CHE 3804 NUCLEAR ENGINEERING

SECTION 3

NEUTRON DIFFUSION

FICK'S LAW

Diffusion of neutrons in a medium is an important aspect of nuclear engineering since neutrons interact with materials in different ways and travel through the medium in a complex manner. If considered to be moving in a completely random manner the neutrons diffuse in the same manner as the molecular diffusion of one gas within another. This diffusion may thus be described in the same way as chemical diffusion. Fick's Law describes such diffusion and states that if the concentration of a solute is greater in one region than another then the solute will diffuse from the region of higher concentration to a region of lower concentration. In the same way neutrons diffuse down a concentration gradient in a medium. In one direction (x-direction) this may be described mathematically as follows where J_x is the neutron flow in the x-direction:

$$J_x = -D\, d\phi/dx$$

The neutron flux ϕ is of course the number of neutrons n multiplied by their velocity v

$$\phi = n\, v$$

The flow J_x is thus proportional to the negative gradient of the number of neutrons n (or flux ϕ). The factor D is known as the *diffusion coefficient*.

Note that the motion of the neutrons is completely random and that they diffuse in all directions as they are scattered by collisions with nuclei. At any location x along the x-direction neutrons will be scattered and will diffuse both up and down the concentration gradient. Since there is a greater number of neutrons to the left (up the concentration gradient) than to the right (down the concentration gradient) more will be scattered and more will diffuse away from the location to the left. Thus more will diffuse down the concentration gradient than up the concentration gradient.

In three dimensions the same situation applies and the flow of neutrons is given by:

$$J = - D \text{ grad } \phi$$

$$J = - D \nabla \phi$$

Here ∇ is the gradient operator. The diffusion coefficient D is given approximately as one third of the transport mean free path λ_{tr}

$$D = \lambda_{tr}/3$$

The transport mean free path in turn is the inverse of the macroscopic transport cross-section Σ_{tr}. *This in turn is related to the macroscopic transport cross-section* Σ_s *and the* average value of the cosine of the angle at which the neutrons are scattered μ. The value of this can be related to the atomic mass number A of the medium.

$$\lambda_{tr} = 1/\Sigma_{tr}$$

$$\Sigma_{tr} = \Sigma_s (1 - \mu)$$

$$\mu = 2/3A$$

From the above it can be seen that, knowing the mass number A of the medium and its macroscopic scattering cross section Σ_s the diffusion coefficient D may be calculated.

Fick's Law is however not an exact relation when applied to the diffusion of neutrons since some are absorbed within the medium. It however can be used when considering most reactor moderating systems which by design are weak neutron absorbers. In particular circumstances Fick's Law cannot be applied as it would give false results. Such circumstances are:

* In a medium that absorbs neutrons strongly
* Close to a neutron source.
* Near the surface of the medium.
* In a medium having a beam of neutrons

Nevertheless Fick's Law may be used in many preliminary calculations regarding the flow of neutrons within a given volume.

NEUTRON BALANCE

In an arbitrary volume of a certain medium containing neutrons the change in the number of neutrons will be equal to the production minus the loss due to absorption or leakage.

$$\partial n/\partial t = \text{Production - Absorption - Leakage}$$

This may be expressed in terms of rate of neutron production s and rates of absorption and leakage

$$\partial n/\partial t = S - \Sigma_a \phi - \text{div } J$$

The leakage in three dimensions div J can be expressed in terms of the Laplacian so that

$$\partial n/\partial t = S - \Sigma_a \phi + D \nabla^2 \phi$$

Under steady state conditions the change in neutron density is zero and the *steady state diffusion equation* is obtained.

$$D \nabla^2 \phi - \Sigma_a \phi + S = 0$$

This equation may be rewritten in the following form:

$$\nabla^2 \phi - (\Sigma_a / D)\, \phi = -S / D$$

Another parameter the *diffusion length* L may now be defined as follows:

$$L^2 = D / \Sigma_a$$

The steady state diffusion equation may thus be written as

$$\nabla^2 \phi - \phi / L^2 = -s / D$$

BOUNDARY CONDITIONS

Fick's Law is not valid near the surface of a medium. This is because, once neutrons have passed through the surface of a medium, none are scattered back. Everywhere else in the medium neutrons diffuse both up and down the concentration gradient. Just inside the surface there are none being scattered up the gradient since no scattering which would deflect some back into the medium occurs just outside the medium. The result is a marked decrease in the number of neutrons near the surface. The neutron flux profile is steeper in this region and cannot be predicted by Fick's Law. In order to be able to use Fick's Law compensation is made for this dip in flux and the flux is assumed to decrease to zero at a certain distance beyond the surface. By adding this *extrapolation distance* d to the dimensions of the medium the neutron flux within the medium can be accurately predicted by Fick's Law while the deviation near the surface is neglected. This gives an overall more accurate analysis since the surface effects occur over a relatively short distance.

The extrapolation distance d may be calculated from the transport mean free path λ_{tr} or the diffusion coefficient as follows:

$$d = 0.71\ \lambda_{tr}$$

$$\lambda_{tr} = 3\ D$$

$$d = 2.13\ D$$

Since the extrapolation distance is usually very small compared with the overall reactor dimensions it is often neglected.

DIFFUSION EQUATION

The general diffusion equation in the form given above may be applied to different geometries and solved subject to geometrical constraints. By substituting known parameters into these solutions the neutron flux at any point in a medium may be obtained. One dimensional solutions may be obtained relatively simply but three dimensional solutions are more complex. Two typical solutions will be given namely that of an infinite planar source and that of a point source.

Infinite Planar Source

The steady state diffusion equation is:

$$\nabla^2 \phi - \phi / L^2 = - S / D$$

Since the flux varies in the x-direction only and since there is no source within the diffusion medium itself this reduces to:

$$d^2\phi/dx^2 - \phi / L = 0$$

The solution to this differential equation is:

$$\phi = A\, e^{-x/L}$$

Fick's Law gives the flow of neutrons as:

$$J = -D\, d\phi/dx$$

Differentiating the flux and substituting for $d\phi/dx$ gives:

$$J = A\, (D / L)\, e^{-x/L}$$

But J is also the neutron flow leaving the source divided between the positive and negative x-directions:

$$J = S / 2$$

Equating the values for J gives:

$$S / 2 = A\, (D / L) e^{-x/L}$$

$$A = S\, L / 2\, D$$

The solution of the equation for neutron flux is therefore:

$$\phi = (S\, L / 2\, D)\, e^{-x/L}$$

Point Source

The steady state diffusion equation is:

$$\nabla^2\phi - \phi / L^2 = -S / D$$

Since the flux varies in the r-direction only this may be converted to spherical co-ordinates.

Also since there is no source in the diffusing medium the source term is zero

$$(1 / r^2)\, d/dr\, (r^2\, d\phi/dr) - \phi / L^2 = 0$$

The solution to this differential equation is:

$$\phi = (A / r)\, e^{-r/L}$$

Fick's Law gives the flow of neutrons as:

$$J = -D\, d\phi/dr$$

Differentiating the flux and substituting for $d\phi/dr$ gives:

$$J = D\, A\, \{1 / r\, L) + (1 / r^2)\}\, e^{-r/L}$$

But J is also the neutron flow leaving the source:

$$J = S / 4\pi$$

Equating the values for J gives:

$$S / 4\pi = D\, A\, \{(1 / r\, L) + (1 / r^2)\}\, e^{-r/L}$$

In the limit as r approaches zero this gives:

$$A = S / 4\pi\, D$$

The solution of the equation for neutron flux is therefore:

$$\phi = (S / 4\pi\, D\, r)\, e^{-r/L}$$

Other configurations may be solved in a similar manner but the mathematics is more complex.

DIFFUSION LENGTH

The diffusion length L has already been defined and it has been shown that this can be easily calculated for any medium. In diffusing through a medium a neutron follows a zig-zag path and migrates a certain distance from its origin before being absorbed. The average direct distance that is migrates r is given by:

$$r^2 = 6\,L^2$$

Note that the total distance travelled is much greater due to its zig-zag path. This path is made up of the number of collisions multiplied by the transport mean free path λ_{tr}.

GROUP DIFFUSION METHOD

Neutrons when slowing down pass through a range of energies. Within each energy range there is a certain probability of scattering or absorption within the medium. The general diffusion equation may still be used but the diffusion characteristics change as the neutron energy varies. The parameters for thermal neutrons are therefore not applicable to neutrons at the higher energies occurring immediately after fission.

In order to solve the general diffusion equation accurately the neutron energy range has to be divided up into a number of groups with each group having its particular diffusion characteristics. A series of diffusion equations each with different parameters is therefore obtained and applied to the neutrons within the appropriate energy range.

A complication is that not all neutrons are created with the same energy. The source term must therefore be divided up so that the appropriate number of neutrons enter each energy group. A further complication is that neutrons do not all lose the same amount of energy in a collision so they do not pass progressively from a particular energy group to the next lower energy group. They may miss one or more groups should they lose a large amount of energy in a single scattering collision. The net result is that any particular group will receive neutrons from any higher energy group including the source and lose neutrons to any lower energy group. All such transfers have a certain probability specified by the group transfer cross section into the group $\Sigma_{n\to i}$ and out of the group $\Sigma_{i\to m}$ where i is the group being considered and n and m are respectively the number of higher and lower energy groups.

This type of analysis is quite complicated and suited to computer analysis.

TWO GROUP CALCULATION

Reasonably accurate results may be obtained by considering just two groups of neutrons namely the higher energy group arising from fission and the thermal energy group consisting of those having lost most of their energy. Two diffusion equations are therefore required for fast and thermal neutrons respectively. If fast neutrons enter the medium from an external source and are converted to thermal neutrons within the medium these equation are respectively

$$D_{fast}\,\nabla^2\phi_{fast} - \Sigma_{fast}\,\phi_{fast} = 0$$

$$D_{thermal}\,\nabla^2\phi_{thermal} - \Sigma_{thermal}\,\phi_{thermal} = \Sigma_{fast}\,\phi_{fast}$$

Note that the transfer cross section term of the fast neutron diffusion equation becomes the source term of the thermal neutron diffusion equation.

FICK'S LAW

RATE OF DIFFUSION FLOW IS EQUAL TO NEGATIVE GRADIENT OF CONCENTRATION

FLOW OF NEUTRONS IN X-DIRECTION

$$J_x = -D \frac{d\phi}{dx}$$

D = DIFFUSION COEFFICIENT - UNITS ?

IN THREE DIMENSIONS

$$J = -D \text{ grad } \phi$$

$$J = -D \nabla \phi$$

∇ = GRADIENT OPERATOR

DIFFUSION COEFFICIENT

$$D \approx \frac{\lambda_{tr}}{3}$$

λ_{tr} = TRANSPORT MEAN FREE PATH

$$\lambda_{tr} = \frac{1}{\Sigma_{tr}}$$

Σ_{tr} = MACROSCOPIC TRANSPORT CROSS SECTION

$$\Sigma_{tr} = \frac{1}{\Sigma_s(1-\bar{\mu})}$$

$\bar{\mu}$ = AVERAGE OF COSINE OF SCATTERING ANGLE

$$\bar{\mu} = \frac{2}{3A}$$

NEUTRON BALANCE OR CONTINUITY EQUATION

CHANGE = PRODUCTION − LOSS

$$\frac{\partial n}{\partial t} = \text{PRODUCTION} - \text{ABSORBTION} - \text{LEAKAGE}$$

NUMBER OF NEUTRONS IN VOLUME V

$$\int_V n\,dV$$

RATE OF CHANGE

$$\frac{d}{dt}\int_V n\,dV$$

$$\int_V \frac{\partial n}{\partial t}\,dV \quad \text{①}$$

PRODUCTION RATE

$$\int_V s\,dV \quad \text{②}$$

ABSORBTION RATE

$$\int_V \Sigma_a \phi\,dV \quad \text{③}$$

LEAKAGE RATE

$$\int_V J_n\,dA$$

$$\int_V \text{div } J\,dV \quad \text{④}$$

COMBINING THESE INTO A SINGLE EQUATION GIVES THE GENERAL EQUATION OF CONTINUITY

GENERAL EQUATION OF CONTINUITY OR NEUTRON BALANCE EQUATION IS OBTAINED

$$\int_V \frac{\partial n}{\partial t}\,dV = \int_V s\,dV - \int_V \Sigma_a \phi\,dV - \int_V \text{div } J\,dV$$

CANCELLING THE VOLUME INTEGRALS

$$\frac{\partial n}{\partial t} = s - \Sigma_a \phi - \text{div } J$$

IF NEUTRON DENSITY DOES NOT VARY WITH TIME WE GET STEADY STATE EQUATION OF CONTINUITY

$$0 = s - \Sigma_a \phi - \text{div } J$$

$$\text{div } J + \Sigma_a \phi\,dV - s = 0$$

NOTE THAT FROM PREVIOUS EQUATIONS (S.13)

$$J = -D \nabla \phi$$

$$\text{div } J = -D \nabla^2 \phi$$

THIS CAN BE SUBSTITUTED INTO CONTINUITY OR NEUTRON BALANCE EQUATION

DIFFUSION EQUATION

CONTINUITY EQUATION

$$\frac{\partial n}{\partial t} = s - \Sigma_a \phi - \text{div } J$$

FROM PREVIOUS WORK

$$\text{div } J = -D \nabla^2 \phi$$

SUBSTITUTING GIVES DIFFUSION EQUATION

$$\frac{\partial n}{\partial t} = s - \Sigma_a \phi + D \nabla^2 \phi$$

$$D \nabla^2 \phi - \Sigma_a \phi + s = \frac{\partial n}{\partial t}$$

$$D \nabla^2 \phi - \Sigma_a \phi + s = \frac{1}{v}\frac{\partial \phi}{\partial t}$$

IF NEUTRON FLUX DOES NOT VARY WITH TIME WE GET STEADY STATE DIFFUSION EQUATION

$$D \nabla^2 \phi - \Sigma_a \phi + s = 0$$

$$\nabla^2 \phi - \frac{\Sigma_a}{D}\phi = -\frac{s}{D}$$

IF SOURCE TERM IS ZERO

$$\nabla^2 \phi - \frac{\Sigma_a}{D}\phi = 0$$

$$\nabla^2 \phi - \frac{1}{L^2}\phi = 0$$

WHERE $L^2 = \frac{D}{\Sigma_a}$

L = DIFFUSION LENGTH

INFINITE PLANAR SOURCE

STEADY STATE DIFFUSION EQUATION

$$\nabla^2\phi - \frac{1}{L^2}\phi = -\frac{S}{D}$$

FLUX VARIES IN x-DIRECTION ONLY AND

NO SOURCE IN DIFFUSING MEDIUM

$$\frac{d^2\phi}{dx^2} - \frac{1}{L^2}\phi = 0$$

GENERAL SOLUTION TO THIS EQUATION IS

$$\phi = Ae^{-x/L} + Ce^{x/L}$$

SINCE SECOND TERM INCREASES WITHOUT LIMIT

C MUST BE SET EQUAL TO ZERO

$$\phi = Ae^{-x/L}$$

NET FLOW OF NEUTRONS IN POSITIVE

x-DIRECTION AS x APPROACHES ZERO

$$J = \frac{S}{2}$$

FROM FICK'S LAW

$$J = -D\frac{d\phi}{dx}$$

$$= -D\left(-\frac{1}{L}Ae^{-x/L}\right)$$

$$= \frac{DA}{L}e^{-x/L}$$

COMBINING THESE EQUATIONS

$$\frac{S}{2} = \frac{DA}{L}e^{-x/L}$$

IN THE LIMIT AS x APPROACHES ZERO

$$\frac{S}{2} = \frac{DA}{L}$$

$$A = \frac{SL}{2D}$$

SUBSTITUTING GIVES FLUX AT ANY

DISTANCE FROM THE PLANAR SOURCE

$$\phi = Ae^{-x/L}$$

$$\phi = \frac{SL}{2D}e^{-x/L}$$

POINT SOURCE

STEADY STATE DIFFUSION EQUATION

$$\nabla^2\phi - \frac{1}{L^2}\phi = -\frac{S}{D}$$

NO SOURCE IN DIFFUSING MEDIUM

$$\nabla^2\phi - \frac{1}{L^2}\phi = 0$$

FLUX VARIES ONLY IN r-DIRECTION

CONVERTING TO SPHERICAL CO-ORDINATES

$$\frac{1}{r^2}\frac{d}{dr}r^2\frac{d\phi}{dr} - \frac{1}{L^2}\phi = 0$$

USE NEW VARIABLE $\omega = r\phi$

$$\frac{d^2\omega}{dr^2} - \frac{1}{L^2}\omega = 0$$

GENERAL SOLUTION TO THIS EQUATION IS:

$$\omega = Ae^{-r/L} + Ce^{r/L}$$

SUBSTITUTING $\omega = r\phi$

$$\phi = \frac{A}{r}e^{-r/L} + \frac{C}{r}e^{r/L}$$

SINCE SECOND TERM INCREASES WITHOUT LIMIT

C MUST BE SET EQUAL TO ZERO

$$\phi = \frac{A}{r}e^{-r/L}$$

NET FLOW OF NEUTRONS IN POSITIVE

r-DIRECTION AS r APPROACHES ZERO

$$J = \frac{S}{4\pi}$$

FROM FICK'S LAW

$$J = -D\frac{d\phi}{dr}$$

$$= -D\left(-\frac{A}{r}\frac{1}{L}e^{-r/L} - \frac{A}{r^2}e^{-r/L}\right)$$

$$= DA\left(\frac{1}{rL} + \frac{1}{r^2}\right)e^{-r/L}$$

COMBINING THESE EQUATIONS

$$\frac{S}{4\pi} = DA\left(\frac{1}{rL} + \frac{1}{r^2}\right)e^{-r/L}$$

IN THE LIMIT AS r APPROACHES ZERO

$$\frac{S}{4\pi} = DA$$

$$A = \frac{S}{4\pi D}$$

SUBSTITUTING GIVES FLUX AT ANY

DISTANCE FROM POINT SOURCE

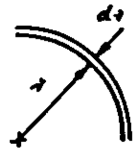
$$\phi = \frac{A}{r}e^{-r/L}$$

$$\phi = \frac{S}{4\pi Dr}e^{-r/L}$$

DIFFUSION LENGTH

POINT SOURCE NEUTRON FLUX

$$\phi_{(r)} = \frac{S e^{-r/L}}{4\pi D r}$$



NEUTRONS ABSORBED IN SHELL

$$dn = \Sigma_a \phi_{(r)} dV$$

SUBSTITUTING FOR $\phi_{(r)}$ AND dV

$$dn = \Sigma_a \left(\frac{S e^{-r/L}}{4\pi D r}\right)\left(4\pi r^2 dr\right)$$

$$= \Sigma_a \left(\frac{S}{D}\right) r e^{-r/L} dr$$

SUBSTITUTING DIFFUSION LENGTH $L^2 = \frac{\Sigma_a}{D}$

$$dn = \frac{S}{L^2} r e^{-r/L} dr$$

PROBABILITY OF ABSORPTION IN SHELL

$$p_{(r)} dr = \frac{1}{L^2} r e^{-r/L} dr$$

AVERAGE DISTANCE FROM SOURCE TO ABSORPTION

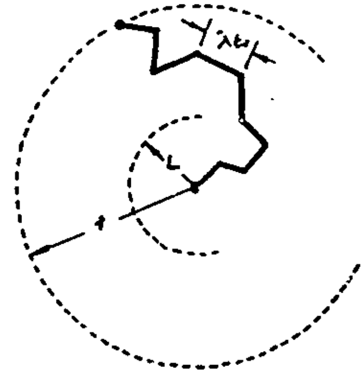
$$\bar{r}^2 = \int_0^\infty r^2 p_{(r)} dr$$

$$= \int_0^\infty \frac{1}{L^2} r^3 e^{-r/L} dr$$

$$= 6 L^2$$

$$L^2 = \frac{1}{6} \bar{r}^2$$

NEUTRON DIFFUSION



$$r^2 = 6 L^2 \qquad r = 2.45 L$$

$$L^2 = D/\Sigma_a$$

$$D = \lambda_{tr}/3$$

$$\lambda_{tr} = 1/\Sigma_{tr}$$

$$\Sigma_{tr} = \Sigma_s(1 - \bar{\mu})$$

$$\bar{\mu} = 2/3A$$

GROUP DIFFUSION METHOD

NEUTRON FLUX IN GROUP g

$$\phi_g = \int_g \phi_{(E)} dE$$

ABSORPTION RATE IN GROUP g

$$R_g = \int_g \Sigma_a(E)\, \phi(E)\, dE$$

$$= \Sigma_{ag} \phi_g$$

TRANSFER RATE OUT OF GROUP g

$$T_{g-h} = \Sigma_{g-h} \phi_g$$

TOTAL TRANSFER RATE

$$T_{g-h_{TOTAL}} = \sum_{h=g+1}^{N} \Sigma_{g-h} \phi_g$$

TRANSFER RATE INTO GROUP g

$$T_{h-g} = \Sigma_{h-g} \phi_h$$

TOTAL TRANSFER RATE

$$T_{h-g_{TOTAL}} = \sum_{h=1}^{g-1} \Sigma_{h-g} \phi_h$$

STEADY STATE DIFFUSION EQUATION

$$D_g \nabla^2 \phi_g - \Sigma_{ag} \phi_g - \sum_{h=g+1}^{N} \Sigma_{g-h} \phi_g + \sum_{h=1}^{g-1} \Sigma_{h-g} \phi_h = -S_g$$

FOR SINGLE GROUP

$$D \nabla^2 \phi - \Sigma_a \phi = -S$$

TWO - GROUP CALCULATION

FAST NEUTRONS ⟶ THERMAL NEUTRONS

FOR FAST NEUTRONS DIFFUSION EQUATION IS :

$$D_1 \nabla^2 \phi_1 - \Sigma_1 \phi_1 = 0$$

SLOWING DOWN DENSITY (FAST ⟶ SLOW)

$$q_T = \Sigma_1 \phi_1$$

FOR SLOW NEUTRONS DIFFUSION EQUATION WITH SOURCE EQUAL TO q_T IS :

$$D \nabla^2 \phi_T - \frac{D}{L_T^2} \phi_T = \Sigma_1 \phi_1$$

$$\nabla^2 \phi_T - \frac{1}{L_T} \phi_T = \frac{\Sigma_1 \phi_1}{D}$$

SOLUTION OF THIS EQUATION IS :

$$\phi_T = \frac{S L_T^2}{4\pi r D (L_T^2 - \tau_T)} \left(e^{-r/L_T} - e^{-r/\sqrt{\tau_T}}\right)$$

WHERE NEUTRON AGE τ_T IS DEFINED AS

$$\tau_T = \frac{D_1}{\Sigma_1}$$

$$\tau_T = \frac{1}{6} \bar{r}^2$$