Mathematics - Course 121

CONFIDENCE LIMITS

Confidence

1. The estimates of the mean time to failure, or the mean wearout life used in the course are point estimates of the true unknown parameters. How much confidence can we have in these figures? We know that statistical estimates are more likely to be close to the true values as the sample size increases; thus there is a close connection between the accuracy of an estimate and the size of the sample from which it was obtained. Only an infinitely large sample size could give a certainty (ie, 100% confidence) that a measured statistical parameter coincides with the true value.

2. Accuracy

   To give an indication of the accuracy of an estimate, we can make a statement like

   "In 9 cases out of 10, the answer will lie between (z_1-a) and (z_1+a)", which can also be expressed:

   "At 90% confidence level, the upper confidence limit is (z_1+a) and the lower confidence limit is (z_1-a)". The confidence interval is therefore (z_1-a) to (z_1+a).

3. Evaluation

   How can these limits be evaluated? Let us assume that we have taken many samples, each of size n and have calculated many sample means m. We would expect the mean of a sample to fall closer to the population mean than, on average, any single observation would. In other words, we would expect sample means to have a smaller spread than single observations. If σ is the standard deviation (sd) of the population, it can be shown that the variance of sample means, σ_m^2, is given by:

   $$\sigma_m^2 = \frac{\sigma^2}{n}$$

   or the sd, $\sigma_m = \frac{\sigma}{\sqrt{n}}$

It is a general rule in statistics that the probability density function (pdf) of sample means drawn from any population approximates to the Normal pdf (see lesson 121.00-7); the larger the sample size, the closer the approximation. This is the principal reason for the importance of the Normal pdf, since in practice we usually work with sample mean parameters.

4. <u>Some Properties of the Normal pdf Recalled from Lesson 121.00-7</u>

Figure 1: Double Sided

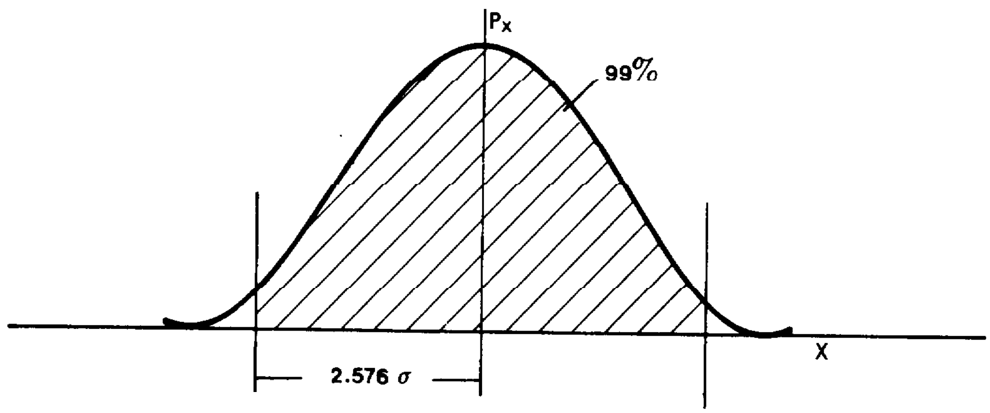


Figure 2: Double Sided

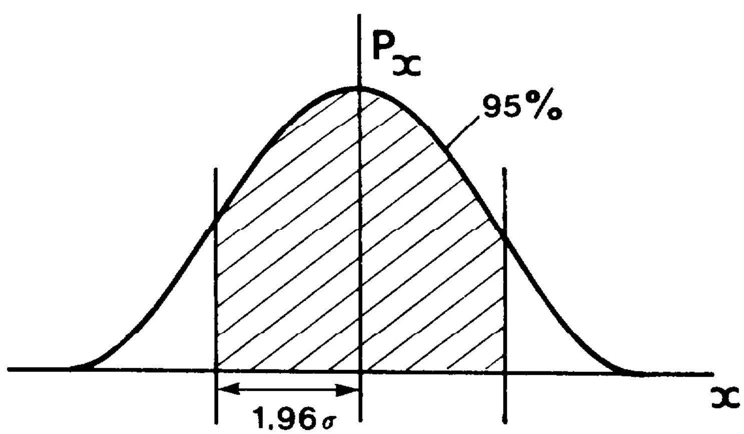


Figure 3: Double-sided

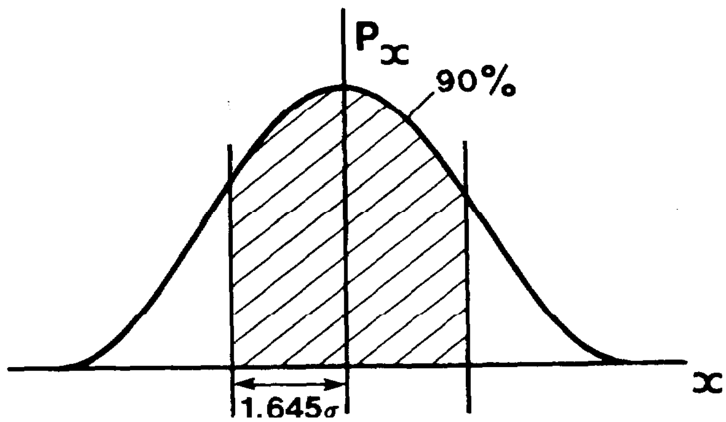


Figure 4: Single-Sided

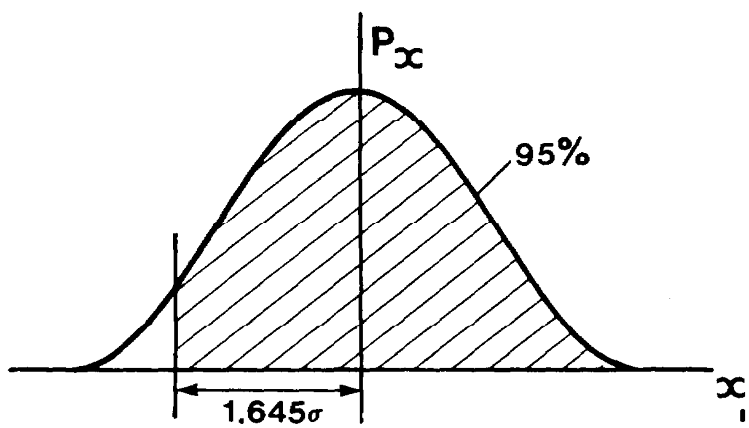
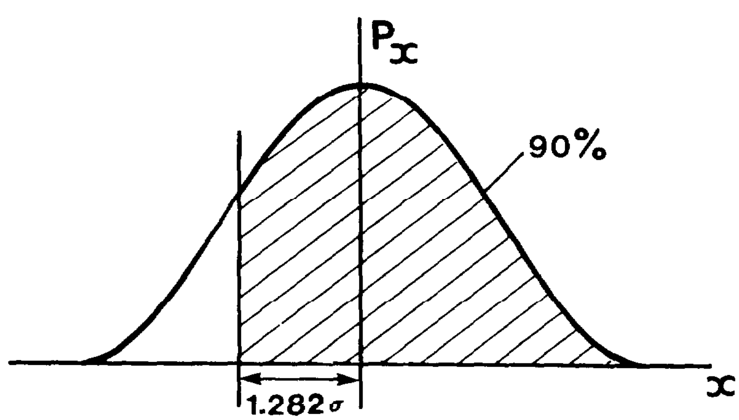


Figure 5: Single-sided



Recall from 121.00-7 the transformation

$$Z = \frac{x-m}{\sigma}$$

so if we plot out the 'm' and transform to the Standardized Normal Distribution, we can write:

$$Z = \frac{m-\mu}{\frac{\sigma}{\sqrt{n}}}$$

where m ≡ sample mean
μ ≡ population mean
σ ≡ sd of sample
n ≡ sample size

eg, F(Z) = 0.95 is the Probability that $\frac{m-\mu}{\frac{\sigma}{\sqrt{n}}}$

-1.96 and +1.96 (Figure 2)

∴ we can say that the probability that the interval

$\left(m - \frac{1.96\sigma}{\sqrt{n}}\right)$ and $\left(m + \frac{1.96\sigma}{\sqrt{n}}\right)$ contains the true value μ is 0.95

Likewise, there is a 99% probability that the interval

$\left(m - \frac{2.576\sigma}{\sqrt{n}}\right)$ and $\left(m + \frac{2.567\sigma}{\sqrt{n}}\right.$ contains the true value μ (Figure 1)

<u>Example</u>

(a) Given a sample of 100 lamps with normally distributed lifetimes, a mean life of 1000 hours and a sd of 81 hours, calculate the 95% two-sided confidence interval for population mean life μ.

The 95% two-sided confidence interval for the population mean life μ is:

$$\left(m - \frac{1.96\sigma}{\sqrt{n}}\right) \text{ to } \left(m + \frac{1.96\sigma}{\sqrt{n}}\right.$$

$$= \left(1000 - \frac{1.96 \times 81}{\sqrt{100}}\right. \text{ to } \left(1000 + \frac{1.96 \times 81}{\sqrt{100}}\right.$$

= 984.1 to 1015.9 hours

(b) If the sample were 1000 lamps, with mean 1000 hours and sd 81 hours, the 95% two-sided confidence interval for the population mean life would be:

$$(m - \frac{1.96\sigma}{\sqrt{n}}) \text{ to } (m + \frac{1.96\sigma}{\sqrt{n}})$$

= 1000 ± 5.02 hours

= 995 to 1005 hours

(c) If the sample were 100 lamps with mean life 1000 hours and sd 81 hours, the 99% two-sided confidence interval for the population mean life would be:

$$(m - \frac{2.576\sigma}{\sqrt{n}}) \text{ to } (m + \frac{2.576\sigma}{\sqrt{n}})$$

= 1000 ± 20.9 hours

= 979.1 to 1020.9 hours

This example illustrates that:

(i) For a given sample size, the confidence interval will decrease as the confidence level is lowered, and vice versa.

(ii) For a given confidence level, the confidence interval will decrease with increased sample size.

5. Sampling From the Exponential Distribution

In order to predict confidence limits from an exponential distribution, it is necessary to invoke the complexity of the χ^2 (Chi-squared) distribution method.

There are slightly different approaches to the use of the χ^2 distribution in estimating confidence limits; here we shall use the method adopted by Billinton[1], which states that the true mean time to failure is equal to or greater than m where

$$m = \frac{2 \times \text{Total test time}}{\chi^2_{(P,\ 2r+2)}}$$

where $P \equiv$ confidence level
$r \equiv$ no. of failures

Example 1 from 121.00-5

(a) Consider 1 component experiencing 3 failures over 1 year's operation. The basic arithmetic MTTF $= \frac{365}{3} =$ 121.7 days.

But at 95% confidence level, we can say that the true MTTF is equal to or greater than m where:

$$m = \frac{2 \times 365}{\chi^2_{(.95,8)}}$$

From Table 1, $\chi^2_{(.95,8)} = 15.5$

$$\therefore \quad m = \frac{2 \times 365}{15.5} = \underline{47.1 \text{ days}}$$

(b) Now consider 1 component experiencing 30 failures over 10 years' operation. The basic arithmetic MTTF $= \frac{365 \times 10}{30} = 121.7$ days

But at 95% confidence level, we can say that the true MTTF is equal to or greater than m where:

$$m = \frac{2 \times 365 \times 10}{\chi^2_{(.95,62)}} = \frac{2 \times 365 \times 10}{80} \approx 91 \text{ days}$$

And if we consider the confidence interval to be (m_{min} to ∞) we can see that by increasing the size of the sample, the confidence interval has been reduced.

How much confidence can we have in the result of a calculation which has assumed a MTTF of 365/3 = 121.7 days? Using the χ^2 approach, and disregarding the effect of the variations in the times to fault discovery, we can ascribe confidence in the answer to be the confidence in the input data.

$$m = \frac{2 \times \text{Total test time}}{\chi^2_{(P,\ 2r + 2)}}$$

$$\therefore \quad \chi^2 (P,\ 2r + 2) = \frac{2 \times \text{Total test time}}{m}$$

$$\therefore \quad \chi^2 (P,\ 8) = \frac{2 \times 365}{121.7} \approx 6$$

From table 1, $\underline{\underline{P \approx 30\%}}$

And if we had 30 failures over 10 years,

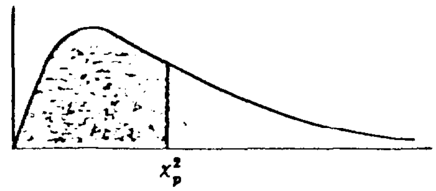
$$\chi^2_{(P,\ 62)} = \frac{2 \times 365 \times 10}{121.7} \simeq 60$$

$\therefore$ $\underline{\underline{P \simeq 50\%}}$ (from Table I)

So that even using the same value of MTTF, the increased sample size has increased our confidence in the result.

Note 1: Power System Reliability Evaluation by Professor Roy Billinton ISBN 0677028709.

TABLE A: Percentile Values (χ^2_p) for the Chi-Square Distribution with v degrees of freedon = 2r +2 (shaded area = p)



ν	$\chi^2_{.995}$	$\chi^2_{.99}$	$\chi^2_{.975}$	$\chi^2_{.95}$	$\chi^2_{.90}$	$\chi^2_{.75}$	$\chi^2_{.50}$	$\chi^2_{.25}$	$\chi^2_{.10}$	$\chi^2_{.05}$	$\chi^2_{.025}$	$\chi^2_{.01}$	$\chi^2_{.005}$
1	7.88	6.63	5.02	3.84	2.71	1.32	.455	.102	.0158	.0039	.0010	.0002	.0000
2	10.6	9.21	7.38	5.99	4.61	2.77	1.39	.575	.211	.103	.0506	.0201	.0100
3	12.8	11.3	9.35	7.81	6.25	4.11	2.37	1.21	.584	.352	.216	.115	.072
4	14.9	13.3	11.1	9.49	7.78	5.39	3.36	1.92	1.06	.711	.484	.297	.207
5	16.7	15.1	12.8	11.1	9.24	6.63	4.35	2.67	1.61	1.15	.831	.554	.412
6	18.5	16.8	14.4	12.6	10.6	7.84	5.35	3.45	2.20	1.64	1.24	.872	.676
7	20.3	18.5	16.0	14.1	12.0	9.04	6.35	4.25	2.83	2.17	1.69	1.24	.989
8	22.0	20.1	17.5	15.5	13.4	10.2	7.34	5.07	3.49	2.73	2.18	1.65	1.34
9	23.6	21.7	19.0	16.9	14.7	11.4	8.34	5.90	4.17	3.33	2.70	2.09	1.73
10	25.2	23.2	20.5	18.3	16.0	12.5	9.34	6.74	4.87	3.94	3.25	2.56	2.16
11	26.8	24.7	21.9	19.7	17.3	13.7	10.3	7.58	5.58	4.57	3.82	3.05	2.60
12	28.3	26.2	23.3	21.0	18.5	14.8	11.3	8.44	6.30	5.23	4.40	3.57	3.07
13	29.8	27.7	24.7	22.4	19.8	16.0	12.3	9.30	7.04	5.89	5.01	4.11	3.57
14	31.3	29.1	26.1	23.7	21.1	17.1	13.3	10.2	7.79	6.57	5.63	4.66	4.07
15	32.8	30.6	27.5	25.0	22.3	18.2	14.3	11.0	8.55	7.26	6.26	5.23	4.60
16	34.3	32.0	28.8	26.3	23.5	19.4	15.3	11.9	9.31	7.96	6.91	5.81	5.14
17	35.7	33.4	30.2	27.6	24.8	20.5	16.3	12.8	10.1	8.67	7.56	6.41	5.70
18	37.2	34.8	31.5	28.9	26.0	21.6	17.3	13.7	10.9	9.39	8.23	7.01	6.26
19	38.6	36.2	32.9	30.1	27.2	22.7	18.3	14.6	11.7	10.1	8.91	7.63	6.84
20	40.0	37.6	34.2	31.4	28.4	23.8	19.3	15.5	12.4	10.9	9.59	8.26	7.43
21	41.4	38.9	35.5	32.7	29.6	24.9	20.3	16.3	13.2	11.6	10.3	8.90	8.03
22	42.8	40.3	36.8	33.9	30.8	26.0	21.3	17.2	14.0	12.3	11.0	9.54	8.64
23	44.2	41.6	38.1	35.2	32.0	27.1	22.3	18.1	14.8	13.1	11.7	10.2	9.26
24	45.6	43.0	39.4	36.4	33.2	28.2	23.3	19.0	15.7	13.8	12.4	10.9	9.89
25	46.9	44.3	40.6	37.7	34.4	29.3	24.3	19.9	16.5	14.6	13.1	11.5	10.5
26	48.3	45.6	41.9	38.9	35.6	30.4	25.3	20.8	17.3	15.4	13.8	12.2	11.2
27	49.6	47.0	43.2	40.1	36.7	31.5	26.3	21.7	18.1	16.2	14.6	12.9	11.8
28	51.0	48.3	44.5	41.3	37.9	32.6	27.3	22.7	18.9	16.9	15.3	13.6	12.5
29	52.3	49.6	45.7	42.6	39.1	33.7	28.3	23.6	19.8	17.7	16.0	14.3	13.1
30	53.7	50.9	47.0	43.8	40.3	34.8	29.3	24.5	20.6	18.5	16.8	15.0	13.8
40	66.8	63.7	59.3	55.8	51.8	45.6	39.3	33.7	29.1	26.5	24.4	22.2	20.7
50	79.5	76.2	71.4	67.5	63.2	56.3	49.3	42.9	37.7	34.8	32.4	29.7	28.0
60	92.0	88.4	83.3	79.1	74.4	67.0	59.3	52.3	46.5	43.2	40.5	37.5	35.5
70	104.2	100.4	95.0	90.5	85.5	77.6	69.3	61.7	55.3	51.7	48.8	45.4	43.3
80	116.3	112.3	106.6	101.9	96.6	88.1	79.3	71.1	64.3	60.4	57.2	53.5	51.2
90	128.3	124.1	118.1	113.1	107.6	98.6	89.3	80.6	73.3	69.1	65.6	61.8	59.2
100	140.2	135.8	129.6	124.3	118.5	109.1	99.3	90.1	82.4	77.9	74.2	70.1	67.3

Source: Catherine M. Thompson, Table of Percentage Points of the χ^2 distribution, Biometrika, Vol. 32 (1941).

R.B. Malcolm