frac{\Delta s}{\Delta t}$$

ie, restating the above in alternative notation,

$$v(t_1) = s^1(t_1) \text{ or } (\frac{ds}{dt})_{t = t_1},$$

where s(t) is the *displacement function.*

The connection between $\frac{ds}{dt}$ of this lesson and $\frac{dy}{dx}$ of lesson 221.20-2, will be obvious from Figure 5, which shows a typical graph of displacement versus time.

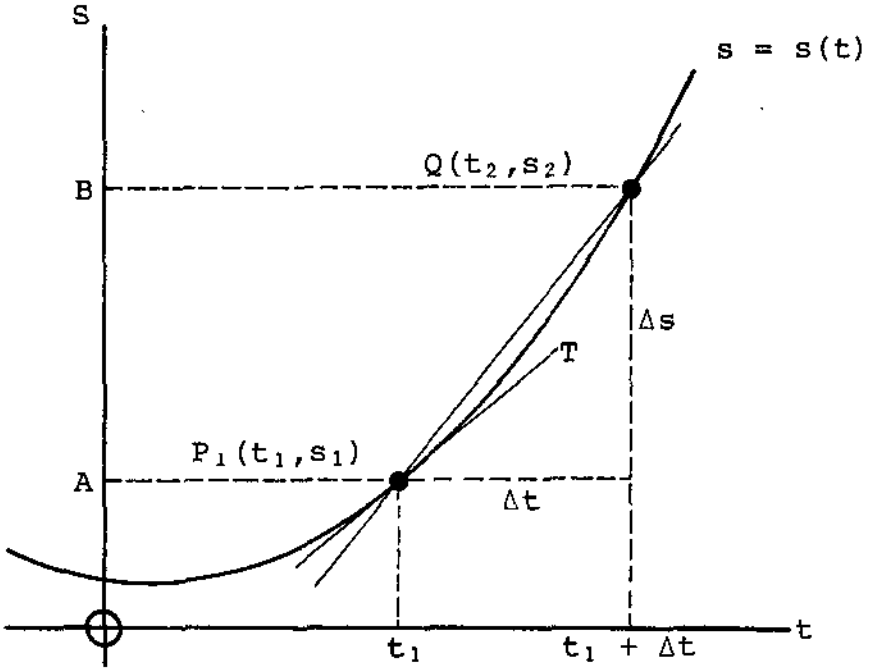


Figure 5

In comparing Figures 4 and 5, note that points A and B appear on the vertical axis, and instants t_1 and $t_1 + \Delta t$ on the horizontal axis of Figure 5.

The trainee should refer back to Figure 4 of lesson 221.20-2, and note its similarity to Figure 5 on previous page.

From Figure 5,

$$\underbrace{\text{instantaneous R/C "s" wrt "t"}}_{\text{instantaneous velocity, by definition}} = \lim_{Q \to P_1} (\text{slope of secant } P_1Q)$$

$$= \text{slope of tangent } P_1T$$

$$= \text{derivative of } s(t) \text{ at } t = t_1$$

Note that in this application "instantaneous" does not appear in inverted commas, because $t = t_1$ does, literally, represent an instant of time.

<u>To Summarize</u>:

$$\boxed{\begin{aligned} \text{average velocity } \overline{v} &= \frac{\Delta s}{\Delta t} \\ \text{instantaneous velocity } v(t) &= \frac{ds}{dt} = \lim_{\Delta t \to o} \frac{\Delta s}{\Delta t} = s^1(t) \\ &= \text{slope of tangent to curve } s = s(t) \end{aligned}}$$

Similar reasoning yields the following results for acceleration "a":

$$\boxed{\begin{aligned} \text{average acceleration } \overline{a} &= \frac{\Delta v}{\Delta t} \\ \text{instantaneous acceleration } a(t) &= \frac{dv}{dt} = \lim_{\Delta t \to o} \frac{\Delta v}{\Delta t} = v^1(t) \\ &= \text{slope of tangent to curve } v = v(t) \end{aligned}}$$

Example 2

Find the velocity and acceleration functions if the displacement function is

$$s(t) = 6t^2 - 4t + 2$$

Calculate the velocity and acceleration at t = 5.

Solution

Velocity function $v(t) = s^1(t)$

$$= \frac{d}{dt}(6t^2 - 4t + 2)$$

$$= \underline{\underline{12t - 4}}$$

Acceleration function $a(t) = v^1(t)$

$$= \frac{d}{dt}(12t - 4)$$

$$= \underline{\underline{12}}$$

Velocity at t = 5, $v(5) = 12(5) - 4$

$$= \underline{\underline{56}}$$

Acceleration at t = 5, $a(5) = \underline{\underline{12}}$

Example 3

If an object is thrown vertically upward with initial velocity v_0 m/s, neglecting air resistance, its displacement upwards from its starting point is given by the function

$$s(t) = v_0t - 4.9t^2 \text{ meters.}$$

Find the time it takes a ball to reach its maximum height if thrown upward with initial velocity of 30 m/s.

Solution

$V_0 = 30 \Rightarrow s(t) = 30t - 4.9t^2$

The ball will be at maximum height when its velocity has fallen to zero. Therefore, proceed by setting the velocity equal to zero, and solving for t:

$$v(t) = s^1(t)$$

$$= 30 - 9.8t$$

$$v(t) = 0 \Rightarrow 30 - 9.8t = 0$$

$$\Rightarrow t = \frac{30}{9.8}$$

$$= 3.1$$

ie, ball reaches maximum height after $\underline{\underline{\text{3.1 seconds}}}$.

Example 4

Two particles have displacement functions $s_1(t) = t^3 - t$ and $s_2(t) = 6t^2 - t^3$, respectively. Find their velocities when their accelerations are equal.

Solution

Differentiate once to get the velocity functions:

$v_1(t) = \frac{ds_1}{dt} = 3t^2 - 1$, and $v_2(t) = \frac{ds_2}{dt} = 12t - 3t^2$

Differentiate again to get the acceleration functions:

$a_1(t) = \frac{dv_1}{dt} = 6t$, and $a_2(t) = \frac{dv_2}{dt} = 12 - 6t$

Set $a_1 = a_2$ and solve for t:

$$6t = 12 - 6t$$

$$\therefore \quad 12t = 12$$

$$\therefore \quad t = 1$$

Substitute t = 1 in v - functions:

$$v_1(1) = 3(1)^2 - 1$$

$$= 2$$

and $v_2(1) = 12(1) - 3(1)^2$

$$= 9$$

ie, particle velocities are 2 and 9 when their accelerations are equal.

ASSIGNMENT

1. Find the slope of the given curve at the given point:

   (a) $y = 8x - 3x^2$ (2,4)

   (b) $y = \frac{8}{x^2}$ (2,2)

   (c) $y = x + \frac{2}{x}$ (2,3)

2. At what point is 2 the slope of the curve $y = 4x + x^2$?

3. Find the equations of tangent and normal to the curve

   (a) $y = x\ (2 - x)^2$ at $x = 1$

   (b) $y = x^3 + 3x^{-1}$ at $x = 1$

4. Find the velocity and acceleration at t = 2 given the displacement function

   (a) $s(t) = 8t^2 - 3t$

   (b) $s(t) = 20 - 4t^2 - t^4$

   (c) $s(t) = \frac{10}{t}\ (t^3 + 8)$

5. A baseball is thrown directly upward with initial velocity 22 m/s. Neglecting air resistance, how high will it rise?

6. Given $f(x) = \frac{x^3}{3} - x^2 - 2x + 1$, find the roots of the equation $f^1(x) = 0$. What significance do these roots have for the curve $y = f(x)$? Plot $y = f(x)$. (See Appendix 3 for methods of solving quadratics).

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Mathematics - Course 221

DIFFERENTIATING EXPONENTIAL FUNCTIONS

I Derivative of $e^{g(x)}$

Recall (lesson 221.20-2) that the derivative of the function f(x) is the 'instantaneous' rate of change of f(x) with respect to x.

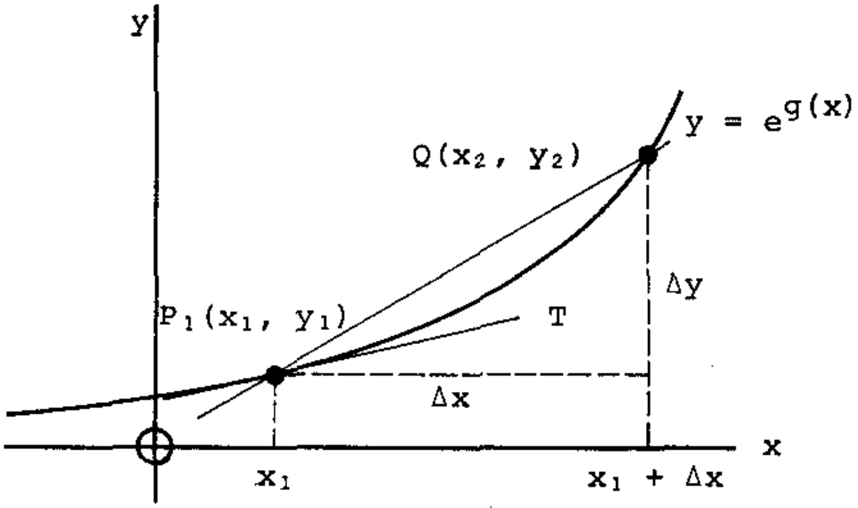

Figure 1

In Figure 1, the 'instantaneous' R/C $f(x) = e^{g(x)}$ wrt x at $x = x_1$ is equivalent to

(1) $\lim\limits_{Q \to P_1}$ (slope of secant P_1Q)

(2) slope of tangent P_1T

(3) $f^1(x_1)$, the derivative of $e^{g(x)}$ evaluated at $x = x_1$.

Recall (lesson 221.20-2) the basic *defining equation of the derivative of f(x):*

$$f^1(x) = \lim_{\Delta x \to o} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

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Applying this equation to $f(x) = e^{g(x)}$ yields

$$f^1(x) = \lim_{\Delta x \to o} \frac{e^{g(x + \Delta x)} - e^{g(x)}}{\Delta x}$$

It can be shown (but is beyond the scope of this course to do so) that the above limit reduces to

$$e^{g(x)} \ g^1(x)$$

Hence the formula for the *derivative of an exponential function* is

$$\boxed{\frac{d}{dx} e^{g(x)} = e^{g(x)} \ g^1(x)}$$

Example 1

$$\frac{d}{dx} e^x = e^x \frac{d}{dx} x$$

$$= e^x$$

Note that e^x equals its own derivative!

Example 2

$$\frac{d}{dx} 6e^{x^2} = 6 \frac{d}{dx} e^{x^2}$$

$$= 6e^{x^2} \frac{d}{dx} x^2$$

$$= 6e^{x^2} (2x)$$

$$= 12xe^{x^2}$$

Example 3

$$\frac{d}{dx} e^{2\sqrt{x}} = e^{2\sqrt{x}} \frac{d}{dx} 2\sqrt{x}$$

$$= e^{2\sqrt{x}} \left(2\frac{d}{dx} x^{½}\right)$$

$$= e^{2\sqrt{x}} (\cancel{2}) \left(\frac{1}{\cancel{2}} x^{-½}\right)$$

$$= \frac{e^{2\sqrt{x}}}{\sqrt{x}}$$

Example 4

$$\frac{d}{dx} (15x^3 - e^{-ax^2}) = \frac{d}{dx} 15x^3 - \frac{d}{dx} e^{-ax^2}$$

$$= 15 \frac{d}{dx} x^3 - e^{-ax^2} \frac{d}{dx} (-ax^2)$$

$$= 15(3x^2) - e^{-ax^2} \left(-a \frac{d}{dx} x^2\right)$$

$$= 45x^2 + 2axe^{-ax^2}$$

Example 5

Given the displacement function

$$s(t) = 5t^2 + 100e^{-0.4t},$$

(a) find the velocity function v(t)

(b) find the acceleration function a(t)

(c) sketch the graphs of s(t), v(t) and a(t) over the interval $0 \leq t \leq 10$

Solution

(a) $v(t) = s^1(t)$

$$= \frac{d}{dt}(5t^2 + 100e^{-0.4t})$$

$$= 10t + 100e^{-0.4t}\frac{d}{dt}(-0.4t)$$

$$= \underline{\underline{10t - 40e^{-0.4t}}}$$

(b) $a(t) = \frac{dv}{dt}$

$$= 10\frac{d}{dt}t - 40\frac{d}{dt}e^{-0.4t}$$

$$= 10 - 40e^{-0.4t}\frac{d}{dt}(-0.4t)$$

$$= \underline{\underline{10 + 16e^{-0.4t}}}$$

t	0	1	2	3	4	6	8	10
s	100	72	65	75	100	189	324	502
v	-40	-17	2	18	32	56	78	99
a	26	20.7	17.2	14.8	13.2	11.5	10.7	10.3

The following are sample calculations of those used to produce the above table of values:

$$s(10) = 5(10)^2 + 100e^{-0.4(10)}$$

$$= 500 + 100e^{-4}$$

$$= 500 + 100\ (0.018)$$

$$= \underline{\underline{501.8}}$$

$$v(10) = 10(10) - 40e^{-0.4(10)}$$

$$= 100 - 40\ (0.018)$$

$$= \underline{\underline{99.3}}$$

$$a(10) = 10 + 16e^{-0.4(10)}$$

$$= 10 + 16\ (0.018)$$

$$= \underline{\underline{10.3}}$$

It was stated in lesson 221.20-3 that velocity is the slope of the s-t curve, and that acceleration is the slope of the v-t curve. Are these statements consistent with the curves of Figure 2?

Note that the slope of the s-t curve is negative at $t = 0$, rises to zero at the curve minimum ($t = 1.9$), and then increases positively to $t = 10$. Note that this is precisely the behaviour of the v-t curve.

Note that the v-t curve rises most sharply at $t = 0$, and gradually settles to a slower, almost linear rate of rise. Accordingly one would expect a positive acceleration in the entire interval $0 \le t \le 10$, and one that would fall from its initial value towards a constant value. This is precisely the behaviour of the a-t curve.

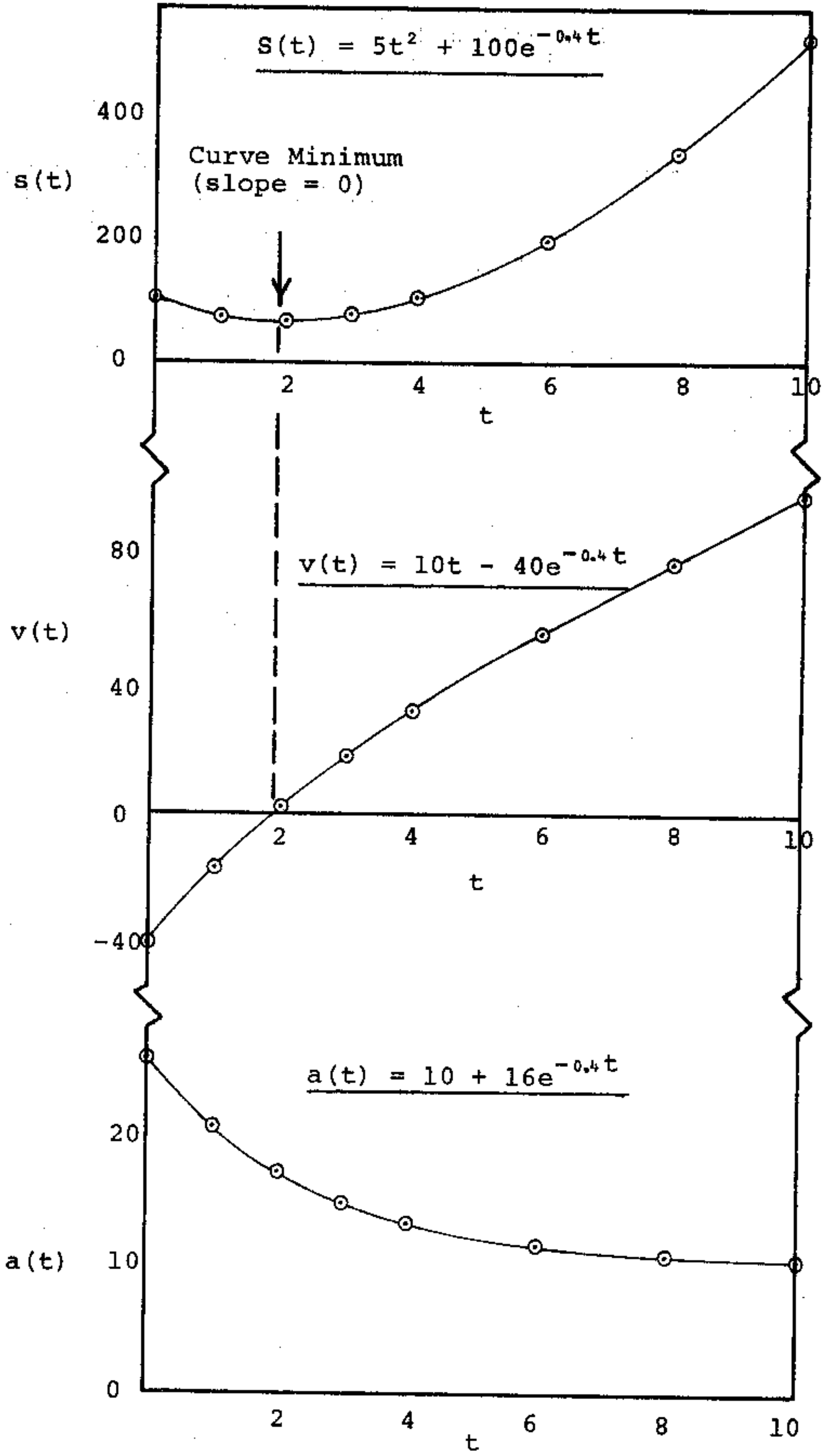

$S(t) = 5t^2 + 100e^{-0.4t}$
400
s(t)
Curve Minimum
(slope = 0)
200
0
2
4
6
8
10
t
80
$v(t) = 10t - 40e^{-0.4t}$
v(t)
40
0
2
4
6
8
10
t
-40
$a(t) = 10 + 16e^{-0.4t}$
20
a(t)
10
0
2
4
6
8
10
t

Figure 2

II Application to Nuclear Decay

The number of radioactive atoms remaining in a radioactive source decays exponentially with time, according to the relation.

$$N(t) = N_0 e^{-\lambda t}$$

where $N(t)$ = number of radioactive atoms remaining after t seconds,

N_0 = number of radioactive atoms at time $t = 0$, and

λ is the decay constant of the radionuclide in s^{-1}

To find the R/C N wrt t, ie, the number of nuclei decaying per second, differentiate the above relation wrt time:

$$\frac{dN}{dt} = \frac{d}{dt} N_0 e^{-\lambda t}$$

$$= N_0 \frac{d}{dt} e^{-\lambda t} \qquad (N_0 \text{ a constant})$$

$$= N_0 \; e^{-\lambda t} \frac{d}{dt} (-\lambda t)$$

$$= N_0 \; e^{-\lambda t} (-\lambda \frac{d}{dt} t)$$

$$= -\lambda N_0 e^{-\lambda t}$$

$$\therefore \quad \boxed{\frac{dN}{dt} = -\lambda N}$$

Note that $\frac{dN}{dt}$ stands for the rate of increase in N. Hence $\frac{dN}{dt}$ is negative (see minus sign on RHS), since N is actually decreasing.

The number of nuclei decaying per unit time is called the *activity* of a source.

Example 6

How many radioactive nuclei are required to make a 5 mCi source of a nuclide whose decay constant equals 7.3×10^{-5} s^{-1}? (1 curie = 3.7×10^{10} dps)

Solution

$$\frac{dN}{dt} = -5 \text{ mCi}$$

$$\Rightarrow \quad -\lambda N = -5 \times 10^{-3} \times 3.7 \times 10^{10}$$

$$\therefore \quad N = \frac{5 \times 10^{-3} \times 3.7 \times 10^{10}}{7.3 \times 10^{-5}}$$

$$= \underline{\underline{2.5 \times 10^{12}}}$$

ie, there are 2.5×10^{12} atoms in a 5 mCi Source.

* * *

If source activity is designated "A",

$$\text{then} \quad A(t) = -\frac{dN}{dt} \qquad \text{(rate of decrease in N)}$$

$$= \lambda N$$

$$= \lambda N_0 e^{-\lambda t} \qquad (\therefore \quad N = N_0 e^{-\lambda t})$$

$$\text{then} \quad A(0) = \lambda N_0 e^0$$

$$= \lambda N_0$$

Let $A_0 = A(0)$

Then $A_0 = \lambda N_0$

and $\boxed{A(t) = A_0 e^{-\lambda t}}$

ie, the activity A(t) obeys the same exponential relationship as N(t).

Example 7

Find the time required for the activity of a source of decay constant 3.5×10^{-4} s^{-1} to decay by a factor of 1000.

Solution

Let required time be t_1.

Then $A(t_1) = A_0 e^{-\lambda t_1}$

ie, $\frac{A(t_1)}{A_0} = e^{-\lambda t_1}$

$\therefore$ $e^{-\lambda t_1} = 0.001$ $\quad (\because \frac{A(t_1)}{A_0} = \frac{1}{1000})$

Taking natural log of both sides,

$$\ell n\ e^{-\lambda t_1} = \ell n\ 0.001$$

$\therefore$ $-\lambda t_1 = \ell n\ 10^{-3}$ (cf lesson 321.10-4)

$$\therefore \quad t_1 = \frac{\ell n\ 10^{-3}}{-\lambda}$$

$$= \frac{-6.91}{-3.5 \times 10^{-4}}$$

$$= \underline{\underline{2.0 \times 10^4 \text{ seconds or } 5.5 \text{ hours}}}$$

Example 8

Prove: $t_{1/2} = \frac{0.693}{\lambda}$, where $t_{1/2}$ is the half-life of a radionuclide, ie, the time required for source activity to decay to one-half its original activity.

Solution

$$A(t_{1/2}) = A_0 e^{-\lambda t_{1/2}}$$

$$\therefore \quad \frac{A(t_{1/2})}{A_0} = e^{-\lambda t_{1/2}}$$

$$\therefore \quad e^{-\lambda t_{1/2}} = 0.5 \qquad \left(\because \ \frac{A(t_{1/2})}{A_0} = 0.5\right)$$

$$\therefore \quad \ell n\ e^{-\lambda t_{1/2}} = \ell n\ 0.5$$

$$\therefore \quad -\lambda t_{1/2} = -0.693$$

$$\therefore \quad t_{1/2} = \frac{0.693}{\lambda}$$

III Application to Reactor Power Growth

Reactor power grows exponentially in time, approximately according to the relation,

$$P(t) = P_0 e^{\frac{\Delta k}{L} t}$$

where $P(t)$ is reactor power at time t,

P_0 is reactor power at $t = 0$,

Δk is the reactivity in units of "k",

L is the mean neutron lifetime in the reactor.

For example, if $P_0 = 100$ W, and $\frac{\Delta k}{L} = 0.05$, the graph of P(t) vs t is shown in Figure 3.

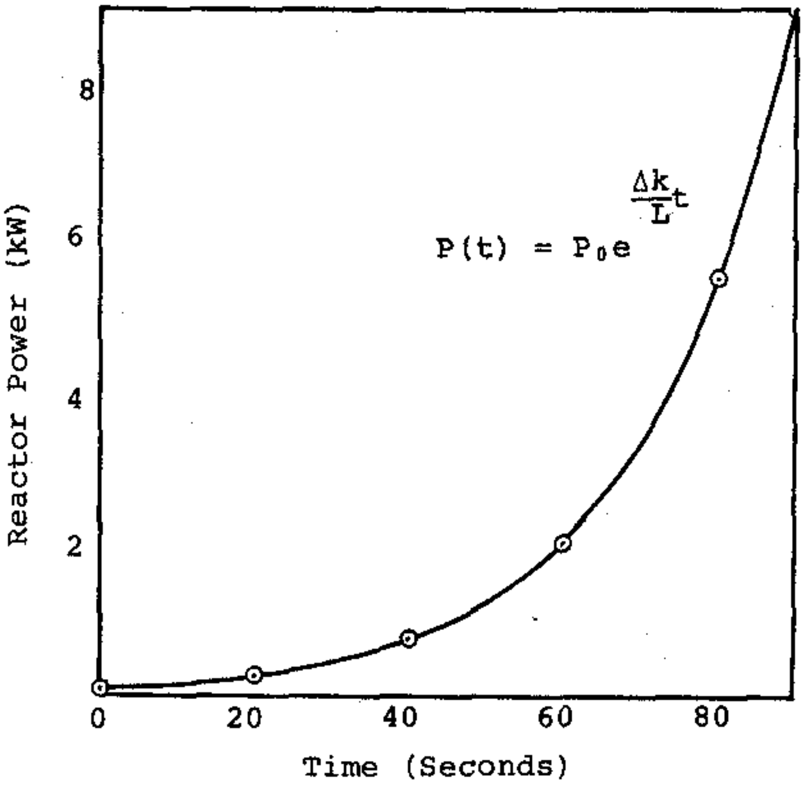


Figure 3

DEFINITION: The *reactor period* "T" is the time required for the power to increase by a factor of e.

Proof that Reactor Period $= \frac{L}{\Delta k}$

$P(T) = eP_0$ by definition of T

ie, $e\cancel{P}_0 = \cancel{P}_0 e^{\frac{\Delta k}{L} T}$

$\therefore$ $\ell n\ e = \ell n\ e^{\frac{\Delta k}{L} T}$

ie, $1 = \frac{\Delta k}{L} T$

$\therefore$ $\boxed{T = \frac{L}{\Delta k}}$

$\therefore$ an alternative form of the power growth equation is

$$\boxed{P(t) = P_0 e^{t/T}}$$

from which it is obvious that each time t increases by T, P increases by a factor of e, consistent with previous definition of T.

Not only does the power P(t) grow exponentially with time, but so also does the <u>rate</u> of growth, $P^1(t)$, as shown below:

$$\frac{dP}{dt} = \frac{d}{dt} P_0 e^{\frac{\Delta k}{L} t}$$

$$= P_0 \frac{d}{dt} e^{\frac{\Delta k}{L} t}$$

$$= P_0 e^{\frac{\Delta k}{L} t} \frac{d}{dt} \frac{\Delta k}{L} t$$

$$= P_0 e^{\frac{\Delta k}{L} t} \frac{\Delta k}{L} \frac{d}{dt} t$$

$$= \frac{\Delta k}{L} \underbrace{P_0 e^{\frac{\Delta k}{L} t}}_{P(t)}$$

$$\therefore \quad \boxed{\frac{dP}{dt} = \frac{\Delta k}{L} P(t) = \frac{1}{T} P(t)}$$

Note that power growth rate $P^1(t)$ is directly proportional to product of reactivity Δk and power P(t). Therefore, given sufficiently high values of Δk and P, P^1 may be so high that rated power is exceeded before the regulation system can arrest power growth.

Thus, for reactor protection, a signal is required to detect dangerously high reactivity valves at low power. Such a signal is one whose output varies as the rate of change of the logarithm of reactor power. This signal is known as "*rate log power*":

$$\frac{d}{dt} (\ell n\ P(t)) = \frac{d}{dt} \ell n\ P_0 e^{\frac{\Delta k}{L} t}$$

$$= \frac{d}{dt} (\ell n\ P_0 + \ell n\ e^{\frac{\Delta k}{L} t})$$

$$= \frac{d}{dt} \ell n\ P_0 + \frac{d}{dt} \frac{\Delta k}{L} t$$

$$= 0 + \frac{\Delta k}{L} \frac{d}{dt} t$$

$$= \frac{\Delta k}{L}$$

$$\therefore \quad \boxed{\text{rate log power, } \frac{d}{dt} (\ell n\ P(t)) = \frac{\Delta k}{L} = \frac{1}{T}}$$

Note that rate log power is proportional to reactivity Δk, independent of reactor power. Hence the reactor can be tripped by this signal at low power, eg, 0.001% full power, long before the *linear rate power*, $P^1(t)$ gets out of hand.

ASSIGNMENT

1. Differentiate:

   (a) e^{x^2-4} (b) $-e^{-x}$

   (c) $-e^{-x^{-1}}$ (d) $2e^{-1/\sqrt{x}}$

   (e) $5e^{\sqrt{t}}\,(t^2-1)$ (f) $\frac{1}{3}\,e^{-1/x^3}$

2. Find (i) $v(t)$ (ii) $a(t)$ (iii) $v(2)$ if

   (a) $s(t) = e^t - t^3$

   (b) $s(t) = e^{-t} + 2t$

3. Plot s - t, v - t, a - t curves for the displacement function of 2(a) above over the time interval $0 \le t \le 3$. Do the slopes of the s - t and v - t curves appear to verify the definitions, $v(t) = s^1(t)$ and $a(t) = v^1(t)$, respectively?

4. If 2.0×10^{19} radioactive nuclei constitute a 5.0 mCi source, what is the decay constant of the radionuclide? (1 curie = 3.7×10^{10} dps)

5. (a) What is the activity of a source consisting of 7.0×10^{13} radioactive nuclei, and having decay constant 2.4×10^{-4} s^{-1}?

   (b) How many radioactive nuclei remain after (i) 20 minutes? (ii) 6 half-lives?

   (c) Calculate the source activity after (i) 20 minutes (ii) 6 half-lives.

   (d) Calculate the half-life of the source.

   (e) How long does the source take to decay to 10 mCi?

6. If $N(t) = N_0e^{-\lambda t}$ and $A = -\frac{dN}{dt}$, prove that (a) $A = \lambda N$

   (b) $A(t) = A_0e^{-\lambda t}$

7. Prove that $t_{1/2} = \frac{\ell n\ 2}{\lambda}$.

8. If $P(t) = P_0 e^{t/T}$, prove that

   (a) $P^1(t) = \frac{1}{T} P(t)$

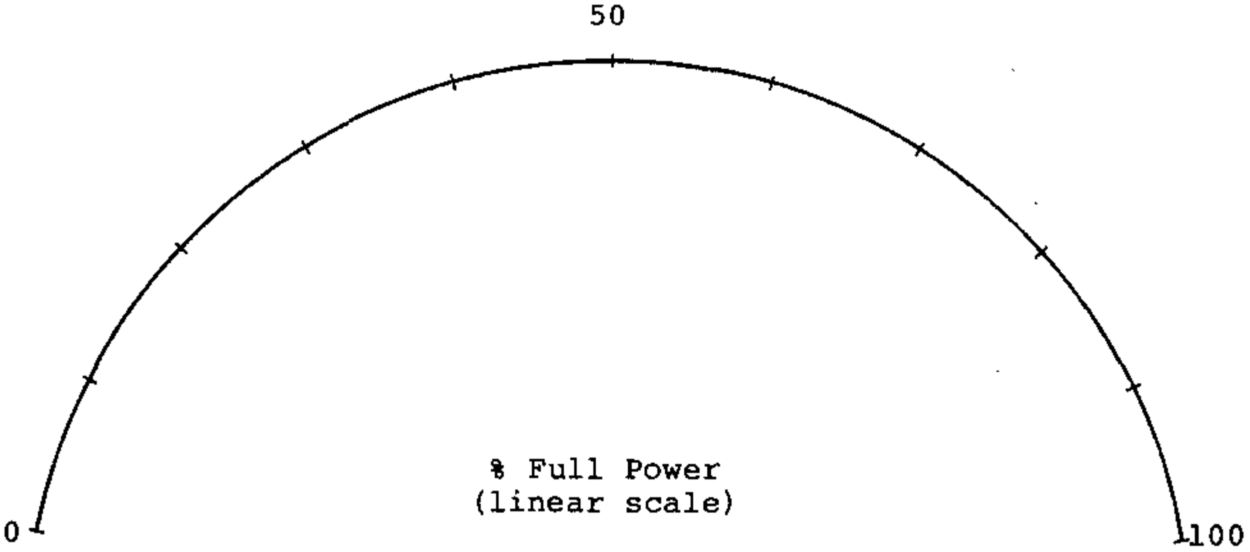
   (b) $\frac{d}{dt} \ell n\ P(t) = \frac{1}{T}$

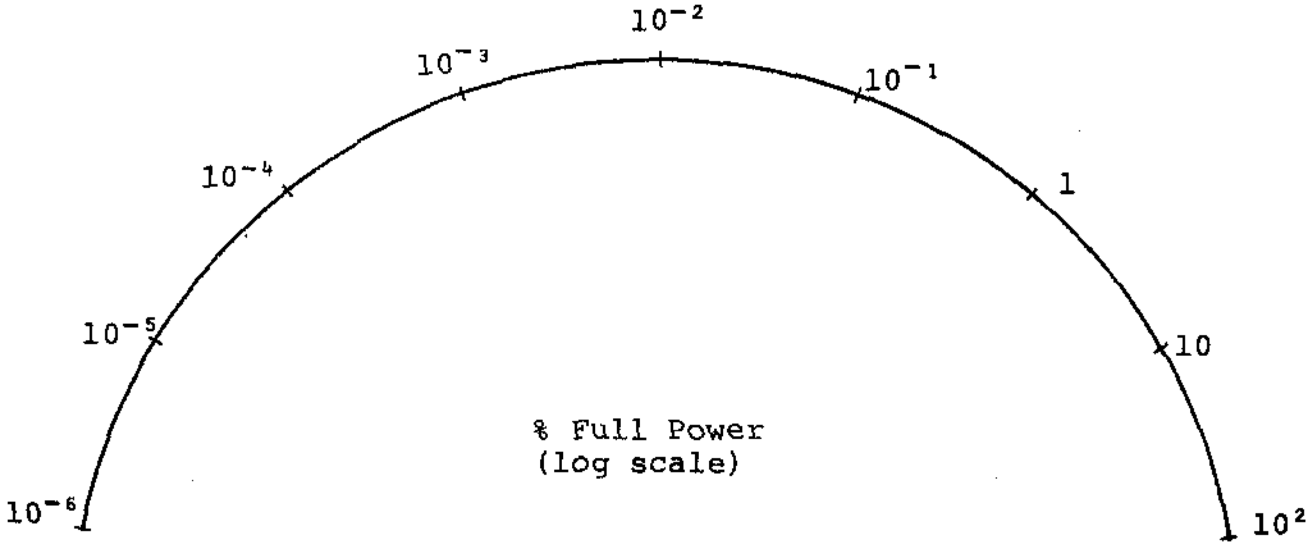
9. Plot a graph of N(t) vs t over the interval $0 \leq t \leq 18$ hours if $N(t) = N_0 e^{-\lambda t}$, where $N_0 = 10^{20}$ and $\lambda = 6.4 \times 10^{-5}\ s^{-1}$.

   (a) on linear paper (b) on log-linear paper.

10. (a) Make a table of values of reactor power P(t) and linear rate, $P^1(t)$ with 20-second increments in t over the interval $0 \leq t \leq 5$ minutes. Assume $P_0 = 100$ W and $\frac{\Delta k}{L} = 0.05$. Express P and P^1 in units of % full power, assuming full power equals 100 MW.

    (b) Show consecutive positions of indicating needles on the following meters, at 20-second intervals.





(c) Describe the needle's motion across each of the above scales, and relate descriptions to the mathematical expressions for linear rate and rate log power.

(d) Which meter is more suitable for monitoring power at low power levels? At high power levels?

(e) Which of the following signals is more appropriate for reactor power control

(i) at low power levels?

(ii) at high power levels?

a signal whose output is proportional to reactor power P, or one whose output is proportional to the logarithm of reactor power, log P?

11. Explain the advantage of a rate log signal for reactor protection.

12. Show that $\frac{d}{dt}$ (log P(t)) = $\frac{\Delta k}{L}$ log e, where log P is the common logarithm of P.

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Mathematics - Course 221

THE DERIVATIVE IN SCIENCE AND TECHNOLOGY

I Some Common Differential Equations

Calculus is to modern science and technology as arithmetic is to accounting. Arithmetic provides the notation and techniques for computing credits, debits and balance on hand; calculus provides the notation and techniques for computing the 'instantaneous' or true rate of change of one physical variable with respect to another, given the mathematical relationship between the two variables. Without calculus, only average rates of change can be calculated, in general. (Calculus provides also the notation and techniques for solving the inverse problem of finding a function, given its rate of change, as will be seen in the next two lessons.)

Differential calculus has been applied previously in this text to the following topics: velocity, acceleration, nuclear decay, and reactor power growth. To illustrate the general applicability of calculus, a few of the most common *differential equations* from the fields of mechanics, electricity, nuclear theory, heat and thermodynamics, and magnetism are listed in Table 1. (A differential equation is simply an equation involving at least one derivative.) Trainees will have seen many of these rate-of-change statements in non calculus form in previous courses.

Note that "t" represents time throughout Table 1.

TABLE 1

Some Common Differential Equations of Science

Differential Equation	Definition of Variables
$F = m \frac{dv}{dt}$ (Newton's Second Law)	F is force m is mass v is velocity
$\omega = \frac{d\theta}{dt}$	ω is angular velocity θ is angular displacement
$\alpha = \frac{d\omega}{dt}$	α is angular acceleration ω is angular velocity
$\tau = I\frac{d\omega}{dt} = I\alpha$	τ is torque I is moment of inertia ω, α as above
$F = - \frac{dE_p}{dr}$	F is force E_p is potential energy in a central force field as a function of r, the distance from the center of force (eg, gravity, Coulomb electric forces)
$P = \frac{dW}{dt}$	P is power W is energy converted or work done
$i = \frac{dq}{dt}$	i is electric current q is charge
$i_C = C \frac{dV_C}{dt}$	i_C is capacitor current flow C is capacitance of capacitor V_C is capacitor voltage

Differential Equation	Definition of Variables
$V_L = L \frac{di_L}{dt}$	V_L is inductor voltage L is inductance i_L is inductor current
$\frac{dN}{dt} = -\lambda N$	N is number of radioactive nuclei remaining at time t λ is the decay constant
$\frac{dA}{dt} = -\lambda A$	A is radioactive source activity λ as above
$\frac{dN}{dx} = -\Sigma x$	Attenuation of nuclear radiation: N is number nuclear projectiles (neutrons, γ's, β's, etc) having penetrated to depth x Σ is macroscopic cross section of attenuating material
$\frac{dP}{dt} = \frac{\Delta k}{L} P$ $\frac{d}{dt} \ell n\ P = \frac{\Delta k}{L}$	P is reactor power Δk is reactivity L is mean neutron lifetime
$C = \frac{1}{m} \frac{dQ}{dT}$	C is specific heat capacity of a substance m is mass of substance Q is quantity of heat stored in substance (in joules) T is temperature of substance (°A)
$\frac{dQ}{dt} = C\Delta T \frac{dm}{dt}$	ΔT is temperature difference (assumed constant in this equation) Q, m, C as above

Differential Equation	Definition of Variables
$H = -kA \frac{dT}{dx}$	H is the heat flow rate through a medium ($\frac{J}{S}$) k is the thermal conductivity of the medium A is the cross sectional area of the medium T is the temperature (^{0}A) x is the penetration depth into the conducting medium
$\frac{dV}{dT} = \frac{nR}{P}$ (Ideal Gas Law)	P is pressure T is temperature (^{0}A) n is number of moles R is gas constant
$V = -N \frac{d\Phi}{dt}$ (Faraday's Law)	V is the voltage across a coil N is number turns in the coil Φ is the magnetic flux through the coil

II Some Instruments Which Differentiate

1) A vibrometer gives the rms velocity of a vibration. This is the average value of the square root of the square of the derivative of the displacement of the pick-up attached to a vibrating object. A second output from the vibrometer gives the acceleration (derivative of the derivative of the displacement) of the vibration.

2) A magnetic phono cartridge delivers a voltage whose amplitude is proportioanl to the time derivative of the displacement of the stylus.

3) An accelerometer is a transducer that provides an output voltage proportional to the acceleration of some object. This voltage is produced across a piezoelectric crystal.

4) An operational amplifier in the differentiating mode produces an output voltage proportional to the derivative of its input (see Figure 1).

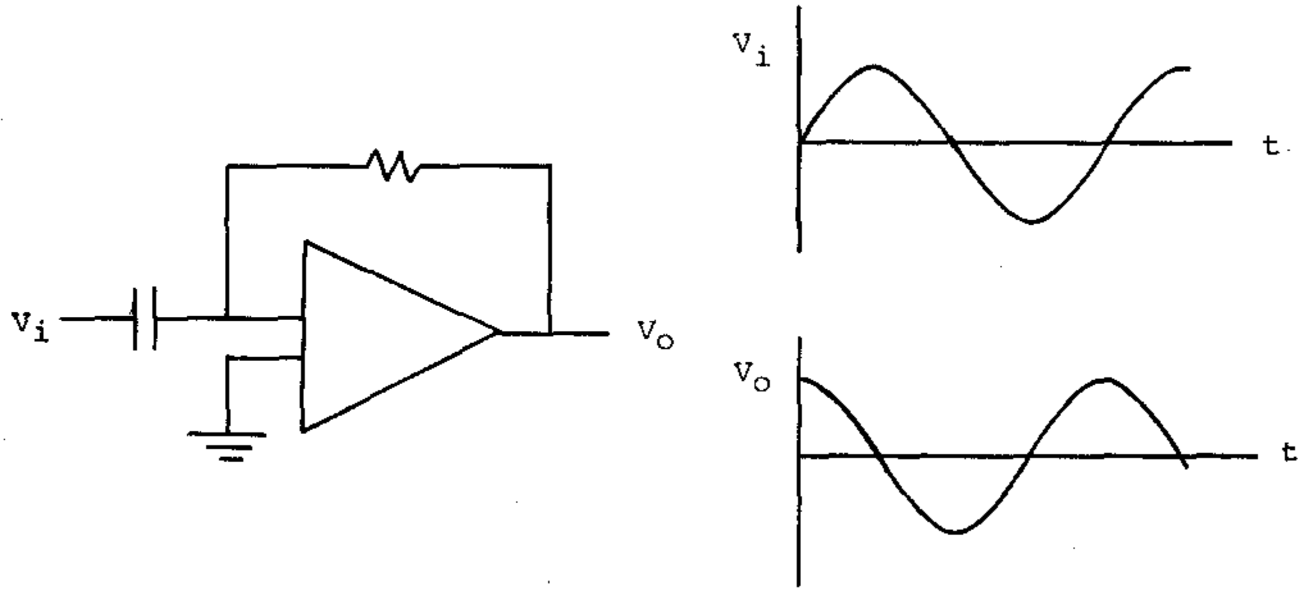


Figure 1

III Derivative Control Mode

Negative feedback is used to control process parameters such as pressure, flow, temperature, reactor power, etc., to set point. The difference between the *set point* and actual *measured value* is called the *error*. For example, suppose the fluid level in the tank of Figure 2 is to be controlled at the set point. The flow demand from the bottom of the tank is variable, and the level is maintained at the set point by manipulating the inflow rate via the inlet control valve. The level measurement is supplied to the controller by the level transmitter (LT). The controller establishes the difference between this measured value and the set point, ie, the error, and sends a control signal to the control valve actuator. This control signal always manipulates the inflow to the tank in such a way as to reduce the error, ie, if the level is below set point, the controller opens the inlet valve, and vice versa.

If the control signal (CS) is directly proportional to the error, e, control is said to be *proportional*, ie,

$(CS)_p = k_p e$ (+ b), where k_p is a constant

and b is the constant equilibrium bias

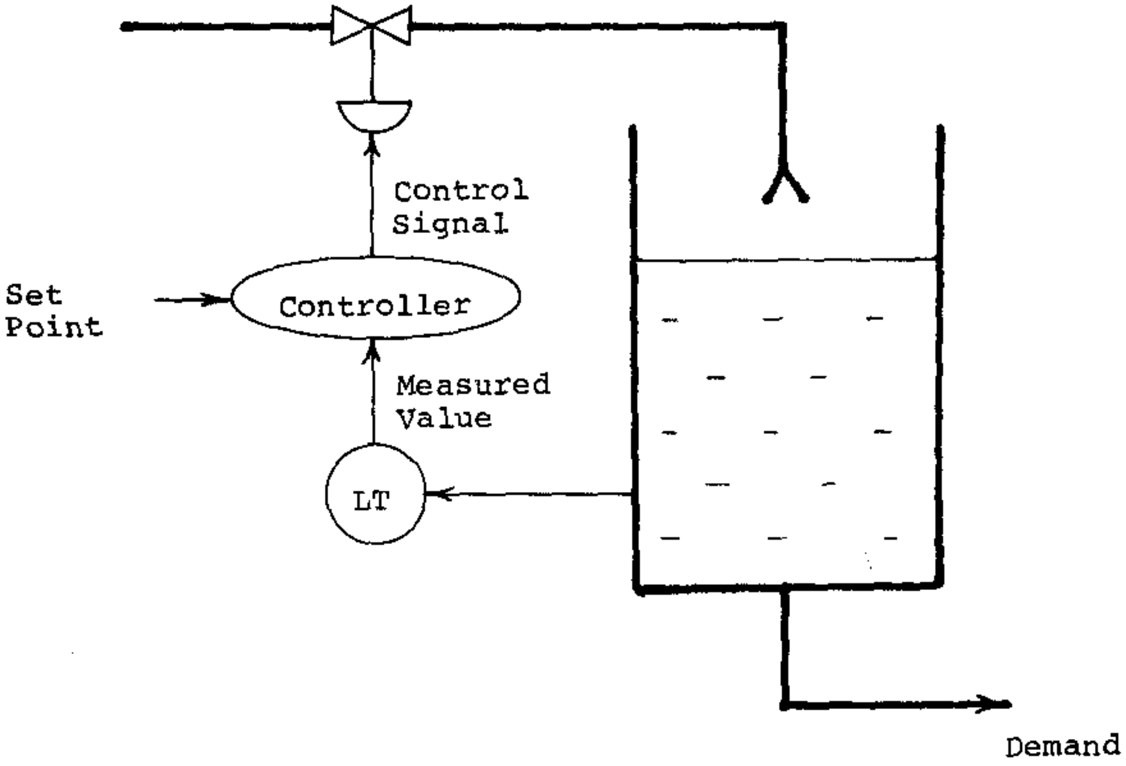


Figure 2 Tank Level Control Loop

The disadvantages of proportional control are:

(1) The controller cannot begin to take corrective action until after an error is established. Because of time lag in the control loop, this often leads to considerable 'overshoot' in corrective action (see Figure 4), with process instability while the controller 'hunts' for the correct control signal to match inlet and outlet flows.

(2) No control signal (other than equilibrium bias) is possible without an error. For example, if demand increases, the level must fall sufficiently below set point that the control signal, $(CS)_P$, can match inflow and outflow. The amount by which the level must remain below set point in order to keep the flows matched (see Figure 4) is called the *offset*.

The first of the above disadvantages can be counteracted with the use of the *derivative control mode*; the second disadvantage is overcome with the use of the *reset control mode* (see 221.30-2, section VIII). A derivative controller's output is proportional to the rate at which the level is straying from set point, ie, to rate of change of the error:

$$(CS)_D = k_D \frac{de}{dt}$$

Graphically speaking, the derivative mode control signal is proportional to the slope of the tangent to the error-time curve. In practice, derivative mode control is usually used in conjunction with proportional mode, so that the control signal is made up of proportional and derivative components:

$$(CS)_{PD} = k_P e + k_D \frac{de}{dt} \quad (+\ b)$$

The rate signal, $\frac{de}{dt}$, can be obtained by passing the error signal e(t) through a differentiating amplifier.

The proportional and derivative components and the total output signal from a proportional-derivative controller are shown in Figure 3 for two hypothetical examples of level fluctuations in the tank of Figure 2. (NB: assume that demand varies such that the level fluctuates as shown in spite of feedback). In Figure 3(a), the tank level drops linearly from set point to a lower value, and in Figure 3(b), the level makes a temporary excursion below set point. Note that a few representative tangents have been drawn on the error curve of Figure 3(b), and that the derivative component of the control signal correlates with the value of the tangent slope at every instant in both Figures 3(a) and 3(b).

Note that in both Figures 3(a) and 3(b), as soon as the level starts to drop, the controller immediately puts out a signal via the derivative mode component to open the inlet valve. The amplitude of this signal is proportional to the rate at which the level is dropping. This is in contrast to the proportional mode component, which becomes significant only after the error grows significantly. Thus derivative control builds an 'anticipatory' feature into the control loop. In fact, an output proportional to the present rate of error growth is actually the same output that would be produced by the proportional mode at some later time (how much later depends on the constant, k_D), assuming that the error were to continue to grow at the same rate in the meantime. Graphically speaking, the derivative mode controller extrapolates a certain number of seconds along the tangent to the error-time curve.

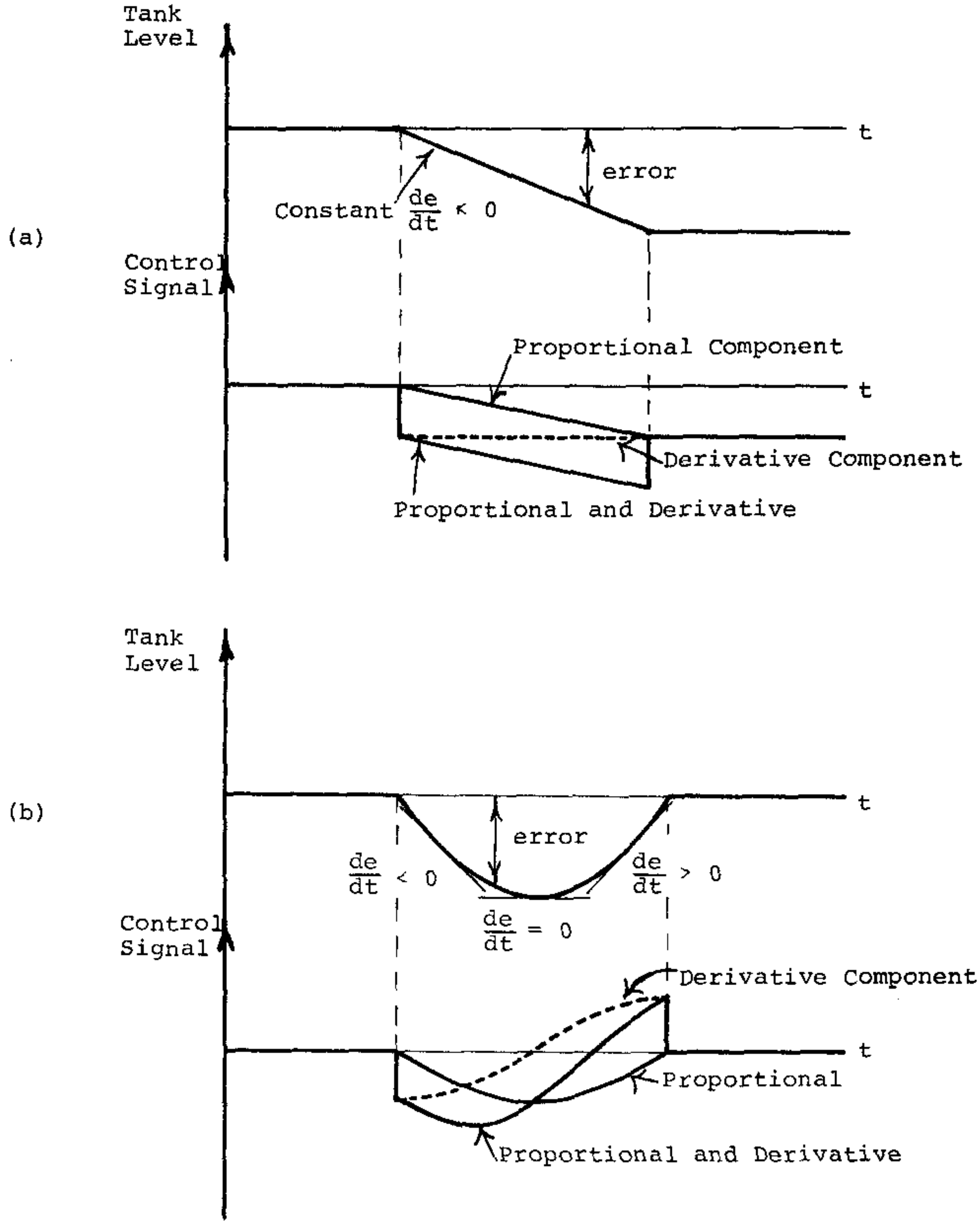


Figure 3 Proportional and Derivative Control Signals for Hypothetical Tank Level Fluctuations

Figure 4 shows typical fluctuations in level following a step increase in demand

(a) without any feedback control - the level drops at a constant rate

(b) with proportional control - the level drops to offset value, overshoots, oscillates, and eventually settles out at offset value

(c) with proportional plus derivative control - the level stabilizes to the same offset value more rapidly, with reduced overshoot.

To summarize, the advantage of derivative mode control is that corrective action is initiated before a large error is established, and faster stabilization is achieved with smaller deviation from set point following process transients. Arguments analogous to the preceeding discussion of tank level control indicate similar advantages to using derivative control mode in controlling reactor power, boiler level, moderator temperature, etc., in CANDU plants.

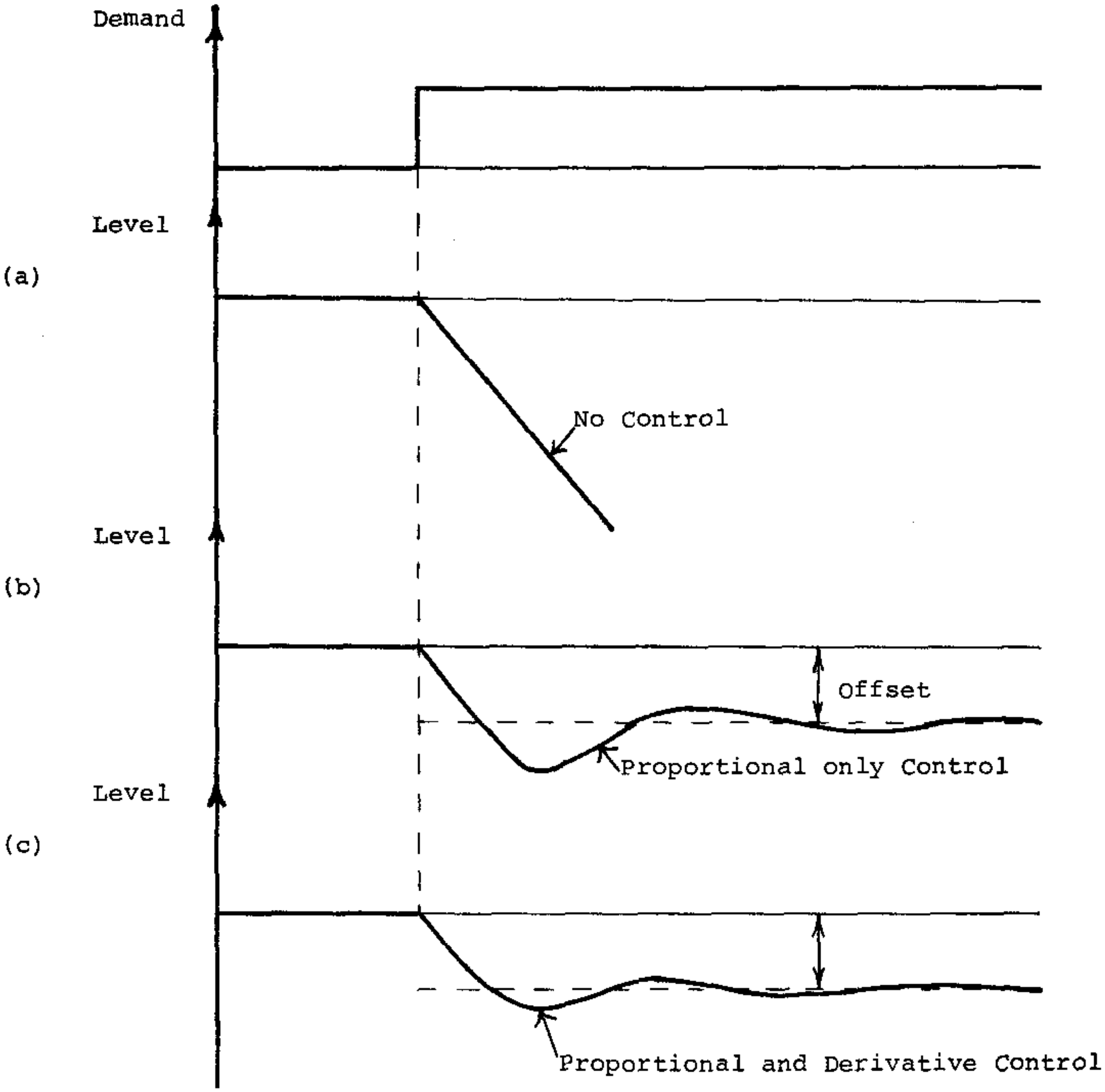


Figure 4 Tank Level Fluctuations Following a Step Demand Increase

ASSIGNMENT

1. Write a verbal rate-of-change statement corresponding to each differential equation in Table 1 of this lesson.

2. Practice writing the differential equations corresponding to the rate-of-change statements given in section 221.40-4 in answer to question #1 above.

3. A solenoid moves a plunger s meters in t seconds according to the relation

$$s(t) = 2t^3 + 0.02t$$

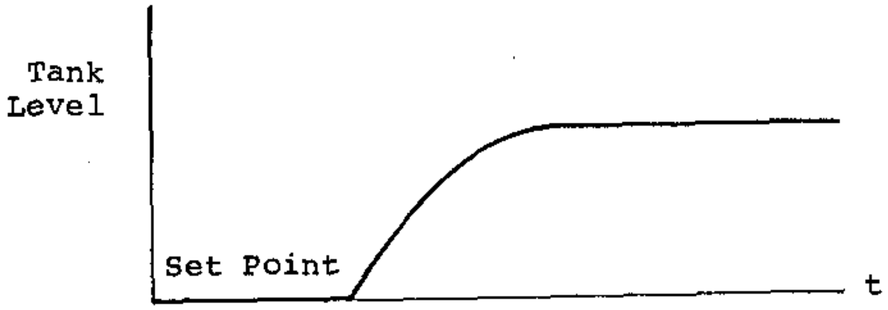
If the plunger mass is 0.05 kg, calculate the force exerted on the plunger at $t = 0.01s$.

4. (a) Write the differential equation corresponding to the following rate-of-change statement: "The voltage V_2 induced in coil #2 equals the product of the mutual inductance M between coils #1 and #2 times the rate of decrease of the current i_1 through coil #1".

   (b) If $M = 2$ henries and $I_1 = 3t^2 - t^3$ amperes, calculate (i) $V_2(t)$, (ii) $V_2(2)$.

5. The following diagram depicts a hypothetical variation in the tank level of Figure 2. On the same time axis, sketch the following for a proportional-derivative controller:

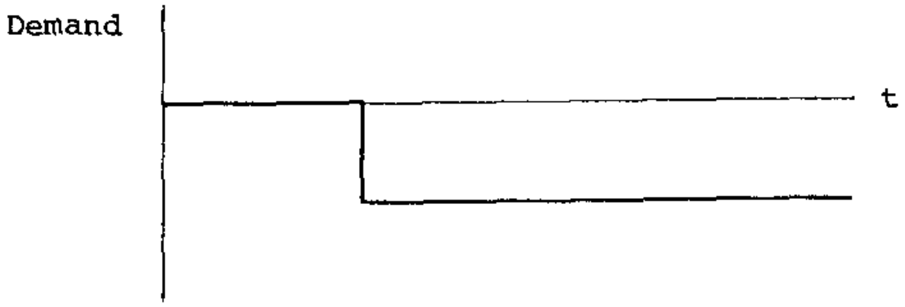
   (a) proportional component of control signal
   (b) derivative component of control signal
   (c) total output signal.



6. The following diagram depicts a step decrease in the demand flow from the tank of Figure 2. On the same time axis, sketch typical corresponding fluctuations in tank level in the following cases:

   (a) no level control
   (b) proportional only level control
   (c) proportional-derivative level control

   Label the offset in (b) and (c). Assume level was at set point prior to demand change.



L.C. Haacke

Mathematics - Course 221

THE INTEGRAL

I The Indefinite Integral

If $F^1(x) = f(x)$, then $f(x)$ is the *derivative* of $F(x)$, and $F(x)$ is an *antiderivative* of $f(x)$. The process of finding $f(x)$ from $F(x)$ is called *differentiation*, whereas the process of finding $F(x)$ from $f(x)$ is called *integration*. Thus differentiation and integration are opposite processes:

$$F(x) \underset{\text{integration}}{\overset{\text{differentiation}}{\rightleftharpoons}} f(x) = F^1(x)$$

Example 1

x^2 is an antiderivative of $2x$ since

$$\frac{d}{dx} x^2 = 2x$$

In fact, any function of the form $F(x) = x^2 + C$ is an antiderivative of $f(x) = 2x$ since

$$\frac{d}{dx} (x^2 + C) = 2x \qquad \left(\frac{d}{dx} C = 0 \right)$$

"C" is called an *integration constant*.

Graphical Significance of Integration Constant

The graphical significance of the *integration constant C* is that $x^2 + C$ represents a *family of curves*, each value of C corresponding to a unique member of the family (see Figure 1). Note that every member of the family has precisely the same slope at any particular x-value, say x_1.

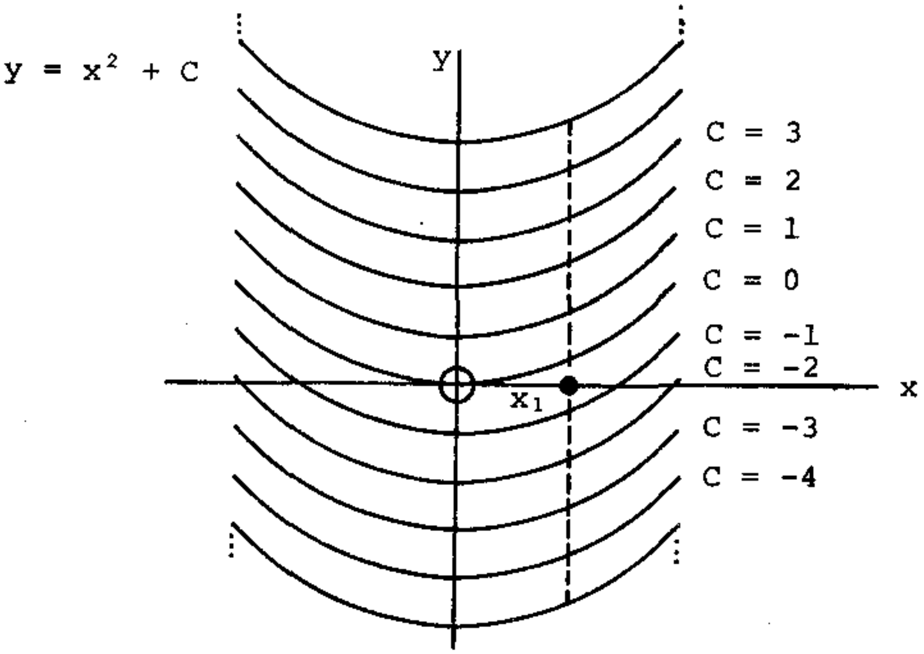


Figure 1

Boundary Condition

Integration may be regarded as the process of finding a function (curve) from its derivative (slope). As Figure 1 illustrates, there is always an infinite family of curves having the given slope.

An integration has a unique solution, however, if a *boundary condition* is imposed.

DEFINITION: A *boundary condition* is the specification of the value of the integral at a particular x-value.

The graphical significance of imposing a boundary condition is that a point is specified on the solution curve, and thus a unique curve is selected from the infinite family as the solution.

Example 2

Find the antiderivative of $f(x) = 2x$, which has the value 5 when $x = 2$.

Solution

The required antiderivative has the form

$$F(x) = x^2 + C \quad \text{(from Example 1)}$$

However, the boundary condition,

$$F(2) = 5 \Rightarrow (2)^2 + C = 5$$

$$\therefore C = 1$$

$$\therefore \quad \underline{\underline{F(x) = x^2 + 1}}$$

Note that the boundary condition in Example 2 selects the curve corresponding to C = 1 in Figure 1.

Integral Notation

"The integral of 2x with respect to x equals x^2 + C" is written symbolically as follows:

$$\underbrace{\int}_{\text{integral sign}} \underbrace{2x}_{\text{integrand}}\ \underbrace{dx}_{\text{differential}} = \underbrace{\underbrace{x^2}_{\text{antiderivative}} + \underbrace{C}_{\text{integration constant}}}_{\text{indefinite integral}}$$

Where:

- the *integral sign* is read "the integral of"
- the *integrand* is the function being integrated
- the *differential* "dx" indicates integration wrt x
- the antiderivative and integration constant together comprise the *indefinite integral* of 2x wrt x.

In general, the derivative of F(x) wrt x equals f(x) if and only if the integral of f(x) equals F(x) + C

ie,

$$\boxed{F^1(x) = f(x) \Longleftrightarrow \int f(x)dx = F(x) + C}$$

Example 3

$$\int 4x^3dx = x^4 + C \quad \text{since } \frac{d}{dx}(x^4 + C) = 4x^3$$

II Displacement, Velocity and Acceleration

$$v(t) = s^1(t) \Longleftrightarrow s(t) = \int v(t)dt$$

$$a(t) = v^1(t) \Longleftrightarrow v(t) = \int a(t)dt$$

Example 4

Find v(t) and s(t) given a(t) = -10, and the boundary conditions, v(0) = 0 and s(0) = 100.

Solution

$$v(t) = \int a(t)dt$$

$$= \int -10dt$$

$$= -10t + C_1$$

But $v(0) = 0 \Rightarrow -10(0) + C_1 = 0$

$$\therefore \quad C_1 = 0$$

$$\therefore \quad \underline{\underline{v(t) = -10t}}$$

$$s(t) = \int v(t)dt$$

$$= \int -10t\ dt$$

$$= -5t^2 + C_2$$

But $s(0) = 100 \Rightarrow -5(0)^2 + C_2 = 100$

$$\therefore \quad C_2 = 100$$

$$\therefore \quad \underline{\underline{s(t) = -5t^2 + 100}}$$

III Integration Formulas

The following is a table of integration formulas with corresponding differentiation formulas studied in lesson 221.20-2.

DIFFERENTIATION FORMULA	CORRESPONDING INTEGRATION FORMULA
$\frac{d}{dx} C = 0$	$\int 0dx = C$
$\frac{d}{dx} x^n = nx^{n-1}$	$\int x^n dx = \frac{x^{n+1}}{n+1} + C,\ n \neq -1$
$\frac{d}{dx} (f(x) \pm g(x)) = \frac{d}{dx} f(x) \pm \frac{d}{dx} g(x)$	$\int (f(x) \pm g(x))dx = \int f(x)dx \pm \int g(x)dx$
$\frac{d}{dx} e^{f(x)} = e^{f(x)} f^1(x)$	$\int e^{f(x)} f^1(x)dx = e^{f(x)} + C$

Example 5

$$\int x^{20} dx = \frac{x^{21}}{21} + C$$

Example 6

$$\int \pi x^5 dx = \pi \int x^5 dx$$

$$= \pi (\frac{x^6}{6} + C)$$

$$= \frac{\pi}{6} x^6 + C_1 \qquad (C_1 = \pi C)$$

Example 7

$$\int(x^3 + \sqrt{x})dx = \int x^3dx + \int\sqrt{x}\ dx$$

$$= \frac{x^4}{4} + C_1 + \frac{x^{3/2}}{3/2} + C_2$$

$$= \frac{1}{4}x^4 + \frac{2}{3}x^{3/2} + C \qquad (C = C_1 + C_2)$$

Example 8

$$\int\frac{x^2 - 1}{\sqrt{x}}\ dx = \int(\frac{x^2}{\sqrt{x}} - \frac{1}{\sqrt{x}})\ dx$$

$$= \int(x^{3/2} - x^{-1/2})dx$$

$$= \int x^{3/2}\ dx - \int x^{-1/2}dx$$

$$= \frac{x^{5/2}}{5/2} + C_1 - (\frac{x^{1/2}}{1/2} + C_2)$$

$$= \frac{2}{5}x^{5/2} + 2\sqrt{x} + C \qquad (C = C_1 - C_2)$$

Note that the integrand in this example was expressed in terms of functions of the form x^n prior to integration, since no method of integrating a quotient of two functions has been given.

Example 9

$$\int 2xe^{x^2}dx = e^{x^2} + C$$

Note that this integral is of the form $\int e^{f(x)}f^1(x)dx$ where $f(x) = x^2$ and $f^1(x) = 2x$.

Example 10

If $v(t) = 10e^{-t} + t$, find $s(t)$ assuming $s(0) = 0$.

Solution

$$s(t) = \int v(t)dt$$

$$= \int (10e^{-t} + t)dt$$

$$= \int 10e^{-t}dt + \int tdt$$

$$= -10 \int e^{-t}(-1)dt + \int tdt$$

$$= -10e^{-t} + \frac{t^2}{2} + C$$

$$\text{But } s(0) = 0 \Rightarrow -10e^0 + \frac{0^2}{2} + C = 0$$

$$\text{ie, } -10 + C = 0$$

$$\text{ie, } \quad C = 10$$

$$\therefore \quad \underline{\underline{s(t) = -10e^{-t} + \frac{t^2}{2} + 10}}$$

Note that "-10" rather than 10 was factored out of the first integrand in line 4 of this solution so as to leave a factor of (-1) in the integrand, which is of the form $e^{f(t)}f^1(t)$ where $f(t) = e^{-t}$ and $f^1(t) = -1$.

IV Area Under a Curve

Let $A(x_1)$ represent the area under the curve $y = f(x)$ from $x = a$ to $x = x_1$. Then $A(x_1 + \Delta x)$ represents the area from $x = a$ to $x = x_1 + \Delta x$, and $A(x_1 + \Delta x) - A(x_1)$ represents the area under the curve between x_1 and $x_1 + \Delta x$, as labelled in Figure 2.

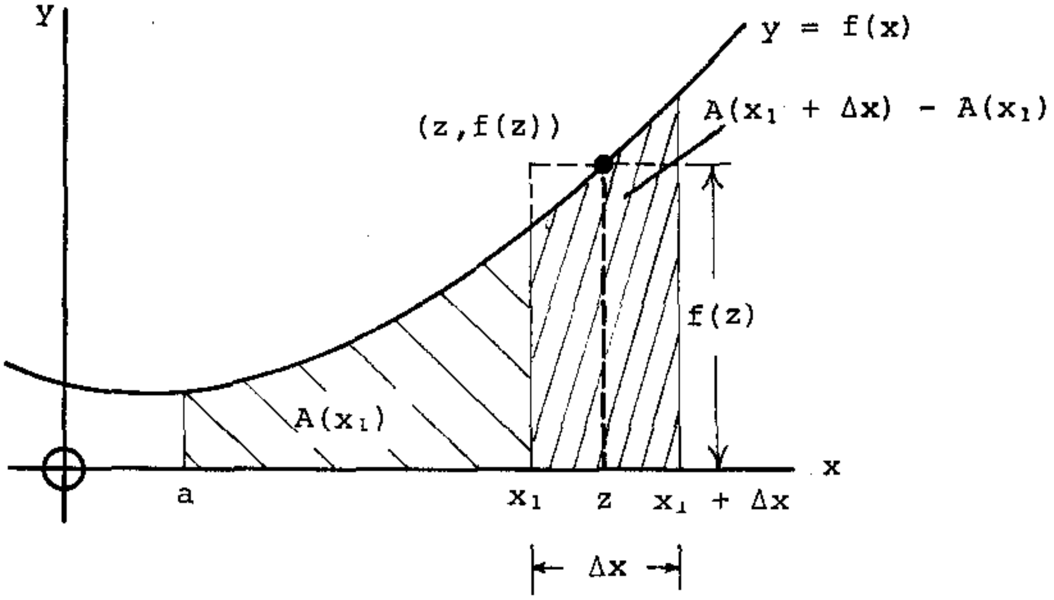


Figure 2

Obviously, for some value of x, say x = z, $x_1 \leq z \leq x_1 + \Delta x$, the area of the rectangle Δx units wide by f(z) units high exactly equals the area $A(x_1 + \Delta x) - A(x_1)$ under the curve,

ie, $A(x_1 + \Delta x) - A(x_1) = f(z)\Delta x$

$$\therefore \quad \frac{A(x_1 + \Delta x) - A(x_1)}{\Delta x} = f(z)$$

$$\therefore \quad \lim_{\Delta x \to 0} \frac{A(x_1 + \Delta x) - A(x_1)}{\Delta x} = \lim_{\Delta x \to 0} f(z)$$

The LHS of this equation is the derivative, $A^1(x_1)$, by definition (see lesson 221.20-2).

Furthermore as $\Delta x \to 0$, $z \to x_1$ (see Figure 2).

$$\therefore \quad A^1(x_1) = f(x_1)$$

Finally, since the value of x_1 is arbitrary, it can be replaced by the variable x.

Thus $A^1(x) = f(x)$

ie, the derivative of the area function equals the curve function.

$$\therefore \quad \boxed{A(x) = \int f(x)dx = F(x) + C}$$

Example 11

Find the area under the curve y = 5 from x = 0 to (a) $x = x_1$ (b) x = 1 (c) x = 10.

Solution

(a) $A(x) = \int f(x)dx$

$= \int 5dx$

$= 5x + C$

But $A(0) = 0 \Rightarrow 5(0) + C = 0$

$\therefore \quad C = 0$

$\therefore \quad \underline{\underline{A(x_1) = 5x_1}}$

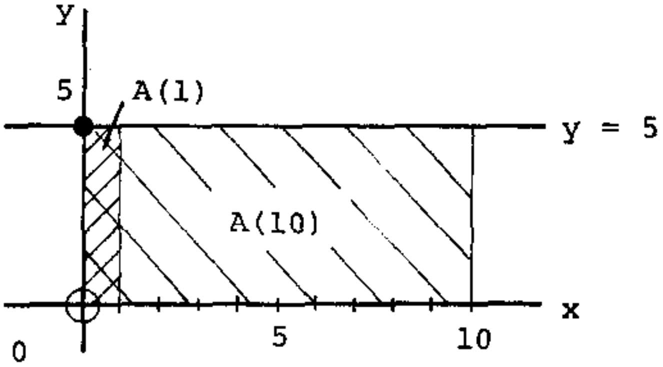


Figure 3

(b) $A(1) = 5(1) = \underline{\underline{5}}$

(c) $A(10) = 5(10) = \underline{\underline{50}}$

Note, with reference to Figure 3, that the above areas are rectangular, and that the same answers are obtained by using "area = length x width".

Example 12

Find the area under the curve y = 2x from x = 0 to (a) $x = x_1$ (b) x = 1 (c) x = 10.

Solution

$$A(x) = \int f(x)dx$$

$$= \int 2x dx$$

$$= x^2 + C$$

But $A(0) = 0 \Rightarrow 0^2 + C = 0$

$\therefore \quad C = 0$

$\therefore \quad A(x) = x^2$

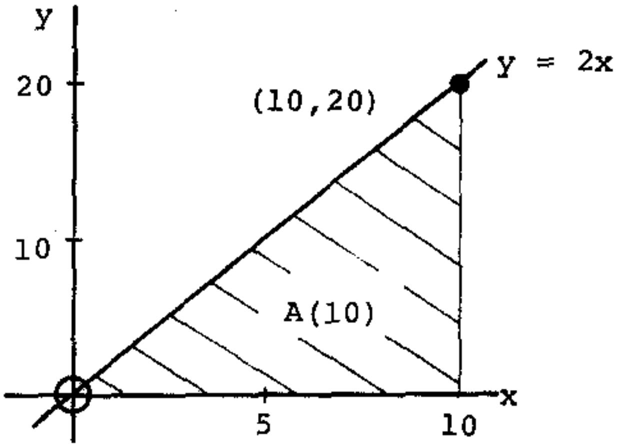


Figure 4

(a) $\therefore \quad \underline{\underline{A(x_1) = x_1^2}}$

(b) $A(1) = 1^2 = \underline{\underline{1}}$

(c) $A(10) = 10^2 = \underline{\underline{100}}$

Note, with reference to Figure 4, that the above areas are triangular, and that the same answers are obtained using "area = $\frac{1}{2}$ base times height".

Example 13

Find the area under the curve $y = x^2$ from $x = 0$ to (a) $x = 5$ (b) $x = 10$.

Solution

$$A(x) = \int x^2 dx$$

$$= \frac{x^3}{3} + C$$

But $A(0) = 0 \Rightarrow \frac{0^3}{3} + C = 0$

$\therefore \quad C = 0$

$\therefore \quad \underline{\underline{A(x) = \frac{x^3}{3}}}$

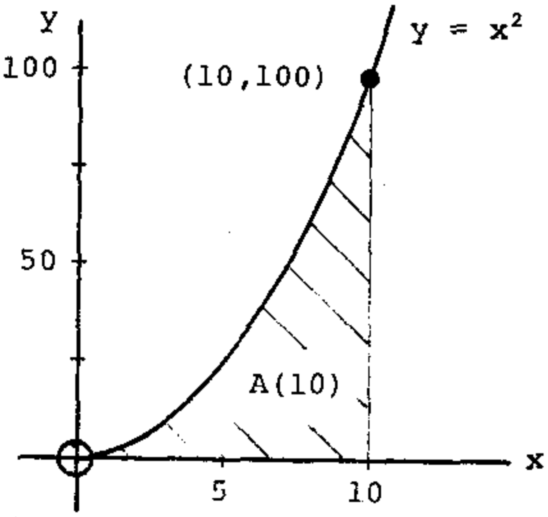


Figure 5

(a) $A(5) = \underline{\underline{\frac{125}{3}}}$

(b) $A(10) = \underline{\underline{\frac{1000}{3}}}$

V The Definite Integral

The procedure for calculating areas illustrated in examples 11 to 13 above can be 'streamlined' considerably by using the definite integral notation, which effectively 'builds in' the boundary condition.

Recall that the area under the curve $y = f(x)$ from $x = a$ up to $x = x$ is given by

$$A(x) = \int f(x)dx = F(x) + C$$

$$\text{But} \quad A(a) = 0 \Rightarrow F(a) + C = 0$$

$$\therefore \quad C = -F(a)$$

$$\therefore \quad A(x) = F(x) - F(a)$$

$\therefore$ the area under the curve between $x = a$ and $x = b$,

$$A(b) = F(b) - F(a)$$

The *definite integral* notation $\int_a^b f(x)dx$ is used to represent $F(b) - F(a)$.

ie,

$$\boxed{A(b) = \int_a^b f(x)dx = F(b) - F(a)}$$

In this notation "a" and "b" are called the *lower limit* and *upper limit*, respectively, of the integration.

Examples 11 and 13 will now be redone using definite integrals:

Example 14

Find the area under the curve $y = 5$ between $x = 0$ and (a) $x = x_1$ (b) $x = 1$ (c) $x = 10$.

Solution

(a)
$$A(x_1) = \int_0^{x_1} 5dx$$

$$= 5x \Big|_0^{x_1}$$

$$= 5x_1 - 5(0)$$

$$= \underline{\underline{5x_1}}$$

The notation $F(x)\Big|_a^b$ is used above to indicate the explicit form of the antiderivative, which is to be evaluated between the limits a and b.

(b)
$$A(1) = \int_0^1 5dx$$

$$= 5x \Big|_0^1$$

$$= 5(1) - 5(0)$$

$$= \underline{\underline{5}}$$

(c)
$$A(10) = \int_0^{10} 5dx$$

$$= 5x \Big|_0^{10}$$

$$= 5(10) - 5(0)$$

$$= \underline{\underline{50}}$$

Example 15

Find the area under the curve $y = x^2$ from $x = 0$ to (a) $x = 5$ (b) $x = 10$.

Solution

(a) $$A(5) = \int_0^5 x^2 dx$$

$$= \frac{x^3}{3}\Big|_0^5$$

$$= \frac{5^3}{3} - \frac{0^3}{3}$$

$$= \underline{\underline{\frac{125}{3}}}$$

(b) $$A(10) = \int_0^{10} x^2 dx$$

$$= \frac{x^3}{3}\Big|_0^{10}$$

$$= \frac{10^3}{3} - \frac{0^3}{3}$$

$$= \underline{\underline{\frac{1000}{3}}}$$

ASSIGNMENT

1. Differentiation and integration are opposite processes, yet the derivative is unique whereas the integral is not, in general. Explain why this is so.

2. Why is a boundary condition necessary to obtain a unique solution to an integral? Explain with reference to graphical significance of boundary condition.

3. Evaluate the following indefinite integrals:

   (a) $\int -3x\,dx$ (b) $\int (e^t + t^3)\,dt$

   (c) $\int (2x^2 + 3x - 5)\,dx$ (d) $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$

   (e) $\int (4x^3 - \sqrt[3]{x})\,dx$ (f) $\int (2te^{-t^2} + t^{3/2})\,dt$

4. Find displacement and velocity functions s(t) and v(t), respectively, given acceleration function a(t) and boundary conditions as follows:

   (a) $a(t) = 0$, $v(0) = 0$, $s(0) = 0$

   (b) $a(t) = 2$, $v(0) = 10$, $s(0) = 14$

   (c) $a(t) = 2t$, $v(0) = V_0$, $s(0) = 0$

   (d) $a(t) = e^{-t}$, $v(0) = 10$, $s(0) = -10$

5. Find the function which gives the vertical displacement of a projectile relative to ground level, neglecting air resistance. The acceleration due to gravity is $9.8\ m/s^2$ downwards. Assume the projectile is given initial velocity V_0 m/s vertically upwards from ground level.

6. Find the area under the curve $y = f(x)$ between a and b where

   (a) $f(x) = 2x$, $a = 1$, $b = 10$

   (b) $f(x) = \sqrt{x}$, $a = 4$, $b = 16$

   (c) $f(x) = x^2 + 4$, $a = -2$, $b = 5$

   (d) $f(x) = xe^{-x^2}$, $a = 0$, $b = 2$

7. Evaluate the following definite integrals:

   (a) $\int_1^9 \sqrt{x}\,(x - 1)dx$

   (b) $\int_{-5}^0 (-5x + 3)dx$

   (c) $\int_0^2 (10 + t - e^t)dt$

L.C. Haacke

Mathematics - Course 221

APPLICATION OF THE INTEGRAL AS AN INFINITE SUM

I Area Under a Curve

Suppose the x-interval from x = a to x = b is *partitioned* into n subintervals of width

$$\Delta x = \frac{b - a}{n}$$

Then the area under the curve $y = f(x)$ between $x = a$ and $x = b$ is approximately the same as the area of the n inscribed rectangles corresponding to these n subintervals, as shown in Figure 1.

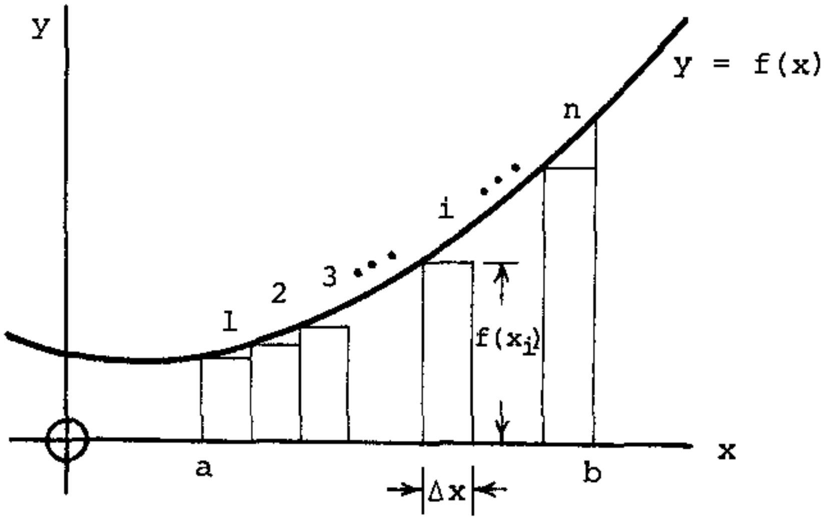


Figure 1

In Figure 1, if x_i is the x-value at the left boundary of the i th rectangle, then $f(x_i)$ is the height of the i th rectangle, and the total area A_r of the n rectangles is

$$A_r = f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + \ldots + f(x_n)\Delta x$$

This sum can be abbreviated using *sigma notation* as follows:

$$A_r = \sum_{i=1}^{n} f(x_i)\Delta x$$

The RHS of this equation is read "the sum of $f(x_i)\Delta x$ as i takes all (integral) values from 1 to n inclusive".

As can be seen from Figure 2, which shows the interval of Figure 1 partitioned into n = 4, 8, 16 subintervals, the larger the number n of rectangles, the closer A_r is to the true area A under the curve. In fact,

$$A = \lim_{n\to\infty} A_r$$

$$\text{ie,}\qquad A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

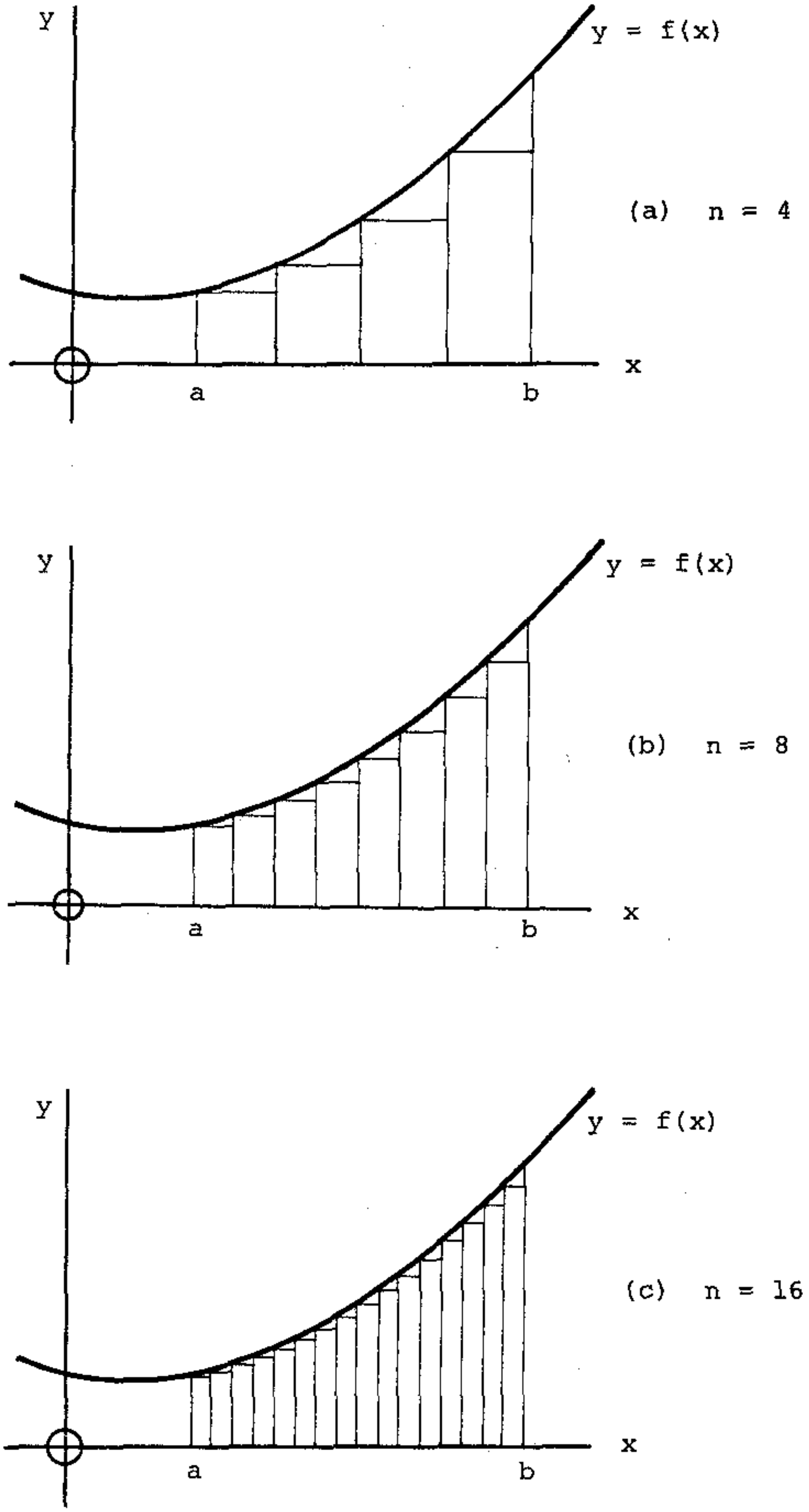
But, from lesson 221.30-1,

$$A = \int_a^b f(x)\,dx$$

$$\therefore \qquad \boxed{\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x,\ \Delta x = \frac{b-a}{n}}$$

ie, the definite integral can be regarded as the sum of an infinite number of infinitely narrow rectangles. Note that as i ranges from 1 to n in the sum, x ranges from a to b in the integral. In fact, the integral sign was originally introduced as a stretched "S" standing for "sum".

This notion of the definite integral as an infinite sum of vanishingly small increments is extremely useful in a wide range of applications, a few of which will be discussed in this lesson. The above result is often referred to as *The Fundamental Theorem.*



<u>Figure 2</u>

Example 1

Find the area enclosed between the curves of $y = x^2$ and $x = y^2$.

Solution

A large, clear diagram showing a representative rectangular slice of the required area is mandatory in the solution to such problems.

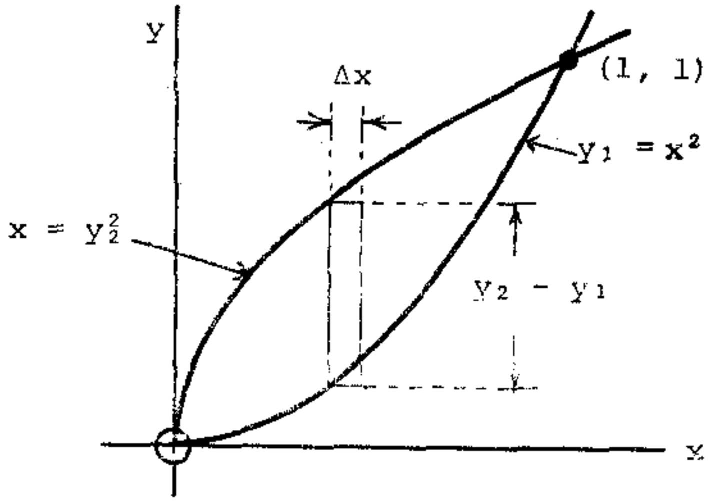


Figure 3

Clearly, from Figure 3, the limits of integration will be the x-values where the two curves intersect. These values are found by solving the two equations $x = y_2^2$ and $y_1 = x^2$:

$$y_1 = y_2 \quad \Rightarrow \quad x^2 = \sqrt{x}$$

$$\text{ie, } x^4 = x$$

$$\text{ie, } x^4 - x = 0$$

$$\text{ie, } x(x^3 - 1) = 0$$

$$\therefore x = 0 \text{ or } x = 1$$

$\therefore$ the curves intersect at (0, 0) and (1, 1).

The required area, $A = \lim_{n\to\infty} \Sigma(y_2 - y_1)\Delta x$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} (\sqrt{x_i} - x_i^2)\Delta x, \quad \Delta x = \frac{1 - 0}{n}$$

$$= \int_{x=0}^{1} (\sqrt{x} - x^2)\,dx \quad \text{(by Fundamental Theorem)}$$

$$= \left[\frac{2}{3}x^{3/2} - \frac{x^3}{3}\right]_0^1$$

$$= \frac{2}{3} - \frac{1}{3} - (0 - 0)$$

$$= \underline{\underline{\frac{1}{3}\text{ square units}}}$$

To illustrate the versatility of this technique the same area will be calculated again, this time using horizontal slices as shown in Figure 4:

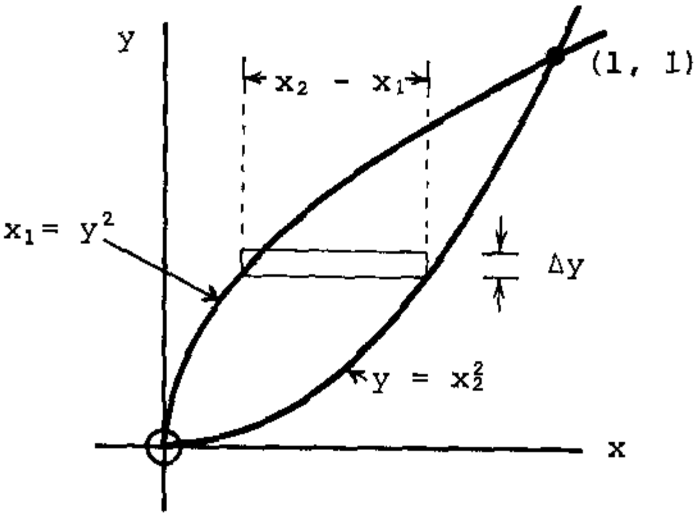


Figure 4

A = sum of horizontal slices

$$= \lim_{n\to\infty} \Sigma(x_2 - x_1)\Delta y$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} (\sqrt{y_i} - y_i^2)\Delta y, \quad \Delta y = \frac{1 - 0}{n}$$

$$= \int_{y=0}^{1} (\sqrt{y} - y^2)\, dy$$

$$= \left[\frac{2}{3} y^{3/2} - \frac{y^3}{3}\right]_0^1$$

$$= \frac{2}{3} - \frac{1}{3}$$

$$= \frac{1}{3} \text{ square units as before.}$$

Example 2

Find the area bounded by the parabola $y = x^2 - 8x + 7$, the x-axis, and the lines x = 2 and x = 6.

Solution

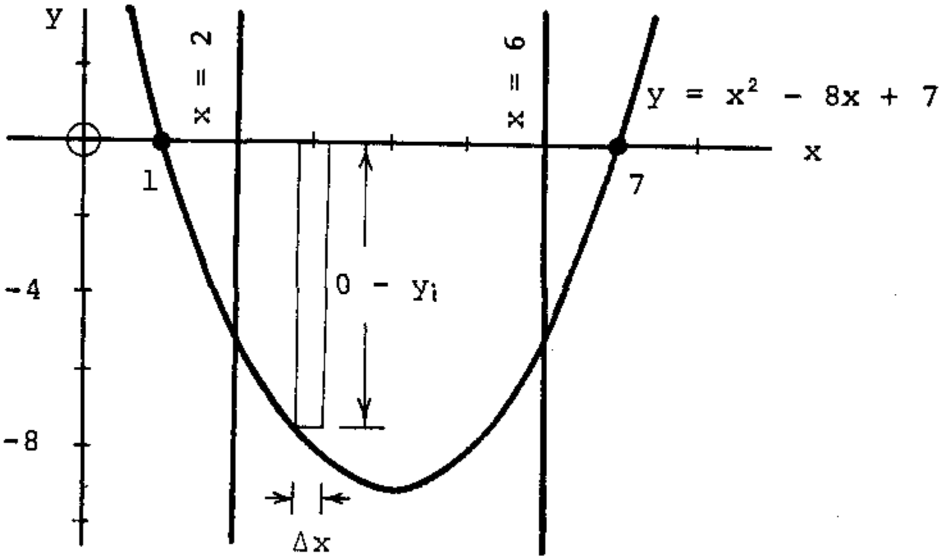


Figure 5

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} (0 - y_i)\Delta x \quad \text{(see Figure 5)}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} (-x_i^2 + 8x_i - 7)\Delta x, \quad \Delta x = \frac{6-2}{n}$$

$$= \int_{x=2}^{6} (-x^2 + 8x - 7)dx \quad \text{(by Fundamental Theorem)}$$

$$= \left[-\frac{x^3}{3} + 4x^2 - 7x \right]_2^6$$

$$= \left[-\frac{6^3}{3} + 4(6)^2 - 7(6) \right] - \left[-\frac{2^3}{3} + 4(2)^2 - 7(2) \right]$$

$$= 30 - \left[-\frac{2}{3} \right]$$

$$= \underline{\underline{30\tfrac{2}{3} \text{ square units.}}}$$

II Calculating Averages

Recall that calculating an average involves dividing a sum of parts by the number of parts (cf lesson 321.30-1).

Example 3

Find the average of the following lengths: 2.10 m, 95 cm, 123 cm, 4.20 m.

Solution

$$\text{Average length} = \frac{2.10 + 0.95 + 1.23 + 4.20}{4}$$

$$= \underline{\underline{2.12 \text{ m}}}$$

It is often necessary to calculate a <u>weighted</u> average, as in Example 4.

Example 4

During a two-year period, the annual inflation rate averaged 8.4% during the first year. During the second year, the rate averaged 9.2% over the first quarter, 9.8% over the second quarter, and 10.2% over the second half. Calculate the mean rate over the two-year period.

Solution

The variation of the inflation rate R is shown in Figure 6.

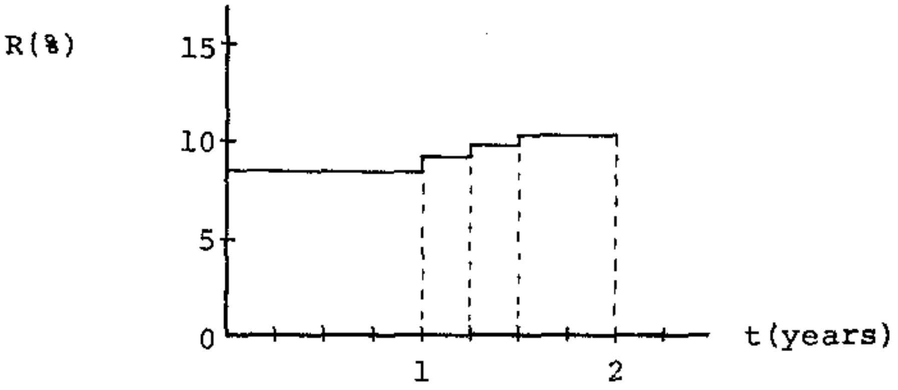


Figure 6

$$\text{Average Inflation Rate } \overline{R} = \frac{1.00 \times 8.4 + 0.25 \times 9.2 + 0.25 \times 9.8 + 0.50 \times 10.2}{2.00}$$

$$= \underline{\underline{9.1\%}}$$

Note that the above calculation is equivalent to dividing the area under the graph of Figure 6 by the total time interval:

$$\overline{R} = \frac{\text{area under R - t graph}}{\text{total t-interval}}$$

This latter approach is useful in calculating averages of continuously varying quantities.

Example 5

The radiation dose rate from a certain radioactive source decays exponentially with time according to the formula,

$$D(t) = D_0 e^{-\lambda t},$$

where, D_0 = 500 mrem/h is the dose rate at t = 0, and
$\lambda = 7.6 \times 10^{-3}\ s^{-1}$ is the decay constant of the source activity.

Find (a) the average dose rate during the first 5 hours.

(b) the total dose received during the first 5 hours.

Solution

t (hours)	0	1	2	3	4	5
D(t) (mrem/h)	500	380	289	220	167	127

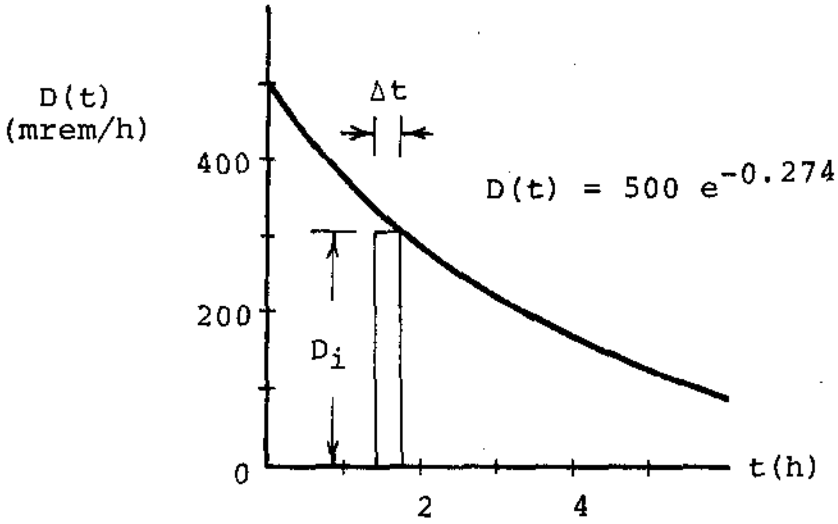


Figure 7

(a) Average dose rate $= \dfrac{\text{total dose received}}{\text{total time}}$

$$= \frac{\lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} D_i \Delta t}{5}$$

$$= \frac{1}{5} \lim_{n\to\infty} \sum_{i=1}^{n} 500e^{-\lambda t_i} \Delta t$$

$$= -\frac{1}{5\lambda} \int_{t=0}^{5} 500e^{-\lambda t}(-\lambda)\,dt \quad \text{(by Fundamental Theorem)}$$

$$= \frac{100}{-\lambda} e^{-\lambda t}\Big|_{0}^{5}$$

$$= \frac{100}{-\lambda} (e^{-0.274 \times 5} - e^{\circ})$$

$$= \frac{100}{0.274} (1 - e^{-1.37})$$

$$= \underline{\underline{2.7 \times 10^2 \text{ mrem/h}}}$$

(b) Total dose $= $ average dose rate x 5

$= 272 \times 5$

$= 1.4 \times 10^3$ mrem

$= \underline{\underline{1.4 \text{ rem}}}$

Example 6

The number N(t) of radioactive nuclei surviving to time t seconds is given by the formula

$$N(t) = N_0 e^{-\lambda t} ,$$

Where N_0 is the number of radioactive nuclei at $t = 0$ and λ is the decay constant.

Find the mean lifetime of a nucleus.

Solution

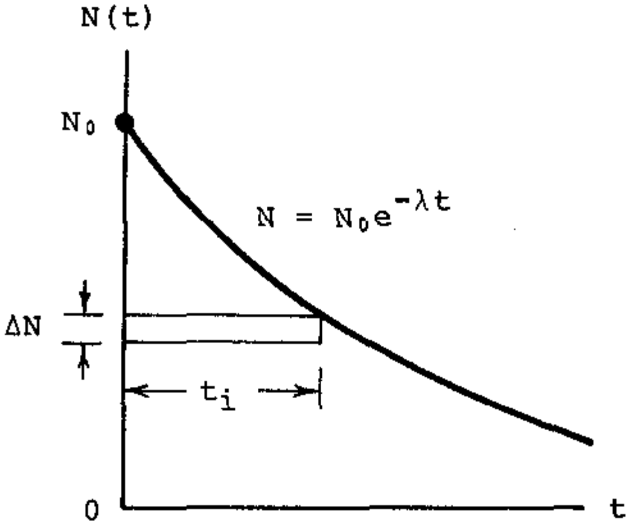


Figure 8

$$\text{Average nuclear lifetime} = \frac{\text{total lifetime of all nuclei}}{\text{total number of nuclei}}$$

$$= \frac{\lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} t_i \Delta N}{N_0}$$

$$= \frac{1}{N_0} \int_{N=0}^{N_0} t dN$$

The integration techniques necessary to evaluate this integral have not been covered in this text. Nevertheless, the answer is that the mean nuclear lifetime, T, equals $1/\lambda$. Thus an alternative form of the given equation is

$$N(t) = N_0e^{-t/T},$$

from which it is obvious that each time one mean nuclear lifetime passes, the number of surviving radioactive nuclei drops by a factor of e.

III Work

Recall from elementary mechanics that work W equals the product of force F times displacement s:

$$W = Fs$$

Example 7

The work done in lifting a 2 kg mass 5 m against gravity,

$$\begin{aligned} W &= F_g s \\ &= mgs \qquad (\text{weight } F_g = mg) \\ &= 2 \times 9.8 \times 5 \quad (g = 9.8 \text{ m/s}^2) \\ &= \underline{\underline{98 \text{ J}}} \end{aligned}$$

In Example 7, the force is constant. When the force is not constant, in general, the work must be calculated by integration.

Example 8

The force F required to stretch a spring varies as the spring extension x,

ie, $F = kx$, where k is the *spring constant*.

Find the amount of energy stored in a spring stretched 0.40 m, assuming $k = 2.0 \times 10^2 \text{ Nm}^{-1}$.

Solution

The extension force F is plotted versus extension x in Figure 9.

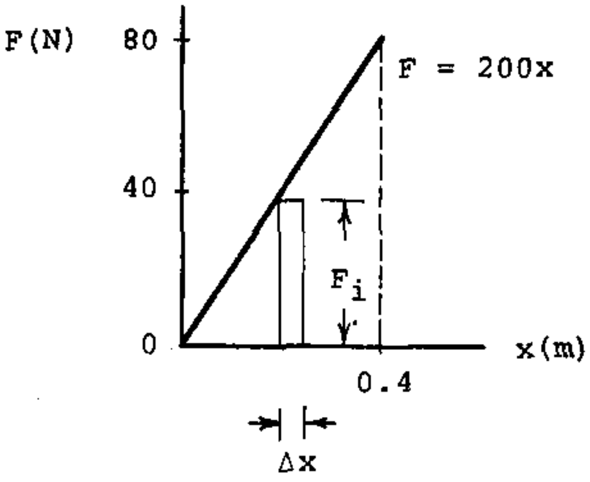


Figure 9

The amount of work represented by the area of the i th rectange in Figure 9 is

$$\Delta W_i = F_i \Delta x \quad \text{(Force } F_i \text{ acts through } \Delta x\text{)}$$

Total work W done in stretching the spring to x = 0.4 m equals total area under F - s graph from x = 0 to x = 0.4,

ie,
$$W = \lim_{n \to \infty} \sum_{i=1}^{n} \Delta W_i$$

$$= \lim_{n \to \infty} \sum_{i=1}^{n} F_i \Delta x$$

$$= \int_0^{0.4} F dx \quad \text{(by Fundamental Theorem)}$$

$$= \int_0^{0.4} kx dx$$

$$= \frac{1}{2} kx^2 \Big|_0^{0.4}$$

$$= \frac{k}{2}\,(0.4^2 - 0^2)$$

$$= \frac{2.0 \times 10^2}{2}\,(0.16)$$

$$= \underline{\underline{16\ J}}$$

$\therefore$ 16 J of elastic potential energy is stored in the stretched spring.

IV Fluid Pressure

Recall from fluid mechanics that the pressure P exerted by a fluid of density ρ kg/m^3 at a depth of h meters is given by the formula.

$$P = \rho g h \quad \text{pascals,}$$

where $g = 9.8$ m/s^2 is the acceleration due to gravity.

Example 9

For the trapezoidal hydraulic dam of Figure 10 (dimensions in meters) find the

(a) total force exerted against the dam face

(b) average pressure exerted against the dam face

(c) center of pressure $(0, \bar{y})$ exerted against dam face, where

$$\bar{y} = \frac{M_x}{F_t}\ ,\ \text{and}$$

$M_x = \int yPdA$ is called the *moment of force about the x-axis*, and

$F_t = \int PdA$ is the total force against the dam,

ie, the same moment would exist if F_t were applied at $(0, \bar{y})$.

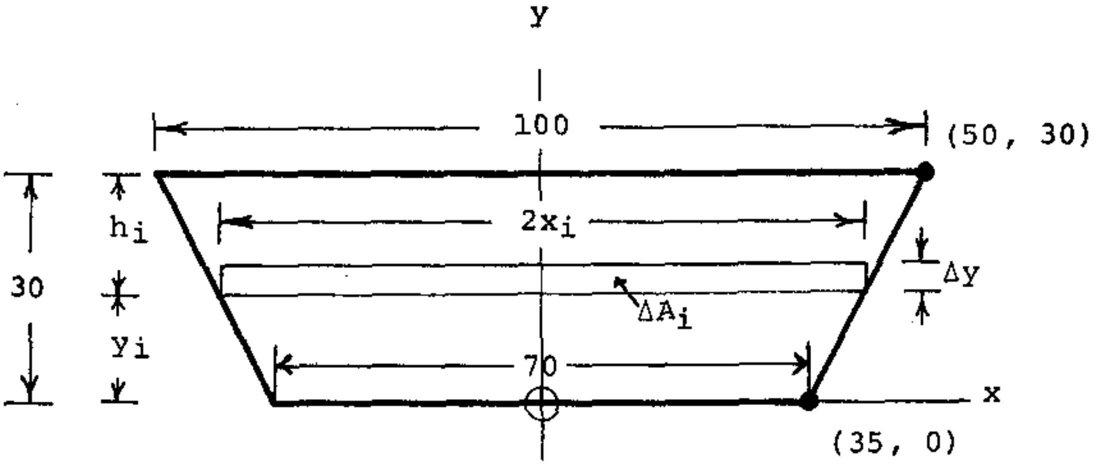


Figure 10

Solution

Total force F_t against dam equals sum of forces ΔF_i against rectangular increments,

$$\text{ie,}\quad F_t = \lim_{n\to\infty} \sum_{i=1}^{n} \Delta F_i$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} P_i \Delta A_i \qquad (\Delta F_i = P_i \Delta A_i)$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} (\rho g h_i)(2x_i \Delta y) \quad (P_i = \rho g h_i;\ \Delta A_i = 2x_i \Delta y)$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \rho g(30 - y_i)\ 2x_i \Delta y \qquad (h_i = 30 - y_i)$$

$$= \int_{y=0}^{30} \rho g(30 - y)\ 2x dy \quad \text{(by Fundamental Theorem)}$$

In order to evaluate this integral, one must express x in terms of y. This is done by first finding the equation of the line representing the right boundary of the dam in Figure 10, using the two-point form:

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}, \quad \text{where } (x_1, y_1) = (35, 0)$$

$$\text{and } (x_2, y_2) = (50, 30)$$

ie, $\frac{y - 0}{x - 35} = \frac{30 - 0}{50 - 35}$

ie, $30(x - 35) = 15y$

ie, $x = \frac{y}{2} + 35$

Substituting for x in the above integral, one gets

$$F_t = \int_0^{30} \rho g(30 - y)\ 2(\tfrac{y}{2} + 35)dy$$

$$= \rho g\int_0^{30} (30 - y)(y + 70)dy$$

$$= \rho g\int_0^{30} (2100 - 40y - y^2)dy$$

$$= \rho g\left[2100y - 20y^2 - \frac{y^3}{3}\right]_0^{30}$$

$$= 10^3 \times 9.8 \left[2100 \times 30 - 20(30)^2 - \frac{30^3}{3}\right]$$

$$= \underline{\underline{3.5 \times 10^8 \text{ N}}}$$

(b) Average pressure on dam $\overline{P} = \frac{\text{total force on dam}}{\text{total area of dam}}$

$$= \frac{3.53 \times 10^8 \text{ N}}{\frac{1}{2}(100 + 70)30 \text{ m}^2}$$

$$= 1.4 \times 10^5 \text{ Pa}$$

$$= \underline{\underline{0.14 \text{ MPa}}}$$

Note: (1) Dam area was calculated using formula for area of a trapezoid,

$A = \frac{1}{2}(s_1 + s_2)h$, where s_1, s_2 are the lengths of the parallel sides, and h is the perpendicular distance between them.

(2) For comparison with above result for $\overline{P}$, the pressure at half-depth (15 m) is

$$\begin{aligned} \rho gh &= 10^3 \times 9.8 \times 15 \\ &= 0.15 \text{ MPa} \end{aligned}$$

$\overline{P}$ is slightly less than this because the dam widens towards the top, ie lower pressures are exerted on the wider slices. If the dam had vertical boundaries, $\overline{P}$ would equal the pressure at half depth.

(c) $\overline{y} = \frac{M_x}{F_t}$,

where $F_t = 3.53 \times 10^8$ Pa from (a), and

$$\begin{aligned} M_x &= \int_{y=0}^{30} yPdA \\ &= \int_0^{30} y\rho gh\ 2xdy \\ &= \rho g\int_0^{30} y(30 - y)(y + 70)dy \\ &= \rho g\int_0^{30} (2100y - 40y^2 - y^3)dy \\ &= \rho g\left[1050y^2 - \frac{40}{3}y^3 - \frac{y^4}{4}\right]_0^{30} \\ &= 10^3 \times 9.8\left\{1050(30)^2 - \frac{40}{3}(30)^3 - \frac{30^4}{4}\right] \\ &= 3.75 \times 10^9 \text{ Nm} \end{aligned}$$

$$\therefore \quad \overline{y} = \frac{3.75 \times 10^9 \text{ Nm}}{3.53 \times 10^8 \text{ Nm}^{-2}}$$

$$= \underline{\underline{11 \text{ m}}}$$

$\therefore$ centre of pressure against dam is at (0, 11) relative to axes of Figure 10.

V Other Applications of the Fundamental Theorem

Applications of the Fundamental Theorem are numerous in science and technology. A few more examples are listed below, but it is beyond the scope of this course to treat them in detail.

(1) Finding the volume of 3-dimensional entities by partitioning the volume into increments such as cubes, slices, or shells.

(2) Finding the center of mass of 3-dimensional entities of either homogeneous or variable densities.

(3) Finding the average neutron flux in reactor cores of various geometries.

(4) Finding moments of inertia of plane or 3-dimensional entities about an axis of rotation

(5) Finding surface area of 3-dimensional figures.

VI Familiar Instruments Which Integrate

By considering the integrating functions of certain familiar instruments, the trainee will consolidate his concept of integration, and better his chances of understanding the same function performed by other (perhaps less familiar) instruments.

(1) The watt-hour meter integrates the power P flowing to the consumer with respect to time, thus obtaining a record of electrical energy consumed, ie, it is effectively finding the area under the P-t curve (see Figure 11).

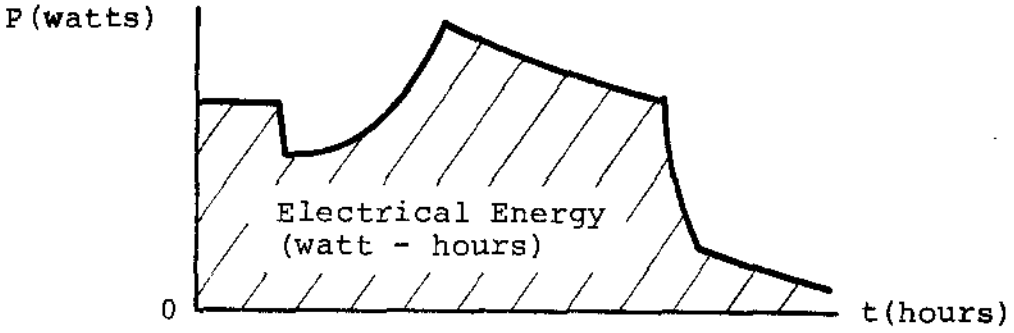


Figure 11

(2) A natural gas meter integrates the volumetric flow rate to the consumer, thus recording total volume consumed (see Figure 12).

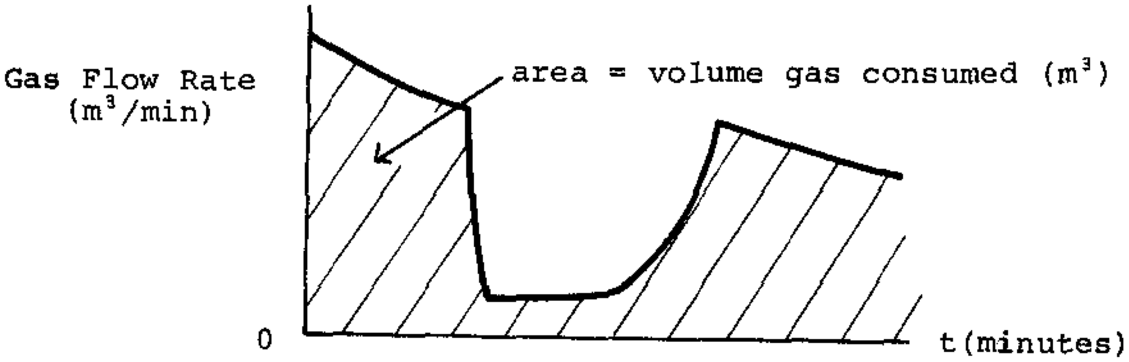


Figure 12

(3) The meter on a service station gasoline pump integrates flow rate of gasoline flowing into the consumer's gas tank, and records the total volume purchased, ie, it gives the area under the curve of Figure 13.

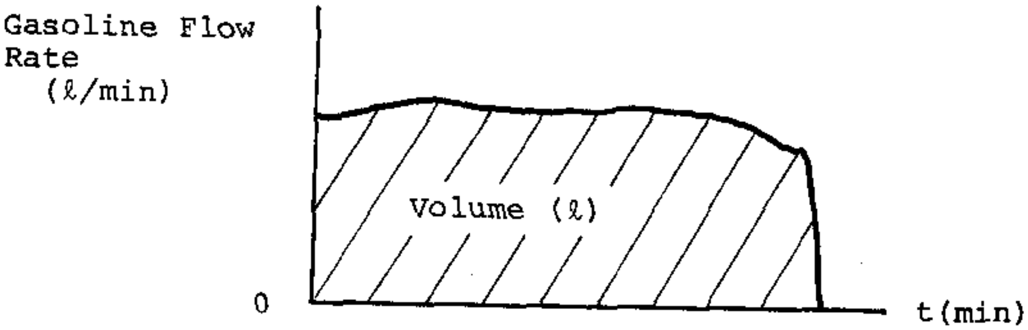


Figure 13

(4) Whereas the speedometer on a car gives the rate of change of distance with time, the odometer integrates this rate to give the total distance, ie, it gives the area under the speed-time curve (see Figure 14).

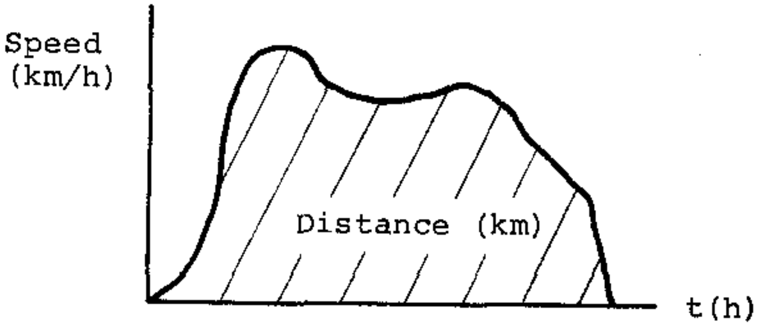


Figure 14

(5) A radiation dosimeter integrates the dose rate to record the total dose received, ie, it gives the area under the dose-rate-time graph (see Figure 15).

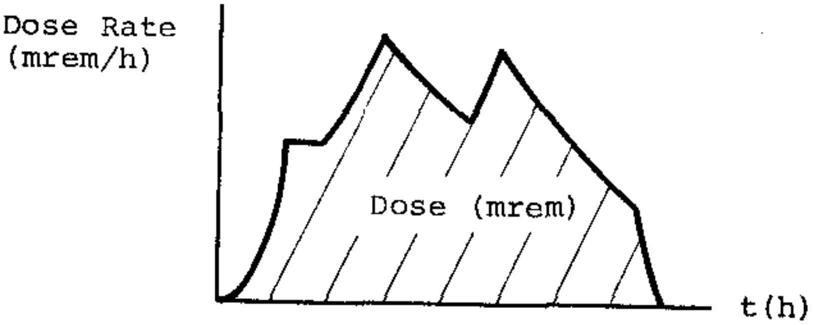
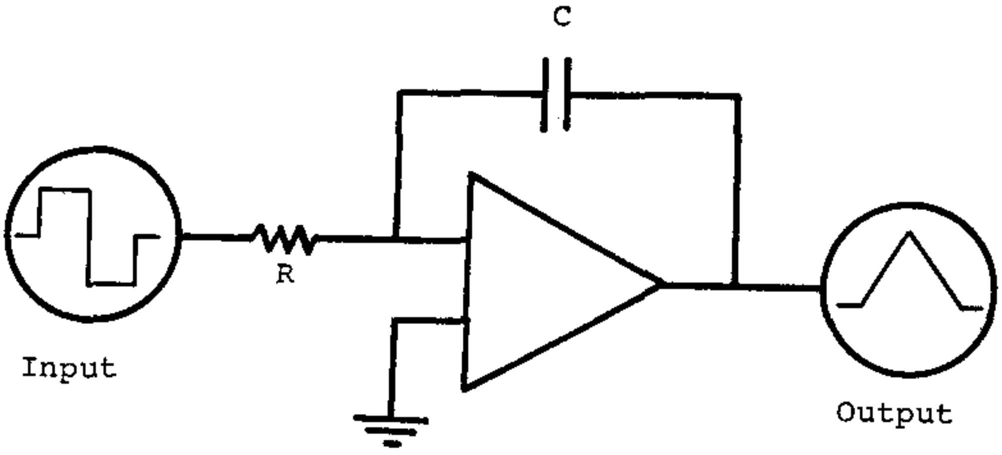


Figure 15

(6) An <u>operational amplifier</u> in the integral mode yields an output which is proportional to the integral of its input (see Figure 16).



<u>Figure 16</u>

VII Integration in Science and Technology

Just as differential calculus provides the notation for writing and techniques for deriving the rate of change of one physical quantity with respect to another, given the one as a function of the other, so integral calculus provides the notation for writing and techniques for deriving a function from its rate of change. Every differential equation listed in Lesson 221.20-5 can be rewritten as an integral,

eg, $v = \frac{ds}{dt} \Rightarrow s = \int v dt$,

$$\frac{dP}{dt} = \frac{\Delta k}{L} P \Rightarrow P = \int \frac{\Delta k}{L} P dt,$$

$$\frac{dV}{dT} = \frac{nR}{P} \Rightarrow V = \int \frac{nR}{P} dT, \text{ etc.}$$

VIII Reset (Integral) Control Mode

Recall (lesson 20-5) that the control signal, $(CS)_P$, from a proportional controller is proportional to the error, e (difference between measured value and set point of controlled parameter):

$(CS)_P = k_P e$ (+b), k_P a constant and

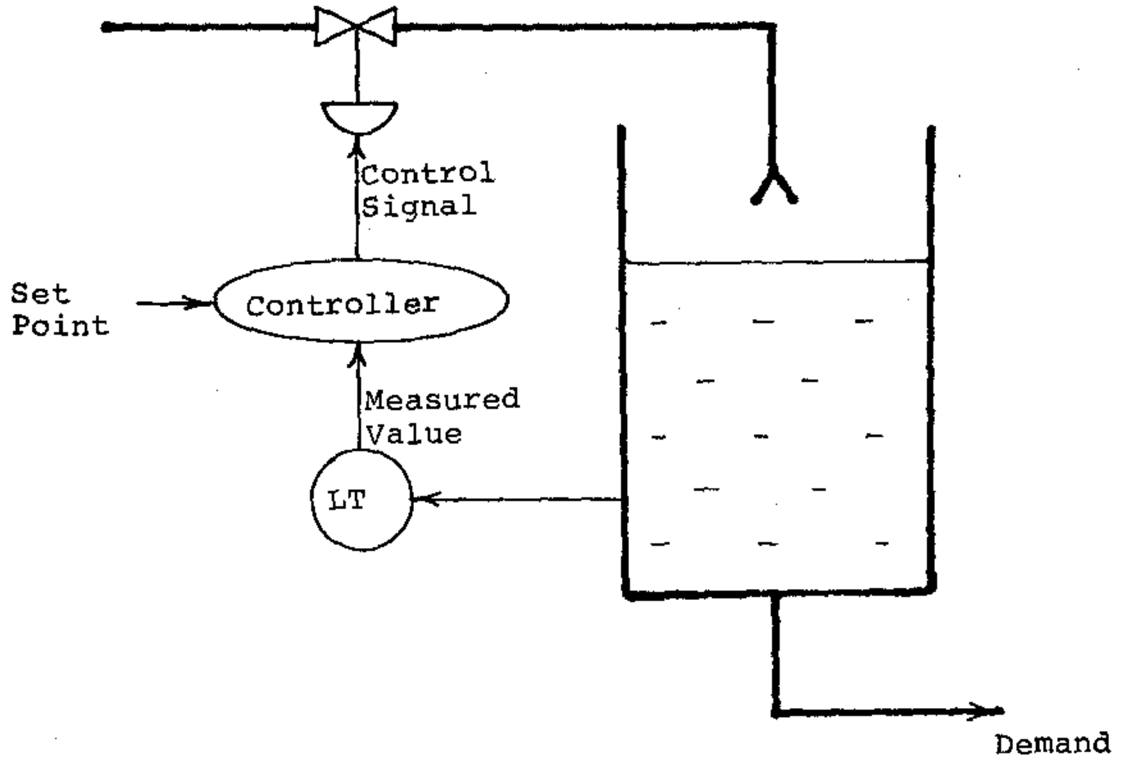
b is the equilibrium bias

One disadvantage of proportional control is *offset* the deviation from set point required merely to generate the required corrective control signal. For example, the tank level in Figure 17 would be offset from set point following any change from equilibrium demand (see lesson 20-5).

This undesirable offset can be eliminated with the use of *reset control*, for which controller output $(CS)_I$ is proportional to the integral of the error signal, e(t):

$(CS)_I = k_I \int edt$, k_I a constant

Reset control drives the error to zero, ie, returns tank level to set point, because $(CS)_I$ keeps changing and varying inflow until e = o. The required integral signal for reset control, $\int edt$, is obtained by passing the error signal through an integrating amplifier.



<u>Figure 17</u>   Tank Level Control Loop

In practice, proportional and reset control are usually used together:

$$(CS)_{PI} = k_p e + k_I \int edt \ (+b)$$

Figure 18 illustrates the concept of proportional and integral control signals for two hypothetical fluctuations in the tank level of Figure 17. (NB: assume that demand varies such that the level fluctuates as shown in spite of feedback.) In Figure 18(a) the level drops linearly (from set point) to a new value; in Figure 18(b), the level fluctuates temporarily below set point. The reader should convince himself that the value of the reset component of the control signal at any instant is proportional to the area under the error-time curve up to that instant in both Figures 18(a) and 18(b).

Figure 19 illustrates typical fluctuation in the tank level of Figure 17 for proportional-only control, and for proportional-plus-integral control, following a step increase in demand. Note that offset is eliminated with the introduction of reset control. Similarly, reset control can be used to eliminate offset in controlling reactor power, boiler level, moderator temperature, etc., in a CANDU station.

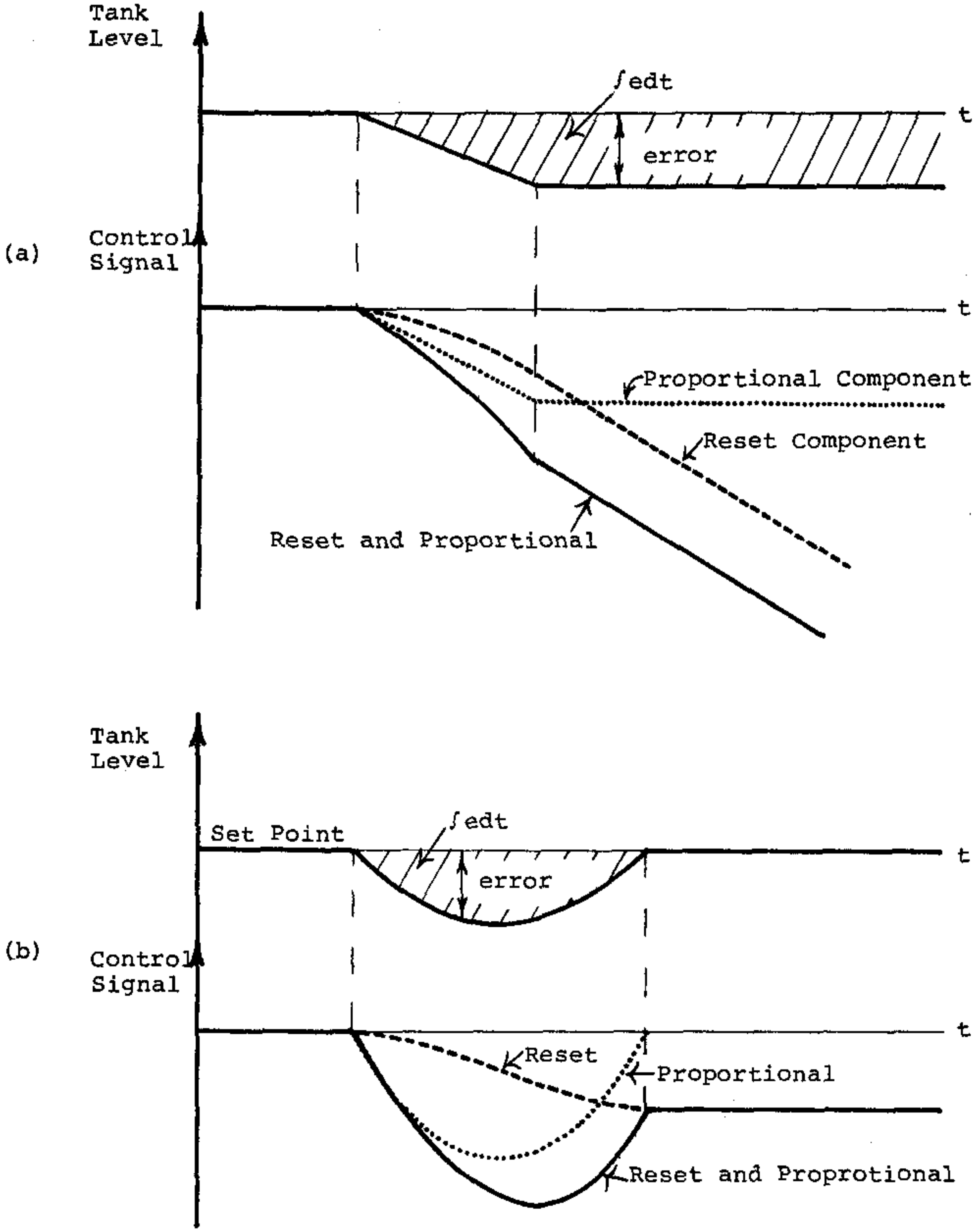


Figure 18 Proportional and Reset Control Signals for Hypothetical Tank Level Fluctuations

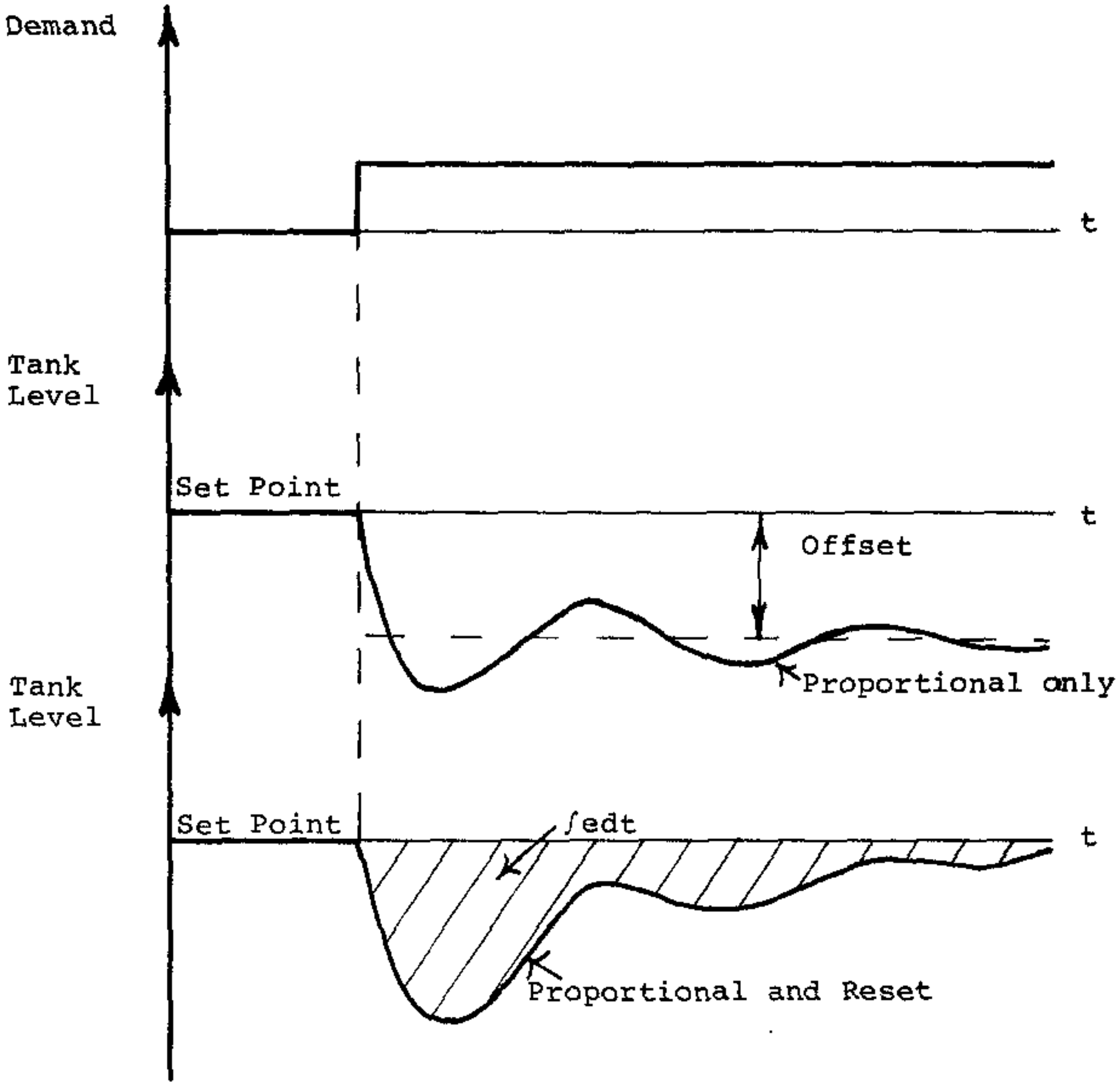


Figure 19 Tank Level Fluctuations Following Demand Increase

ASSIGNMENT

1. Find the areas bounded by the following. Make large, clear diagrams showing the representative rectangular area increment in each case.

   (a) the x-axis and the curve $y = x^3 - 6x^2 + 8x$

   (b) the parabolas $y = 6x - x^2$ and $y = x^2 - 2x$

   (c) the parabola $y^2 = 4x$ and the line $y = 2x - 4$.

2. The radiation dose rate D(t) mrem/h after t hours is given for a certain work location by the expression

$$D(t) = 600e^{-0.8t}$$

   (a) Plot a graph of D(t) vs t for $0 \leq t \leq 5$

   (b) Calculate (i) the total dose
   (ii) the average dose rate

   received during the first 4 hours.

3. The voltage V(t) after t seconds, across a capacitor of capacitance C farads, as it discharges through a resistor of resistance R ohms, is given by

$$V(t) = V_0e^{-t/RC},$$

   where V_0 is the capacitor voltage at t = 0.

   Recall that electrical power P dissipated in a resistor is given by

$$P = \frac{V^2}{R}$$

   (a) Find a mathematical expression for the average power dissipated in a resistance R which drains a capacitance C for T seconds.

   (b) Suppose a capacitor voltage is restored to 6.0 volts every 2 ms, and the capacitor discharges through a 20 Ω resistor during the 2 ms. If the capacitance is 100 μf, select an appropriate power rating for the resistor from the following:

   $\frac{1}{8}$W, $\frac{1}{4}$W, $\frac{1}{2}$W, 1W, 2W, 5W, 10W.

4. A lead ball is dropped by a parachutist while descending at 6 m/s, from a height of 1000 m above the ground. The acceleration due to gravity is 9.8 m/s^2. Neglecting air resistance, find the

   (a) velocity and displacement functions

   (b) time required for the ball to reach the ground

   (c) average velocity of the ball during its descent

   (d) average height of the ball above ground during its descent

5. Find the amount of work done in stretching a spring from an extension of 0.15 m to an extension of 0.35 m if the spring constant $k = 2.4 \times 10^2\ Nm^{-1}$.

6. The boundary of the parabolic dam illustrated below follows the curve

$$y = \frac{x^2}{100} ,$$

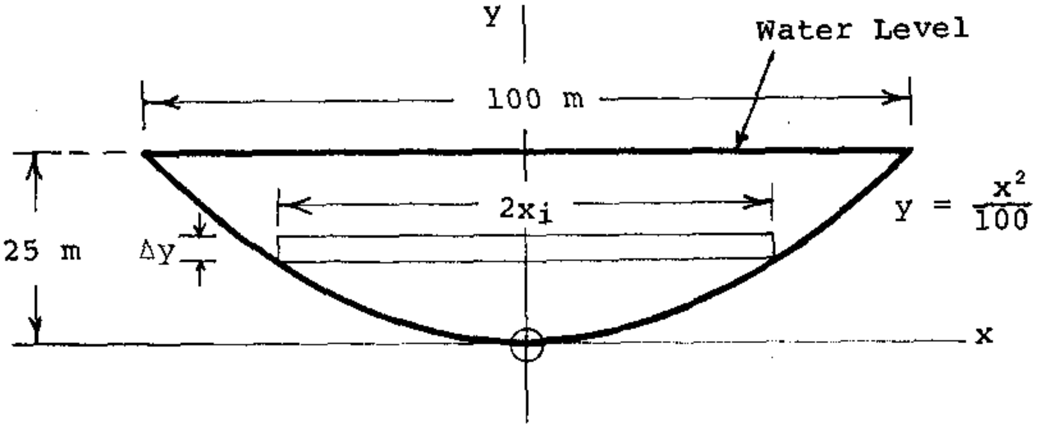
relative to axes as indicated.

Find the

(a) total force exerted against the dam

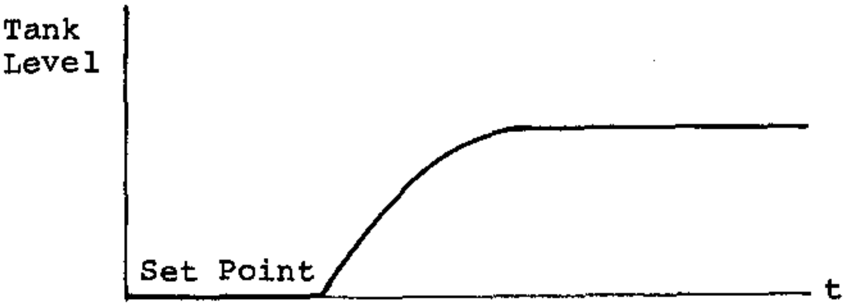
(b) average pressure exerted against the dam

(c) center of pressure exerted against the dam



7. The following diagram depicts a hypothetical variation in the tank level of Figure 17. On the same time axis, sketch the following for a proportional-reset controller:

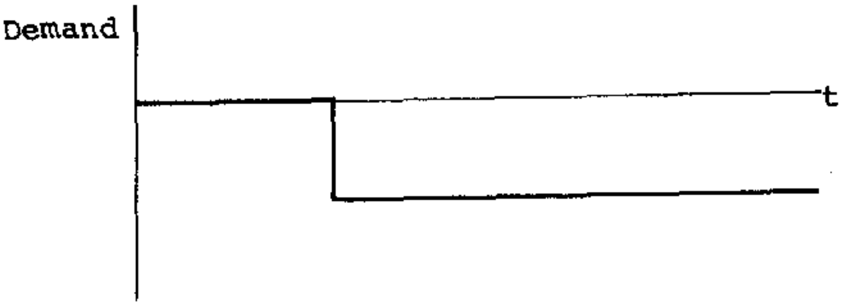
   (a) proportional component of control signal
   (b) reset component of control signal
   (c) total output signal.



8. The following diagram depicts a step decrease in demand flow from the tank of Figure 17. On the same time axis, sketch typical corresponding fluctuations in tank level in the following cases:

   (a) no level control
   (b) proportional only level control
   (c) proportional-reset level control.

   Assume level was at set point prior to demand change. Label the offset in (b).



L.C. Haacke

Mathematics - Course 221

APPENDIX 1: REVIEW EXERCISES

Review Exercise #1: Reliability

1. A system of 12 dousing valves, tested monthly, has developed 10 failures of individual valves in 8 years' operation. Calculate the unreliability of an individual valve.

2. Calculate the annual risk of a nuclear incident at a reactor, which, during 9 years' operation, developed the following faults:

   (a) 3 unsafe failures of the regulating system.

   (b) 50 complete failures of the protective system, failures of which are detected and corrected at the beginning of each shift.

3. At a certain nuclear generating station, three independent divisions of equipment protect against nuclear accidents:

   (i) process equipment with a failure frequency of 0.3 per annum,

   (ii) protective equipment with unreliability of 2×10^{-3}, and

   (iii) containment equipment with unreliability of 5×10^{-3}.

   Calculate the annual risk (frequency) of

   (a) an incident consisting of process failure combined with simultaneous failure of either protective or containment systems.

   (b) simultaneous failure of all three systems.

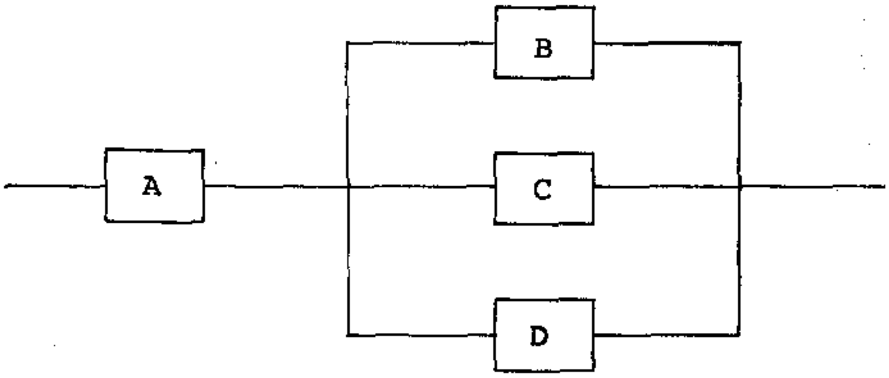
4. Monthly testing of 6 safety switches has revealed 8 failures of individual switches during 15 years' operation.

   (a) Calculate the unreliability of a switch.

   (b) How, without altering the equipment, could the unreliability in (a) be decreased by a factor of about 30?

4. (c) How often should the switches be tested if the permitted unreliability of a switch is 10^{-2}?

5.



In the above system, a system failure consists of a failure of either component A, or a failure of at least two of B, C, D.

Calculate the unreliability of the system, given component reliabilities,

$$Q_A = 0.05\text{, and}$$
$$Q_B = Q_C = Q_D = 0.1.$$

6. A pump designed for continuous operation has failed 6 times in 5 years' operation, with total down time of 124 hours. Calculate the unavailability of

(a) the pump

(b) a system of three such pumps in a 3 x 50% parallel arrangement.

Review Exercise #2 (Lessons 221.20-1, 421.40-2, 321.10-3)

1. Find the equation of the following lines in the xy-plane:

   (a) passing through (2,4) and (3,1)
   (b) passing through (-4,5) with slope $-\frac{4}{3}$
   (c) with y-intercept -5 and slope $\frac{2}{5}$
   (d) passing through (4,-1) and parallel to $2x - 5y + 8 = 0$
   (e) passing through (6,-2) and perpendicular to $5x + 3y - 2 = 0$
   (f) passing through (-5,-3) with angle of inclination 135°

2. Given $f(x) = x^2 -2x + 5$ evaluate (a) $f(0)$
   (b) $f(-5)$
   (c) $f(x + a)$

3. Plot $y = x^2 - 2x + 5$ and label the roots on the graph.

4. Find the slope and y-intercept of the following lines:

   (a) $6x - 5y + 8 = 0$
   (b) $\frac{2}{15}x + \frac{4}{3}y - 1 = 0$

5. For each line in Question 4 state the change in

   (a) x corresponding to an increase of y by 2
   (b) y corresponding to an increase of x by 5
   (c) y corresponding to a decrease in x of $\frac{1}{2}$

Review Exercise #3 (Lessons 221.20-2, 3 and 321.10-4)

1. If $s(t) = t^2 + 2t$

   (a) plot $s(t)$ vs t

   (b) find the average velocity over the first 2 seconds
   ($t = 0$ to $t = 2$)

   (c) find the formula for the instantaneous velocity, $v(t)$, at time t, using

$$v(t) = \lim_{\Delta t \to o} \frac{s(t + \Delta t) - s(t)}{\Delta t}$$

   (d) find $v(2)$. (Compare with (b))

2. Find $v(t)$ and $a(t)$, using differentiation formulas if

   (a) $s(t) = -2\sqrt{t}$ (find $v(4)$)

   (b) $s(t) = t^3 - 4t$ (find $v(5)$)

3. Differentiate with respect to x:

   (a) $5x^3 - 2x + 13$

   (b) $4\sqrt[3]{x^2}$

   (c) $-x^2 + \frac{2}{x^2}$

   (d) $\frac{x^2 - x}{\sqrt{x}}$

4. Find equation of tangent and normal to curve $y = x^2 - 2x + 3$
   at (a) $x = -1$ (b) $x = 1$

5. The decay constant λ for a radionuclide is 0.010s^{-1}

   (a) Find the activity of a 10 curie source after 2.0 minutes.

   (b) There are 3.7×10^{13} radioactive nuclei to start with. How many remain after 2 minutes?

6. Given that a radioactive source has decay constant $\lambda = 1.2 \times 10^{-5}\ \text{s}^{-1}$, find the

   (a) activity of a 10 Ci source after 20 hours

   (b) half-life of the source

   (c) time for a 10 Ci source to decay to 1 Ci

   (d) number of radioactive nuclei in a 10 Ci source

Review Exercise #4: Practice in Solving Quadratics

1. At what points is 2 the slope of the curve

$$y = \frac{x^3}{3} + x^2 - 13x + 10?$$

2. At what instants does the velocity equal zero if

$$s(t) = 2t^3 + \frac{11t^2}{2} - 10t - 5?$$

3. At what x values are the slopes of the two curves equal?

(a) $y_1 = x^3; \quad y_2 = x^2 + x$

(b) $y_1 = 3x^3 + \frac{19}{2}x^2; \; y_2 = 12 + 15x - \frac{x^3}{3}$

4. At what instants are the velocities equal for the following displacement functions?

$$s_1(t) = t^3 + \frac{5}{2}t^2 - 3t \; ; \quad s_2(t) = \frac{t^3}{3} + 3t^2 + 4$$

Review Exercise #5: (To end of lesson 221.20-4)

1. Using data tables, find $\ell n\ x$ for $x = .001, .01, .1, .5, 1$ $2, 4, 6, 8, 10$. Use your table of values to plot a graph of $\ell n\ x$ versus x.

2. Find (a) $\ell n\ e^{2\cdot 5}$ (b) $\log \frac{1}{10,000}$

   (c) $\log 10^{4/t}$ (d) $\ell n\ e^{-\lambda t}$

3. If $A(t) = A_0 e^{-\lambda t}$ prove that half-life $t_{\frac{1}{2}} = \frac{0.693}{\lambda}$

4. The decay constant for a radionuclide is $3.8 \times 10^{-5}\ s^{-1}$. Find the

   (a) half-life

   (b) number of half-lives to decay from 8 Ci to 1 Ci.

   (c) time for activity to die down to 1% original

   (d) time for activity to die from 2 Ci to 100 μCi

   (e) number of radioactive nuclei in a 2 Ci source
   (1 Ci = 3.7×10^{10} dps)

5. Given $\Delta k = .008$, $L = 0.25$, $P_0 = 10$ watts, $P(t) = P_0 e^{\frac{\Delta k}{L}t}$

   (a) find $P(t)$, $P^1(t)$ at $t = 20$ minutes

   (b) If rated power were 1000 MW, find the time to reach rated power.

6. Given $P(t) = P_0 e^{\frac{\Delta k}{L}t}$ show that rate log power is directly proportional to $\frac{\Delta k}{L}$. 'Explain the importance of a rate log meter to reactor operation.

7. Differentiate with respect to variable x or t as applicable:

   (a) $\frac{6}{\sqrt{x}}$ (b) $2x^3 - 11x^2 + 14$ (c) $-\sqrt{x} + \frac{2}{x}$

   (d) e^{2x^2} (e) $100e^{-4t}$ (f) $e^{2/t}$

Review Exercise #6: (To end of lesson 221.20-4)

1. Differentiate: (a) x^{15} (b) $\frac{-2}{\sqrt{x}}$ (c) $\frac{x^2-x}{\sqrt[3]{x}}$

   (d) $e^{\sqrt{x}}$ (e) e^{-1/x^2}

2. Find v(t) and a(t) given

   (a) $s(t) = 50t - 9.8t^2$ (b) $s(t) = e^t + t$

   (c) $s(t) = 2t^3 - 14t^2 + 5t - 8$

3. Find equation of tangent and normal to $y = x^2 - 6x - 16$ for

   (a) $x = -1$ (b) $x = 3$

4. Find equation of the line

   (a) through (-2,5) and (6,-1)
   (b) through (4,1) and having y-intercept of -3.

5. Find slope and y-intercept of $4x + 5y - 13 = 0$.

6. Find the roots of $f(x) = x^2 - 6x - 16$.

7. Given $\Delta k = .005$, $L = .12$ s, $P_0 = 50$ W, find, using $P(t) = P_0 e^{\frac{\Delta k}{L}t}$,

   (a) reactor power after 2 minutes
   (b) time for reactor to gain from 1% to 100% power.

8. Find (a) $\ell n\ e^{-t^2}$ (b) $\log 10^{\sqrt[4]{t}}$

9. Given $t_{\frac{1}{2}} = 10$ minutes, find

   (a) decay constant (b) time for source to decay from 0.5 Ci to 10 μCi

10. Find the activity of a source consisting of 2.0×10^{15} radioactive atoms if the decay constant is $6.5 \times 10^{-4}\ s^{-1}$ in

    (a) dps (b) curies

Review Exercise #7: Integration Problems (Lesson 221.30-1)

1. Find the area under the following curves between the indicated limits:

   (a) $y = 2x + 1$, $x = 0$ to $x = 5$

   (b) $y = \sqrt{x}$, $x = 4$ to $x = 9$

   (c) $y = \sqrt[3]{x}^2 + 3x + 2$, $x = 8$ to $x = 27$

2. Find (i) $v(t)$ (ii) $s(t)$ given the following:

   (a) $a(t) = -2$, $v(0) = 6$, $s(0) = 0$

   (b) $a(t) = 2\sqrt{t}$, $v(0) = 0$, $s(0) = 100$

   (c) $a(t) = -t + 3$, $v(0) = v_0$, $s(0) = 0$

3. (a) Given $\frac{dy}{dx} = 4x + 5$ find y

   (b) Given $\frac{ds}{dt} = t^{3/2}$ find $s(t)$

   (c) Given $\frac{dv}{dt} = 6t$ find $v(t)$

4. Find $v(t)$ and $s(t)$ given

   (a) $a(t) = -9.8 \text{ m/s}^2$ $v(0) = v_0$, $s(0) = 0$

   (b) $a(t) = 0 \text{ m/s}^2$ $v(0) = v_0$, $s(0) = 0$

   (c) $a(t) = \sqrt[3]{t} \text{ m/s}^2$ $v(0) = v_0$, $s(0) = 10$ m

5. Integrate:

   (a) $x^2 - 2$ (b) $2t^3 - 4t$

   (c) $\sqrt{x}$ (d) t^{-5}

   (d) $\frac{2}{x^2}$ (f) $\frac{5}{\sqrt{x}} + 14$

6. Find the displacement function $s(t)$ if the velocity function is

   (a) $v(t) = 2t - 3$

   (b) $v(t) = 3\sqrt{t} + 4$

7. Find $v(t)$ and $s(t)$, given

   (a) $a(t) = -5 \text{ m/s}^2$, $v(0) = 10$ m/s, $s(0) = 0$

   (b) $a(t) = 2t^2$, $v(0) = 0$, $s(0) = 0$

Review Exercise #8: (To end of lesson 221.30-1)

1. Given $f(x) = x^3 + x^2 - 17x + 15$. Graph $y = f(x)$, and label the roots of $f(x) = 0$.

2. Find the roots of $f^1(x) = 0$, given $f(x)$ as in Question #1. What is the significance of these roots to the curve $y = f(x)$?

3. Find the equation of the tangent and normal to $y = f(x)$ of Question #1 at $x = 1$.

4. Evaluate

   (a) $e^{\ell n\ 0.1}$ (b) $10^{\log t^2}$ (c) $\ell n\ e^{-2/t}$

   (d) $\log 10^y$ (e) $-\ell n\ e^{-0.4}$ (f) $10^{\log 100}$ (g) $e^{\ell n\ \lambda t}$

5. A source consisting of 8.6×10^{13} radioactive atoms is decaying at the rate of 7.5×10^9 dps. Find

   (a) the decay constant
   (b) the half-life
   (c) the time required for the activity to die to 1 μCi.

6. If the half-life of a radionuclide is 8.4 minutes, find

   (a) the decay constant

   (b) the time for source activity to decrease by a factor of 1000.

7. Differentiate:

   (a) $x^7 - 6x^3 + \sqrt[3]{x}$ (b) $\frac{x^3 - 1}{\sqrt{x}}$

   (c) $\sqrt[3]{x^2}$ (d) $x^{2/5} + \frac{a}{x}$

   (e) e^{-2/t^2} (f) e^{x^2-4}

   (g) $\sqrt{x}(x^3 - \frac{1}{x})$

8. If reactor power builds up from 100 W to 1000 MW is 5.0 minutes and the mean time between neutron generations is .1 seconds, find the reactivity.

9. If $N(t) = N_0 e^{-\lambda t}$, prove that $\frac{dN}{dt} = -\lambda N$.

10. Find the equation of the line

    (a) parallel to $5y - 2x + 3 = 0$ and passing through the origin

    (b) perpendicular to $5y - 2x + 3 = 0$, and having the same y-intercept

    (c) having an angle of inclination of 45° and the same x-intercept as $5y - 2x + 3 = 0$.

11. (a) Given $f^1(x) = -2x + 0.4$, find $f(x)$ if $f(0) = -7$.

    (b) Given acceleration $a(t) = \frac{2}{\sqrt{t}}$, $v(0) = 1$, $s(0) = 4$, find $v(t)$, $s(t)$.

    (c) The R/C of $g(x)$ with respect to x is $-\frac{a}{x^2} - 10$. Find $g(x)$.

    (d) y increases 3 times as fast as x. If $y = -5$ when $x = 0$, find y as a function of x.

12. Find the area under the curve

    (a) $y = x^3$ from $x = 1.5$ to $x = 5$

    (b) $y = e^x$ from $x = 0$ to $x = 3$

    (c) $y = \sqrt[3]{x}$ from $x = 1$ to 8

Review Exercise #9: (To end of Lesson 221.30-2)

1. Plot $y = 1.5^x$, $-5 \le x \le 10$.

2. Plot A vs t where $A(t) = A_0e^{-\lambda t}$, $A_0 = 100$ Ci, and $\lambda = 0.01\ s^{-1}$

   (a) using semi-log graph paper for $10 \le t \le 1000$ s.

   (b) using linear graph paper for $0 \le t \le 1000$ s.

   (Note the advantages/disadvantages of logarithmic graph paper.)

3. The force F to extend a spring varies directly as the extension x in meters.

   ie, $F = kx$, where k is called the spring constant

   (a) Prove that the work done in stretching the spring x meters equals $\frac{1}{2}kx^2$.

   (b) How much work is done in stretching a spring by 0.25 m if its spring constant is 1.2×10^4 N/m?

4. The force of gravity, F_g, on a satellite of mass M_s varies inversely as the square of its distance x from the earth's centre, ie,

$$F_g = \frac{GM_eM_s}{x^2}$$

   where G is the universal gravitation constant, and M_e is the earth's mass.

   (a) Prove that the work done by a rocket to lift a satellite d meters above the earth's surface (neglecting friction) is

$$W = \frac{GM_eM_sd}{R_e(R_e + d)}$$

   where R_e is the earth's radius.

   (b) How much energy must the rocket provide to free the satellite from earth's gravity altogether?

5. Translate the following rate-of-change statements to differential equations:

   (a) The torque T on a wheel equals the product of the wheel's moment of inertia I times the time rate of change of the wheel's angular velocity, ω.

(b) The voltage V across an inductor equals the product of the inductance L times the rate of change of the current i with respect to time.

6. For a poison-injection shut down of a reactor, gadolineum (Gd) is injected into the moderator at a concentration of 20 mg Gd/kg D_2O. This Gd must be removed before the reactor can be restarted. Moderator cleanup is achieved by cycling the moderator through ion exchange columns, and the concentration C(t) of Gd remaining in the moderator after t hours is given by the expression.

$$C(t) = 20\ e^{-0.35t} \text{ mg Gd/kg } D_2O$$

Find:

(a) the time required to reduce the concentration to 0.8 mg Gd/kg D_2O, at which the reactor can be restarted.

(b) an expression for the <u>rate</u> at which Gd is removed, as a function of time.

(c) the total reduction in Gd concentration during the first 10 hours.

(d) the average rate of Gd concentration during the first 10 hours

(e) the instantaneous rate of Gd removal at half-time (t = 5 h). Why is this rate different from that of (d)?

(f) the average concentration during the first 10 hours.

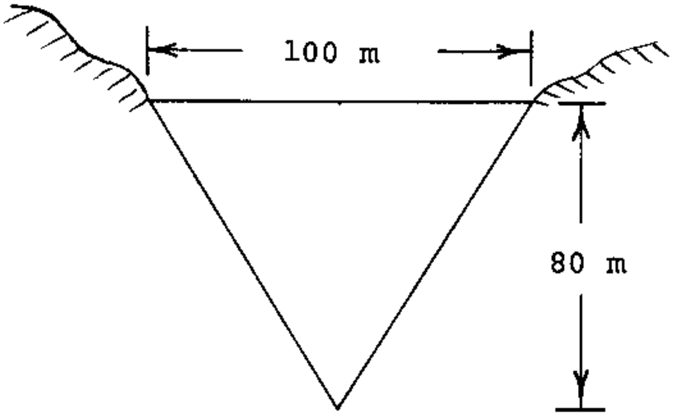
7. For the V-shaped hydraulic dam illustrated below, assuming water level coincides with top of dam, find:

(a) the total force exerted by the water against the face of the dam

(b) the average pressure of the water against the dam face

(c) the center of pressure exerted by the water.

7.



L.C. Haacke

Mathematics - Course 221

APPENDIX 2: QUESTIONS BEARING DIRECTLY ON COURSE 221 CONTENT FROM RECENT AECB NUCLEAR GENERAL EXAMINATIONS

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1. Question #6, October, 1978

Neutron power (linear N), logarithm of neutron power (log N), rate of change of neutron power (linear rate) and rate of change of logarithm of neutron power (rate log) are four types of neutron power signals used for CANDU reactor control.

(a) Which of these signals are used for reactor regulation:

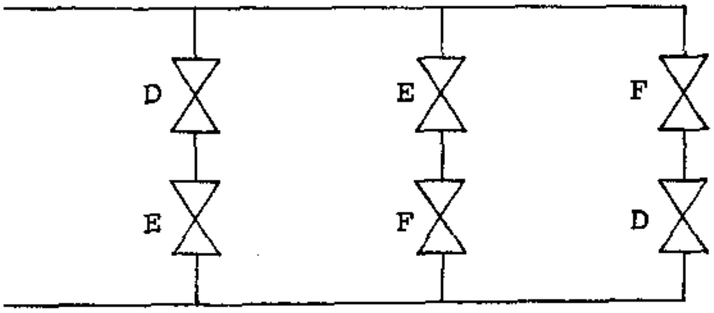
- i) at low power?
- ii) at high power?

In each case, explain why the signals you selected are required in order to provide adequate reactor regulation.

(b) Of the four signals listed previously, linear N, linear rate and rate log are used for CANDU reactor protection. Which one(s) of these signals is (are) more likely to respond to dangerous conditions and to activate the protective system(s) when the reactor is:

- i) at low power? Explain.
- ii) at high power? Explain.

2. Question #7, October, 1978



The above diagram is a schematic representation of the typical dump valve arrangement for a reactor with moderator dump. The opening and closing of valves D, E and F are controlled by channels D, E and F respectively. During five years of reactor operation, the electronics of channels D, E and F were tested three times a week and four unsafe failures of individual channels were found.

(Cont'd)

2. Question #7, October, 1978 (Cont'd)

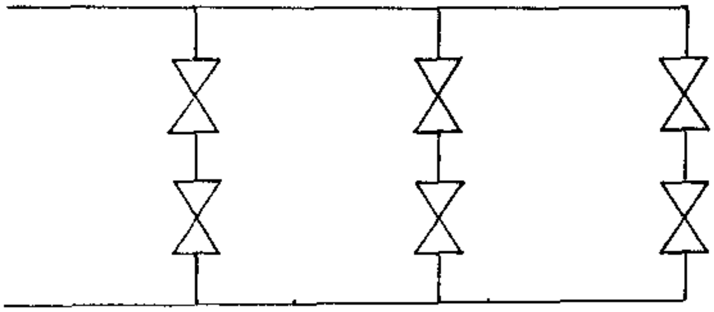
   (a) Calculate the unreliability of a dump channel.

   (b) If the correct operation of one dump line is sufficient to achieve an efficient dump,

      i) list the various combinations of channel failures that will cause dump system to fail;

      ii) calculate the unreliability of the dump system due to dump channel failures.

3. Question #5, June, 1978



The above diagram is a schematic representation of the typical dump valve arrangement for a reactor with moderator dump. In five years of operation of this reactor, six failures (to open) of individual valves were found. The dump valves are tested twice a week.

   (a) Calculate the unreliability of:

      i) a dump valve,

      ii) a dump line.

   (b) Suppose that you have a dump system consisting of a single dump line. Give and briefly discuss one advantage and one disadvantage of using two dump valves in that line instead of one.

4. Question #7, June, 1977

Give and explain four advantages that result from using triplicated instruments arranged in two-out-of-three tripping circuit instead of a simple circuit actuated by a single instrument.

5. Question #11, June, 1977

In a control room we usually find meters which indicate neutron power on a linear and logarithmic scale.

(a) Draw two simple instrument dials, one showing a linear scale with values of neutron power from 0 to 100% and the other a logarithmic scale with values of neutron power from 10^{-5}% to 100%.

(b) Given that 200 megawatts is 100% power, mark and identify the positions on each scale which correspond to the following power levels:

i) 50% power

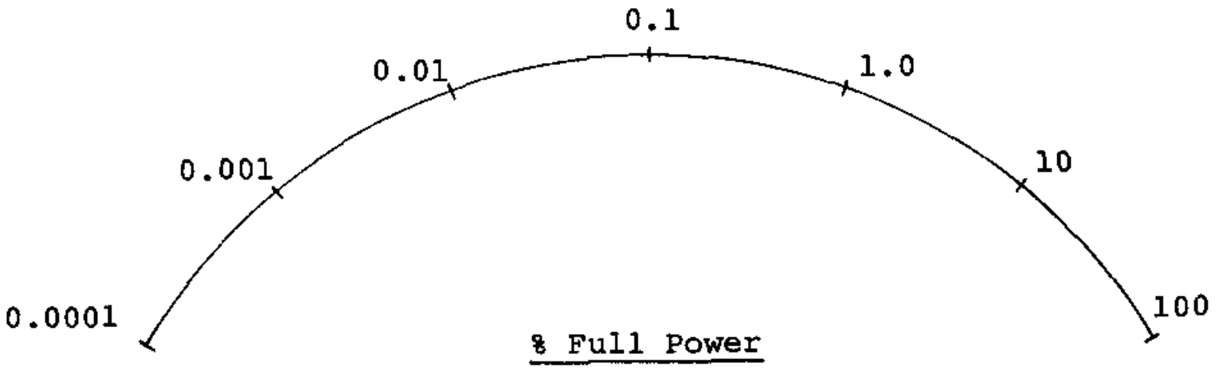
ii) 400 watts

iii) 10 kilowatts

6. Question #8, October, 1976

Give and explain three reasons why reactor safety systems should be tested routinely.

7. Question #1, June, 1975



The above diagram represents the face of an instrument which indicates neutron power on a logarithmic scale. Given that 100 megawatts is 100% power, mark the positions on the scale which correspond to the following power levels:

(a) 50% power

(b) 0.005 megawatts

(c) 500 watts

NOTE: Mark the positions on the above diagram.

* * * * *

NOTE: Recent AECB nuclear general examinations have contained many more questions impinging on course 221 content - questions regarding nuclear decay rates, rate of fission product buildup, variation of reactor power with time following reactivity changes, etc. No doubt such topics can be discussed _qualitatively_ without any use of calculus (and _qualitative discussion_ is all the AECB requires), but a _quantitative_ treatment of such topics certainly does _involve the use_ of calculus. Thus a background knowledge of calculus concepts can hardly fail to quicken one's insight into such topics, and to aid one's ability to discuss them definitively, even at the descriptive level.

L.C. Haacke

Mathematics - Course 321

APPENDIX 3: SOLVING QUADRATIC EQUATIONS

I Introduction

A *quadratic function* is a function of the form

$$f(x) = ax^2 + bx + c,$$

where a, b, c are real constants.

A *quadratic equation* is an equation of the form

$$ax^2 + bx + c = 0$$

II Roots of a Quadratic Equation

The *roots of a quadratic equation* are the x-values which *satisfy* the equation. Therefore, to *solve* a quadratic equation is to find its roots.

The roots of the quadratic equation,

$$ax^2 + bx + c = 0$$

are given by the formula,

$$\boxed{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$

The quantity b^2-4ac is called the *discriminant*, designated "D". The value of D determines the number and nature of roots of the quadratic, as summarized in the following table:

Value of $D = b^2-4ac$	Number and Nature of Roots	Figure
>0	2 real roots	1,2
0	1 real root	3,4
<0	2 complex roots	5,6

III Graphical Solution of Quadratic Equations

The graph of a quadratic function $y = ax^2 + bx + c$ is a *parabola*, and the roots of the corresponding quadratic equation, $ax^2 + bx + c = 0$, are the x-coordinates of the parabola's x-intercepts, since $y = 0$ at these x-values.

The "ax^2" term dominates the value of $ax^2 + bx + c$ for large values of x, and therefore, the sign of "a" governs whether the parabola opens upward or downward, as summarized in the following table:

Value of "a"	Parabola Opens	As in Figures
> 0	upward	1, 3, 5
< 0	downward	2, 4, 6

The graphical significance of "a" and "D" values is summarized in Figures 1 to 6.

IV Examples

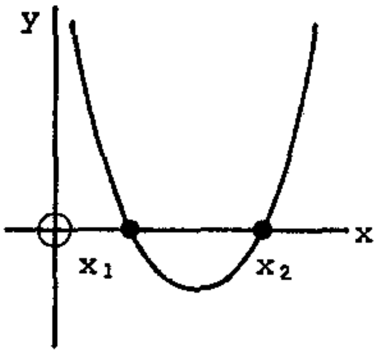
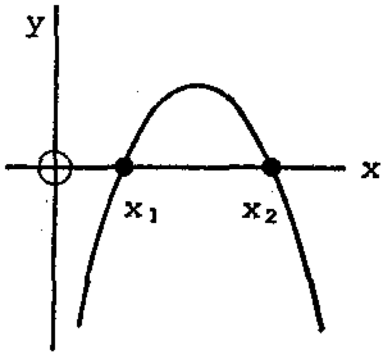
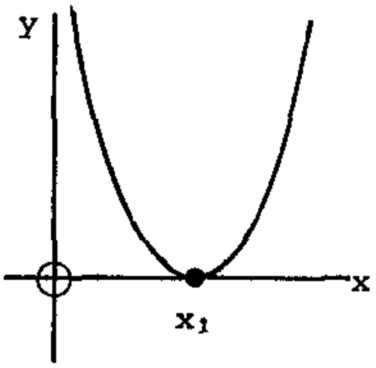
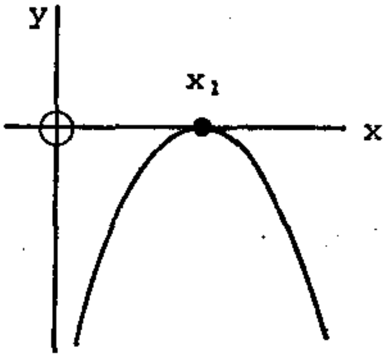
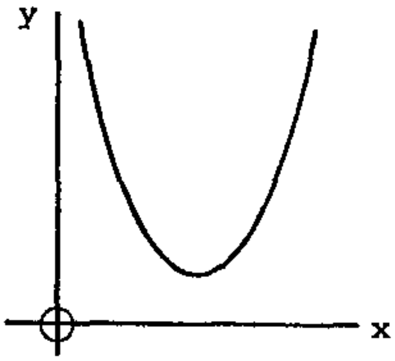
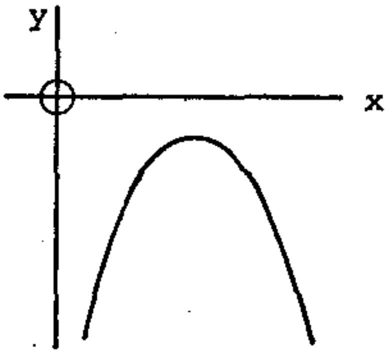
(1) $y = 2x^2 - 3x - 4$
(2) $y = -x^2 + 6x - 9$
(3) $y = x^2 + x + 2$

(a) Graph each of the above quadratic functions.

(b) From the graphs, find the roots of the corresponding quadratic equations.

(c) Calculate the roots by formula.

Summary of Graphical Significance of Values of "a" and "D" = $b^2 - 4ac$"

<table>
<tr><th>a > 0</th><th>a < 0</th></tr>
<tr><td>Figure 1</td><td>Figure 2</td></tr>
<tr><td colspan="2">2 real roots x_1, x_2
(D > 0)</td></tr>
<tr><td>
</td><td>
</td></tr>
<tr><td>Figure 3</td><td>Figure 4</td></tr>
<tr><td colspan="2">1 real root, x_1
(D = 0)</td></tr>
<tr><td>
</td><td>
</td></tr>
<tr><td>Figure 5</td><td>Figure 6</td></tr>
<tr><td colspan="2">2 complex roots
(D < 0)</td></tr>
<tr><td>
</td><td>
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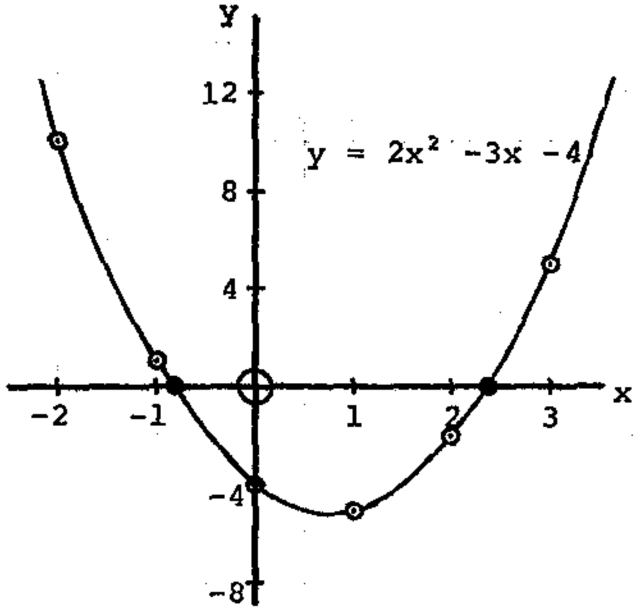
Solutions to Examples:

1. (a)

x	-2	-1	0	1	2	3
y	10	1	-4	-5	-2	5



(b) From graph, roots of $2x^2 - 3x - 4$ are approximately -0.8, 2.3.

(c) a = 2, b = -3, c = -4

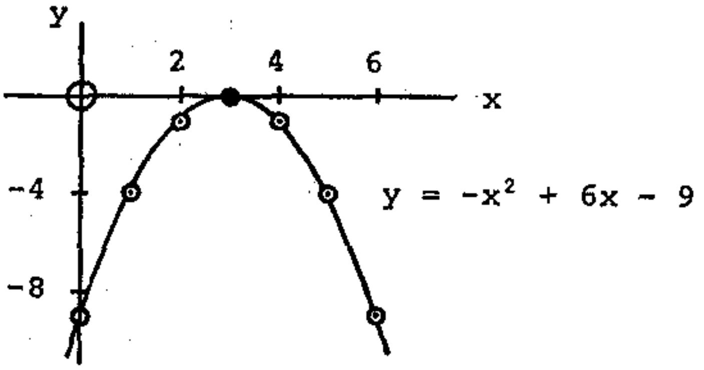
$$\therefore \text{ roots are } x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(2)(-4)}}{2(2)}$$

$$= \frac{3 \pm \sqrt{41}}{4}$$

= 2.35 or -0.85 (to 2 D.P.)

2. (a)

x	0	1	2	3	4	5	6
y	-9	-4	-1	0	-1	-4	-9



(b) The graph indicates that $-x^2 + 6x - 9 = 0$ has only one root, namely 3.

(c) a = -1, b = 6, c = -9

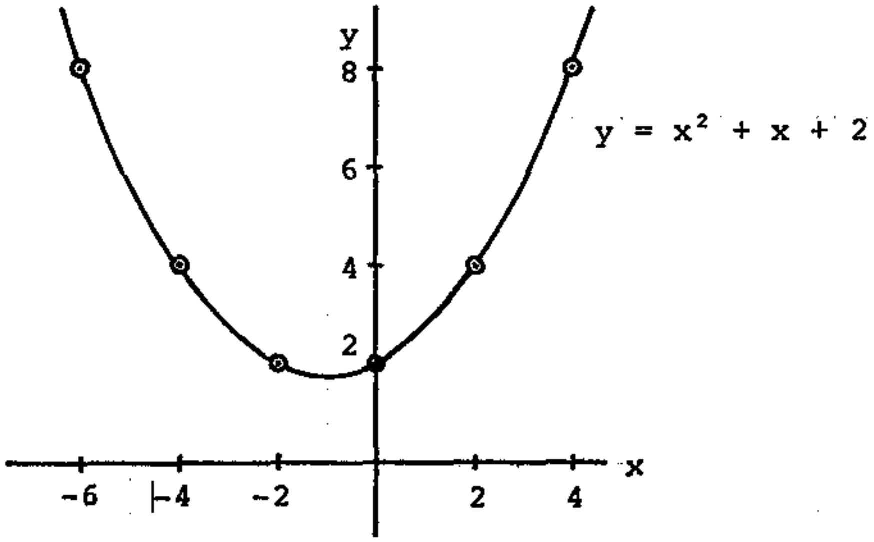
$$\therefore \text{ roots are } x = \frac{-6 \pm \sqrt{6^2 - 4(-1)(-9)}}{2(-1)}$$

$$= \frac{-6 \pm \sqrt{0}}{-2}$$

$$= 3$$

3. (a)

x	-3	-2	-1	0	1	2
y	8	4	2	2	4	8



(b) Since the parabola does not intersect the x-axis, $x^2 + x + 2 = 0$ has no real roots.

(c) $a = 1, b = 1, c = 2$

$\therefore$ roots are $x = \dfrac{-1 \pm \sqrt{1^2 - 4(1)(2)}}{2(1)}$

$$= \frac{-1 \pm \sqrt{-7}}{2}$$

Since $b^2 - 4ac < 0$, roots are complex.

V Other Methods of Solving Quadratics

Alternative methods for solving quadratic equations include

(1) factoring, and
(2) completing the square.

Trainees are not required to be able to use these alternative methods, but may use such methods at their discretion.

ASSIGNMENT

Solve each of the following quadratic equations

(a) by graphing the associated quadratic function

(b) by using the formula.

1. $x^2 - 3 = 0$
2. $x^2 - 2x - 8 = 0$
3. $4x^2 - 15x + 9 = 0$
4. $9x^2 - 24x + 16 = 0$
5. $-x^2 + 5x + 2 = 0$
6. $2x^2 + x + 3 = 0$

L. Haacke

Mathematics - Course 221

APPENDIX 4: ANSWERS TO ASSIGNMENTS AND REVIEW EXERCISES

421.10-4 Assignment

1. (a) 10^7 (b) 10 (c) 10^6 (d) 10^{12}

(e) 10^{-8} (f) 10^{-9} (g) 10^{-4} (h) 10^{-4}

2. (a) 10^{-2} (b) 10^{-8} (c) 10^2 (d) 1

(e) -10^{-7} (f) 10^{13} (g) 10^{10} (h) 10^{-13}

(i) 10^9 (j) 10^{-11} (k) 10^{13} (l) -10

3. (a) 100 (b) 0.001 (c) 100,000 (d) 0.000001

(e) 1,000,000 (f) 0.0001

4. (a) 1.65×10^5 (b) 6.93×10^{-3} (c) 3.75×10

(d) 2.5×10^{-2} (e) 2.934×10^3 (f) 1.01×10^{-3}

(g) 1×10^4 (h) 2.0×10^{-4} (i) -2.49×10^2

(j) 9.7×10^{-1} (k) 1.76×10^{-1} (l) 2.7

(m) 9.57×10^4 (n) 1.75×10^{-14} (o) 2.4×10^7

(p) 3.2×10^{12}

5. (a) 2.4×10^3 (b) 5.6×10^{12} (c) -1.1×10^{14}

(d) -4.3×10^3 (e) 4.5×10^{12}

6. (a) 9.3×10^5 (b) 6.9×10^5 (c) 3.4×10^7

(d) 5.5×10^{-12} (e) 5.5×10^2

7. (a) 2.3×10^{-1} (b) 9.4×10^{-5}

421.20-1 Assignment

1\. (a) $-10\frac{5}{6}$ (b) 9 (c) 5

2\. (a) a^{10} (b) $\frac{3}{32}a^{10}$ (c) b^{12} (c) 3^7

(e) m^6 (f) a^{19} (g) $\frac{1}{a}$ (h) b^7

(i) a^{14} (j) $-27a^6$ (k) $\frac{1}{32}x^{20}$ (l) $a^{5/6}$

(m) $a^{1/6}$ (n) $9x^2y$ (o) 1 (p) $x \quad y^{-4}$

3\. (a) 27 (b) $\frac{1}{32}$ (c) $\frac{1}{2}$ (d) $6\frac{3}{4}$

(e) $-\frac{1}{27}$ (f) 6 (g) -2 (h) -32

(i) $1\frac{7}{9}$

4\. (a) 1 (b) $4x^8$ (c) $3^5a^2b^3$ (d) $\frac{18y^2z^2}{x^3}$

5\. 9.1×10^{-28}g

6\. 6.2×10^{18} fissions/s

7\. (a) $11a$ (b) $6x^2 + 6x + 5xy + 5$

(c) $25x + 20y$ (d) $8a + 5b + 12c$

(e) $5k - 2j$ (f) $10a$

(g) $4xy^2 + y$ (h) $4x^2y^3 + 1 - x$

(i) $-5x - y + 4z$

8\. (a) $-2y$ (b) $\frac{1}{3y}$ (c) $5b$ (d) -2

(e) x (f) $-3abc$ (g) $\frac{2y}{x}$ (h) $\frac{1}{2}x^2$

(i) $-24x^2y^4p$ (j) $22p^2qs^2t$

9. (a) $x^2 - 4xy - 32y^2$ (b) $15x^2 + 22x + 8$

(c) $12a^2 - 23ac + 10c^2$ (d) $x^4 - y^2$

(e) 0 (f) $2x + 5y$

(g) $a^2 - 1$ (h) $4x + 5$

(i) $3x - 8$ (j) $-(x^2 + x - 3)$

10. (a) $32m^2nx$ (b) $-36ab^2c^2d$

(c) $-18my + 15ty$ (d) $20h - 30k$

(e) $2x - 23y$ (f) $x^2 + xy$

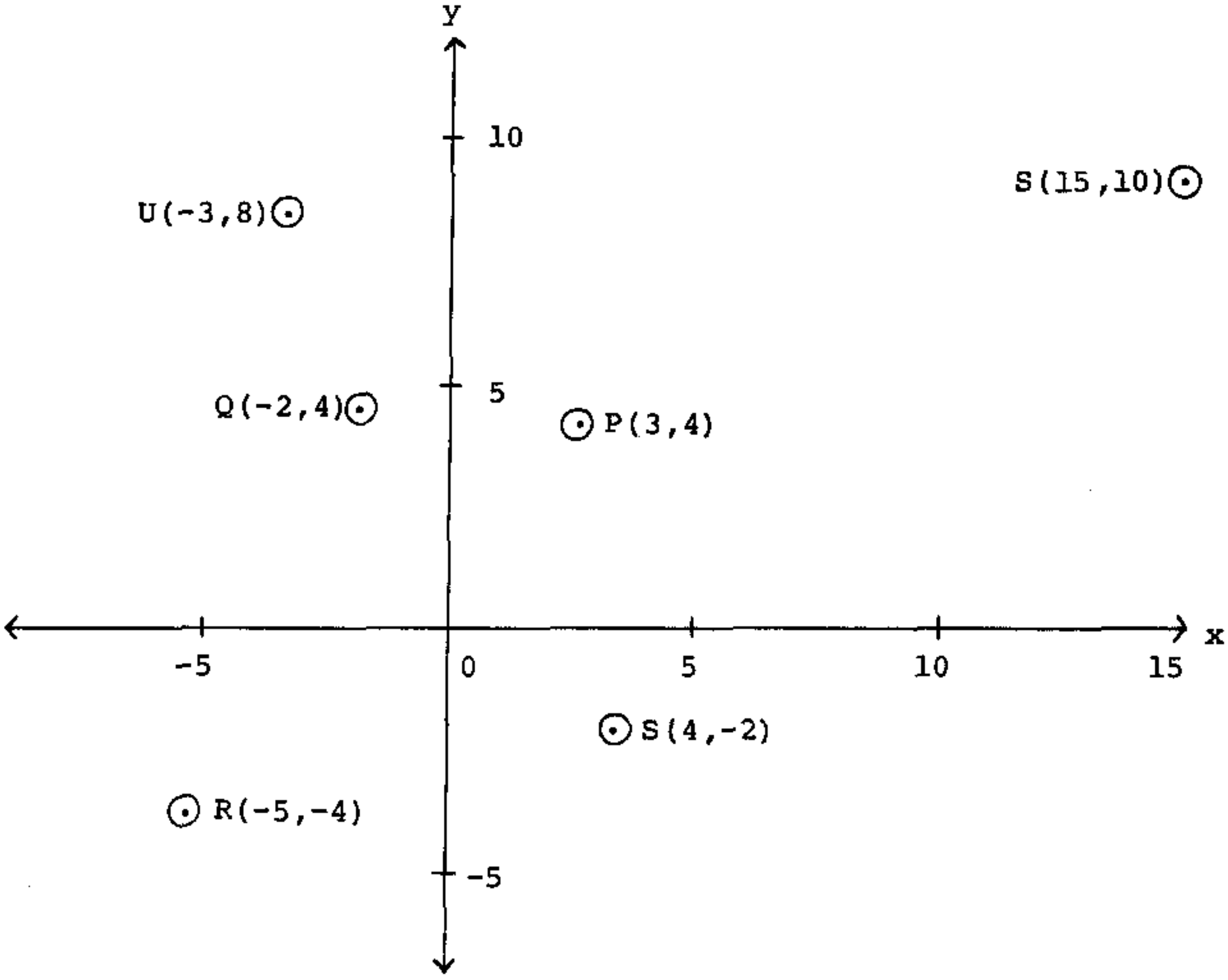
(g) $3c + k$ (h) $-b + 7c$

(i) $-3a - 6x$ (j) $2ab$

(k) $2xy$

421.40-1 Assignment

1.



3.	(a)	12 A	(b)	2.4 Ω
4.	(a)	893 kPa	(b)	12.5 cm

421.40-2 Assignment

1. (a) $C(r) = 2\pi r$

   (b) $d(t,v) = vt$

   (c) $A(r,h) = 2\pi r^2 + 2\pi rh$

2. $f(6) = 9$; $f(0) = -3$; $f(-2) = -7$

3. $H(0) = 9$; $H(1) = 0$; $H(a) = 0$

4. $d(p) = \frac{\sqrt{2p}}{4}$

5. (a) ±2

   (b) no real roots since curve does not cross x-axis

   (c) -2.9, -0.2, 3.1

   (d) 3, -2

   (e) -1.53, -0.33, 1.87

321.10-3 Assignment

1. (a) (i) $R = 1.67\Omega$ (ii) $I = 1.2$ A.
   (b) (i) $R = 2.17\Omega$ (ii) $I = 4.6 \times 10^{-2}$ A.

2. (a) 7.4×10^{-5} s^{-1} (b) 2.6 h

3. (a) $\frac{1}{2}$ (b) 0.6 (c) 9 (d) 4
   (e) -0.2 (f) -10 (g) 10 (h) 8

4. 0.84 kΩ

5. 1.1×10^{2} hr.

6. (a) -3.1 (b) -9.8 (c) 2.6 (d) 1.7×10^{9}
   (e) 1.1 (f) -11 (g) $x = 3^{2.7} = 19$
   (h) $x = 7^{4.8} = 1.1 \times 10^{4}$ (i) $x = 9^{2.1} = 1.0 \times 10^{2}$
   (j) $x = 4^{5.3} = 1.6 \times 10^{3}$ (k) $x = 17^{16.8} = 4.7 \times 10^{20}$
   (l) $x = 6^{7.5} = 6.9 \times 10^{5}$

7. (a) $\frac{3}{4} \log X + \frac{1}{6} \log Y + \frac{5}{4} \log Z$

   (b) $\log X + \frac{15}{2} \log Z - 9 \log Y$

   (c) $\frac{7}{6} \log X - \frac{5}{12} \log Y + \frac{1}{2} \log Z$

321.10-3 Assignment

8. (a) 1339.43 (b) 3.02×10^{-2} (c) 25.31
   (d) 50.12 (e) 5.01×10^{-5} (f) 0.76

9. (a) 9.1×10^{2} (b) 0.46 (c) 1.2 (d) 3.2×10^{3}

321.10-4 Assignment

4. 3.3 minutes

5. 1.55×10^{3} MPC

6. 19.5 s

221.10-1 Assignment

1. 3×10^{-3}

2. 2×10^{-4}

3. (a) 1.4×10^{-3} (b) 1.4×10^{-3}

4. 1.7×10^{-2}

5. 4×10^{-4}

6. 3×10^{-3}

7. every 4 weeks

221.20-1 Assignment

1.

	Slope	Angle of Inclination	Equation
(a)	$\frac{4}{3}$	53.1°	$4x - 3y = 0$
(b)	$\frac{-2}{3}$	146.3°	$2x + 3y - 6 = 0$
(c)	-1	135°	$x + y = 0$
(d)	0	0°	$y - 2 = 0$
(e)	∞	90°	$x + 3 = 0$

2. Slope PQ = slope QR = ½; Q is common to segments PQ, QR
$\therefore$ P,Q, R are collinear.

3.

	Slope	x - intercept	y - intercept
(a)	-1	4	4
(b)	$\frac{5}{4}$	4	-5
(c)	0	none	$\frac{6}{5}$
(d)	undefined	$\frac{-4}{15}$	none

221.20-2 Assignment

1\.

	Tangent slope at (x,f(x))	Tangent slope at x = 2
(a)	$10x - 2$	18
(b)	$-\frac{2}{x^2}$	$-\frac{1}{2}$

2\. (a) $8x^3 - 12x^2$ (b) $\frac{2x}{a^2} - \frac{2a^2}{x^3}$

(c) $-\frac{3}{2} x^{-3/2}$ or $-\frac{3}{2\sqrt{x^3}}$

3\. (a) $2x - 6$ (b) $10x^4 - 3x^2$

(c) $2ax + b$ (c) $\frac{2}{3} x^{-1/3} - x^{-2/3}$

4\. (a) 11 (b) -3

(c) $-\frac{1}{3}$ and 1

221.20-3 Assignment

1. (a) -4 (b) -2 (c) $\frac{1}{2}$

2. (-1,-3)

3.

	Tangent Equation	Normal Equation
(a)	$x + y - 2 = 0$	$x - y = 0$
(b)	$y = 4$	$x = 1$

4.

	v(t)	v(2)	a(t)	a(2)
(a)	$16t - 3$	29	16	16
(b)	$-8t - 4t^3$	-48	$-8 - 12t^2$	-56
(c)	$20t - 80/t^2$	20	$20 + 160/t^3$	40

5. 25m

6. Roots of $f'(x) = 0$ are $x = 2.73, -0.73$; $y = f(x)$ has a local maximum at $x = -0.73$, and a local minimum at $x = 2.73$.

221.20-4 Assignment

1. (a) $2x\ e^{x^2-4}$ (b) e^{-x}

   (c) $-x^{-2}e^{-x^{-1}}$ (d) $x^{-3/2}e^{-x^{-1/2}}$ or $x^{-3/2}e^{-1/\sqrt{x}}$

   (e) $\frac{5}{2}(5t^{3/2} - t^{-1/2})et^{5/2} - t^{1/2}$

   (f) $x^{-4}e^{-1/x^3}$

2. (a) (i) $v(t) = e^t - 3t^2$ (ii) $a(t) = e^t - 6t$

   (iii) $v(2) = e^2 - 12$

   (b) (i) $v(t) = -e^{-t} + 2$ (ii) $a(t) = e^{-t}$

   (iii) $v(2) = 2 - e^{-2}$

3.

t	0	0.25	0.5	1	1.5	2.0	2.5	2.75	3
s	1	1.27	1.52	1.72	1.10	-0.61	-3.44	-5.15	-6.91
v	1	1.10	0.90	-0.28	-2.27	-4.61	-6.57	-7.04	-6.91
a	1	-0.22	-1.35	-3.28	-4.52	-4.61	-2.82	-0.86	2.09

4. $9.3 \times 10^{-12}\ s^{-1}$

5. (a) 0.45 Ci
   (b) (i) 5.2×10^{13} (ii) 1.1×10^{12}
   (c) (i) 0.34 Ci (ii) 7.1 mCi
   (d) 48 minutes
   (e) 4.4 hours

6. see text

7. By definition of $t_{½}$, $0.5A_0 = A_0e^{-\lambda t_{½}}$

$$\therefore \quad \ln 0.5 = -\lambda t_{½}$$

$$\therefore \quad t_{½} = -\frac{\ln 0.5}{-\lambda}$$

But $\ln 0.5 = \ln ½ = \ln 2^{-1} = -\ln 2$

$$\therefore \quad t_{½} = \frac{\ln 2}{\lambda}$$

9.

t	0	1	2	3
N(t)	10^{20}	7.94×10^{19}	6.31×10^{19}	5.01×10^{19}

t	5	10	15	18
N(t)	3.16×10^{19}	9.99×10^{18}	3.16×10^{18}	1.58×10^{18}

10.

t(s)	P(W)	P(%F.P.)	P^1(%F.P./s)
0	100	1×10^{-4}	5×10^{-6}
20	2.7×10^2	2.7×10^{-4}	1.4×10^{-5}
40	7.4×10^2	7.4×10^{-4}	3.7×10^{-5}
60	2.0×10^3	2.0×10^{-3}	1.0×10^{-4}
80	5.5×10^3	5.5×10^{-3}	2.7×10^{-4}
100	1.5×10^4	1.5×10^{-2}	7.4×10^{-4}
120	4.0×10^4	4.0×10^{-2}	2.0×10^{-3}
140	1.1×10^5	1.1×10^{-1}	5.5×10^{-3}
160	3.0×10^5	3.0×10^{-1}	1.5×10^{-2}
180	8.1×10^5	8.1×10^{-1}	4.1×10^{-2}
200	2.2×10^6	2.2	0.11
220	6.0×10^6	6.0	0.30
240	1.6×10^7	16	0.81
260	4.4×10^7	44	2.2
280	1.2×10^8	1.2×10^2	6.0
300	3.3×10^8	3.3×10^2	16.3

(c) The rate at which the needle moves across the linear scale (linear rate) increases exponentially with time--the needle moves imperceptibly for about 3 minutes, then moves ever more rapidly across the scale, covering the final half of the range in just 14 seconds. This is just as one would expect from the mathematical expression

$$P'(t) = \frac{\Delta k}{L} P(t) = \frac{\Delta k}{L} P_0 e^{\frac{\Delta k}{L} t}$$

The rate at which the needle moves across the log scale (rate log P) is constant, ie, the needle advances by the same amount each 20 seconds. This is in agreement with the mathematical expression

$$\frac{d}{dt} \log P(t) = \frac{\Delta k}{L} \log e \quad \text{(cf. question 12)}$$

(d) linear scale more sensitive at high power; log scale at low power

(e) (i) signal output proportional to log P is more sensitive to changes in P at low power ($\lesssim$10% full power, say)

(ii) signal output proportional to P is more sensitive to changes in P at high power

11. see text

221.20-5 Assignment

1. Force F equals mass m times R/C velocity v wrt time t.

   Angular velocity ω equals R/C angular displacement θ wrt time t.

   Angular acceleration α equals R/C angular veloctiy ω wrt time t.

   Torque τ equals moment of intertia I times R/C angular velocity ω wrt time t.

   Force F equals rate of decrease of potential energy E_p wrt distance r from force center.

   Power P equals time rate of energy output (or conversion).

   Electric current i equals rate of flow of charge q.

   Capacitor current i_c equals capacitance C times R/C capacitor voltage V_c wrt time t.

   Inductor voltage V_L equals inductance L times R/C inductor current i_L wrt time t.

   Rate of decrease of number N of radioactive nuclei remaining at time t equals decay constant λ times N.

   Rate of decrease in radioactive source activity A equals decay constant λ times A.

   Rate of decrease in number N of nuclear projectiles equals macroscopic cross section Σ times penetration depth x.

   'Linear rate' power P equals reactivity Δk times power P divided by mean neutron lifetime L.

   'Rate log power' equals reactivity Δk divided by mean neutron lifetime L.

   Specific heat capacity C of a substance equals R/C quantity Q of heat stored in substance wrt temperature T of substance, divided by the mass m of the substance.

   Heat Q flow rate (in a fluid) equals specific heat capacity C times temperature difference ΔT across system times mass m flow rate.

Heat H flow rate equals minus thermal conductivity k times cross sectional area A times R/C (rate of increase) temperature T wrt depth x in conducting medium. (Minus sign indicates directions of heat flow and increasing temperature are opposite.)

R/C of gas volume V wrt temperature T equals number n of moles of gas times gas constant R divided by gas pressure P.

Voltage V induced across a coil equals number N of turns in coil times rate of decrease in magnetic flux Φ linking the coils.

2. See Table 1, lesson 221.20-5.

3. 6.0×10^{-3}N

4. (a) $V_2 = -M\frac{di_1}{dt}$

   (b) (i) $V_2(t) = 6t^2 - 12t$ (ii) $V_2(2) = 0$

221.30-1 Assignment

1. see text

2. see text

3. (a) $-\frac{3}{2}x^2 + C$ (b) $e^t + \frac{1}{4}t^4 + C$

   (c) $\frac{2}{3}x^3 + \frac{3}{2}x^2 - 5x + C$ (d) $2e^{\sqrt{x}} + C$

   (e) $x^4 - \frac{3}{4}x^{4/3} + C$ (f) $-e^{-t^2} + \frac{2}{5}t^{5/2} + C$

4.

	v(t)	s(t)
(a)	0	0
(b)	$2t + 10$	$t^2 + 10t + 14$
(c)	$t^2 + v_0$	$\frac{1}{3}t^3 + v_0t$
(d)	$11 - e^{-t}$	$e^{-t} + 11t - 11$

5. $s(t) = v_0t - 4.9t^2$

6. (a) 99 (b) $37\frac{1}{3}$

   (c) $72\frac{1}{3}$ (d) $\frac{1}{2}(1 - e^{-4})$

7. (a) $79\frac{7}{15}$ (b) 77.5

   (c) $23 - e^2$

221.30-2 Assignment

1. (a) 8 square units (b) $21\frac{1}{3}$ square units

   (c) 9 square units

2. (b) (i) 7.2×10^2 mrem (ii) 1.8×10^2 mrem/h

3. (a) $\frac{V_0{}^2C}{2T}(1 - e^{-2T/RC})$ (b) 1 W

4. (a) $v(t) = -6 - 9.8t$; $s(t) = 1000 - 6t - 4.9t^2$

   (b) 14 seconds

   (c) -73 m/s

   (d) 6.5×10^2 m

5. 12 J

6. (a) 1.6×10^8 N (b) 9.8×10^4 Pa

   (c) at (0,11)

221.40-1 Review Exercises

Review Exercise #1

1. 4×10^{-3} 2. 8×10^{-4}

3. (a) 2×10^{-3} (b) 3×10^{-6}

4. (a) 3.7×10^{-3} (b) test daily

   (c) test every 12 weeks

5. 0.08

6. (a) 3×10^{-3} (b) 2×10^{-5}

Review Exercise #2

1. (a) $3x + y - 10 = 0$ (b) $4x + 3y + 1 = 0$

   (c) $2x - 5y - 25 = 0$ (d) $2x - 5y - 13 = 0$

   (e) $3x - 5y - 28 = 0$ (f) $x + y + 8 = 0$

2. (a) 5 (b) 40

   (c) $x^2 + 2ax - 2x + a^2 - 2a + 5$

3. no real roots

4. (a) slope = $\frac{6}{5}$; y-intercept = $\frac{8}{5}$

   (b) slope = $-\frac{1}{10}$; y-intercept = $\frac{3}{4}$

5. (a) (i) $\frac{5}{3}$ (ii) -20

   (b) (i) 6 (ii) $-\frac{1}{2}$

   (c) (i) $-\frac{3}{5}$ (ii) $\frac{1}{20}$

Review Exercise #3

1. (b) 4 m/s (c) $2t + 2$ m/s (d) 6 m/s

2. (a) $v(t) = -1/\sqrt{t}$; $v(4) = -\frac{1}{2}$; $a(t) = \frac{1}{2}t^{-3/2}$

   (b) $v(t) = 3t^2 - 4$; $v(5) = 71$; $a(t) = 6t$

3. (a) $15x^2 - 2$ (b) $\dfrac{8}{3\sqrt[3]{x}}$

   (c) $-2x - 4x^{-3}$ (d) $\dfrac{3x - 1}{2\sqrt{x}}$

4. (a) tangent $4x + y - 2 = 0$

   normal $x - 4y + 25 = 0$

   (b) tangent $y = 2$

   normal $x = 1$

5. (a) 3.0 Ci (b) 1.1×10^{13}

6. (a) 4.2 Ci (b) 16 h (c) 53 h (d) 3.1×10^{16}

Review Exercise #4

1. $(-5, 58\frac{1}{3})$ and $(3,-11)$
2. $t = -\frac{5}{2}$ and $t = \frac{2}{3}$
3. (a) $x = 1$ and $x = -\frac{1}{3}$

   (b) $x = -\frac{5}{2}$ and $\frac{3}{5}$

4. $t = \frac{3}{2}$ and $t = -1$

Review Exercise #5

2. (a) 2.5 (b) -4 (c) $4/t$ (d) $-\lambda t$

4. (a) 5.1 h (b) 3 (c) 34 h (d) 72 h

(e) 1.9×10^{15}

5. (a) $P = 4.8 \times 10^{17}$ W ; $P^1 = 1.5 \times 10^{16}$ W/S

(b) 9.6 minutes

7. (a) $-3/\sqrt{x}^3$ (b) $6x^2 - 22x$

(c) $-\frac{1}{2\sqrt{x}} - \frac{2}{x^2}$ (d) $4xe^{2x^2}$

(e) $-400\ e^{-4t}$ (f) $-\frac{2}{t^2}\ e^{2/t}$

Review Exercise #6

1. (a) $15x^4$ (b) $x^{-3/2}$

(c) $\frac{5}{3}\ x^{2/3} - \frac{2}{3}\ x^{-1/3}$ (d) $\frac{e^{\sqrt{x}}}{2\sqrt{x}}$

(e) $\frac{2}{x^3}\ e^{-1/x^2}$

2. (a) $v = 50 - 19.6t$; $a = -19.6$

(b) $v = e^t + 1$; $a = e^t$

(c) $v = 6t^2 - 28t + 5$; $a = 12t - 28$

3. (a) tangent: $8x + y + 17 = 0$

normal: $x - 8y - 71 = 0$

(b) tangent: $y + 25 = 0$

normal: $x = 3$

4. (a) $3x + 4y - 14 = 0$

(b) $x - y - 3 = 0$

5. slope = $-\frac{4}{5}$; y-intercept = $\frac{13}{5}$

6. 8, -2

7. (a) 7.4 kW (b) 1.1×10^2 s or 1.8 minutes

8. (a) $-t^2$ (b) $\sqrt[4]{t}$

9. (a) 1.2×10^{-3} s^{-1} (b) 2.6 h

10. (a) 1.3×10^{12} dps (b) 35 Ci

Review Exercise #7

1. (a) 30 (b) $12\frac{2}{3}$ (c) 1162.1

2. (a) $v = -2t + 6$ $s = -t^2 + 6t$

(b) $v = \frac{4}{3}t^{3/2}$ $s = \frac{8}{15}t^{5/2} + 100$

(c) $v = -t^2/2 + 3t + v_0$

$s = -t^3/6 + 3t^2/2 + v_0t$

3. (a) $y = 2x^2 + 5x + C$

(b) $s(t) = \frac{2}{5}t^{5/2} + C$ (c) $v(t) = 3t^2 + C$

4. (a) $v(t) = -9.8t + v_0$; $s(t) = -4.9t^2 + v_0t$

(b) $v(t) = v_0$; $s(t) = v_0t$

(c) $v(t) = \frac{3}{4}t^{4/3} + v_0$; $s(t) = \frac{9}{28}t^{7/3} + v_0t + 10$

5. (a) $\frac{x^3}{3} - 2x + C$ (b) $\frac{x^4}{2} - 2x^2 + C$

(c) $\frac{2}{3} x^{3/2} + C$ (d) $-\frac{x^4}{4} + C$

(e) $-\frac{2}{x} + C$ (f) $10\sqrt{x} + 14x + C$

6. (a) $t^2 - 3t + C$ (b) $2t^{3/2} + 4t + C$

7. (a) $v(t) = -5t + 10$; $s(t) = -\frac{5}{2}t^2 + 10t$

(b) $v(t) = \frac{2}{3}t^3$; $s(t) = t^4/6$

Review Exercise #8

1. Roots are $x = -5, 1, 3$

2. Roots are $x = -2.73, 2.07$.
Significance: $f' = 0 \Rightarrow$ tangent slope $= 0$

$\therefore$ curve $y = f(x)$ has maximum at $x = -2.73$ and minimum at $x = 2.07$.

3. tangent: $12x + y - 12 = 0$

normal: $x - 12y - 1 = 0$

4. (a) 0.1 (b) t^2 (c) $-\frac{2}{t}$ (d) y

(e) 0.4 (f) 100 (g) λt

5. (a) $\lambda = 8.7 \times 10^{-5}\ s^{-1}$ (b) 2.2 h

(c) 39 h

6. (a) $1.4 \times 10^{-3}\ s^{-1}$ (b) 1.4 h

7. (a) $7x^6 - 18x^2 + \frac{1}{3} x^{-2/3}$ (b) $\frac{5}{2} x^{3/2} + \frac{1}{2} x^{-3/2}$

(c) $\frac{2}{3} x^{-1/3}$ (d) $\frac{2}{5} x^{-3/5} - \frac{a}{x^2}$

(e) $\frac{4}{t^3} e^{-2/t^2}$ (f) $2x\, e^{x^2} - 4$

(g) $\frac{7}{2} x^{5/2} + \frac{1}{2} x^{-3/2}$

8. 0.0054

10. (a) $2x - 5y = 0$ (b) $25x + 10y + 6 = 0$

(c) $2x - 2y - 3 = 0$

11. (a) $-x^2 + 0.4x - 7$

(b) $v(t) = 4\sqrt{t} + 1$; $s(t) = \frac{8}{3} t^{3/2} + t + 4$

(c) $\frac{a}{x} - 10x + C$

(d) $y = 3x - 5$

12. (a) 155 (correct to 3 S.F.)

(b) $e^3 - 1$

(c) $11\frac{1}{4}$

<u>Review Exercise #9</u>

3. (b) 3.8×10^2 J

4. (b) GM_eM_s/R_e

5. (a) $T = I \frac{d\omega}{dt}$ (b) $V = L \frac{di}{dt}$

6. (a) 9.2 h (b) $-7.0\ e^{-0.35t}$

   (c) 19 mgGd/kg$D_2$0 (d) 1.9 mgGd/kg$D_2$0 per hour

   (e) 1.2 mgGd/kg$D_2$0, different from (d) since C'(t) is exponential, not linear in time.

   (f) 1.7 mgGd/kgD_2O.

7. (a) 5.3×10^8 N

   (b) 0.13 MPa

   (c) 39 m vertically above "V" bottom

221.40-3 Assignment

1. ±1.73
2. -2, 4
3. $\frac{3}{4}$, 3
4. $\frac{4}{3}$
5. -0.37, 5.37
6. $\frac{-1 \pm \sqrt{-23}}{4}$ (no real roots)

L.C. Haacke