Mathematics - Course 121

THE NORMAL DISTRIBUTION AND APPLICATIONS

I THE NORMAL DISTRIBUTION

If a variable x is normally distributed, then the probability that x lies within the interval (x, x + dx) is $P_x d_x$, where

$$P_x = \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$

where μ is the mean of the distribution - the expectation value of x, and

σ is the *standard deviation* - the square root of the expectation value of $(x - \mu)^2$.

The Normal Distribution is shown in Figure 1. Note that $P_x d_x$ is an incremental slice of the area under the curve representing P_x as a function of x. Thus the probability that x is less than "a",

$$P(x < a) = \int_{-\infty}^{a} P_x dx$$

ie, the sum of all incremental slices from $x = -\infty$ to $x = a$. Obviously, the total area under the curve,

$$\int_{-\infty}^{\infty} P_x dx = 1,$$

since x must take some value.

II STANDARDIZED NORMAL DISTRIBUTION

The transformation

$$z = \frac{x - \mu}{\sigma},$$

transforms the preceding expression for the Normal Distribution into the so called Standardized Normal Distribution

$$P_z = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$$

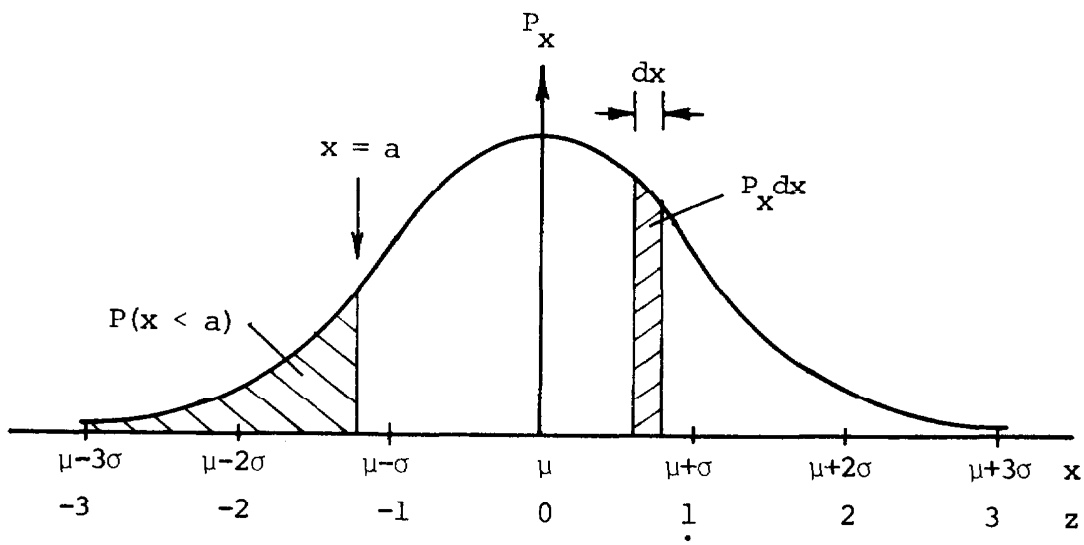


Figure 1

The Normal Distribution

Figure 1 shows corresponding values of x and z along the abscissa. The curve of Figure 1 can represent either P_x vs x or P_z vs z, providing vertical and horizontal Normal Distribution has a mean of zero and standard deviation of 1.

The advantage of transferring to the Standardized Normal Distribution is that one and the same table of values of

$$F(a) \equiv P(z < a) = \int_{-\infty}^{a} P_z dz$$

can be used when working with any normally distributed variable. Table 1 is a table of F(a) versus a.

Notes re Use of Table 1

1. F(a) is not tabulated for a < 0 since, due to the symmetry of the distribution,

   $$F(-a) = 1 - F(a)$$

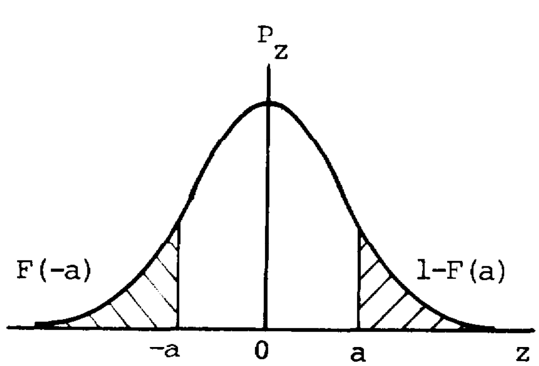
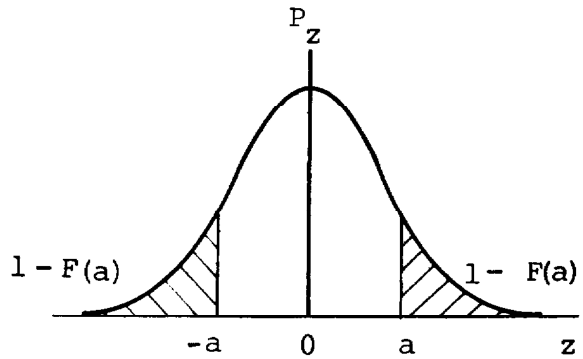


TABLE 1

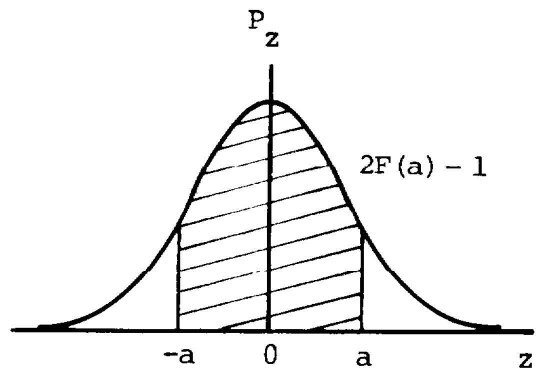
NORMAL DISTRIBUTION TABLE

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998

2. $P(|z - a| > 0) = 2(1 - F(a))$



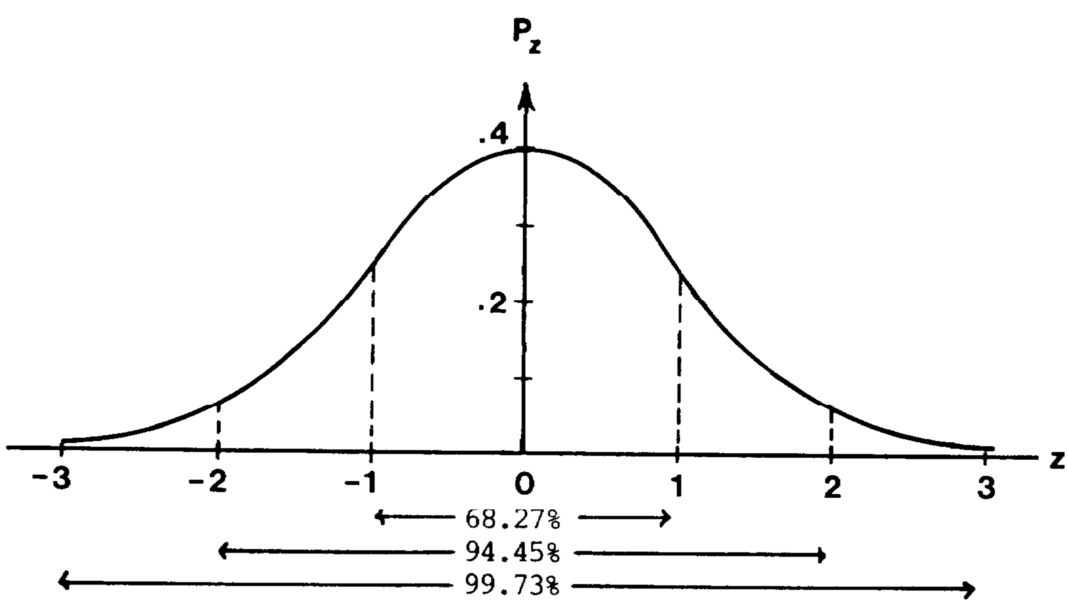
3. $P(-a < z < a) = 2F(a) - 1$



4. a) $P(-1 < z < 1) = 0.6827$

b) $P(-2 < z < 2) = 0.9445$

c) $P(-3 < z < 3) = 0.9973$



Example 1

Suppose the distribution of the diameters of the ball bearings in a certain shipment is approximately normal with the mean μ = 0.500 cm and the standard deviation σ = 0.010 cm. If a ball bearing is effective when its diameter lies between 0.490 and 0.515, find the probability of finding an effective ball bearing in this shipment.

Solution

$$P(0.490 < x < 0.515)$$

$$= P\left(\frac{0.490 - 0.500}{0.010} < z < \frac{0.515 - 0.500}{0.010}\right)$$

$$= P(-1.00 < z < 1.50)$$

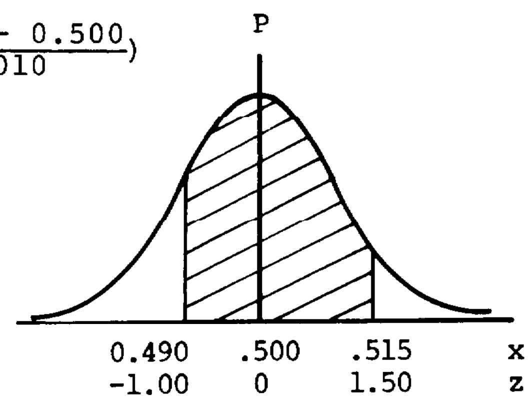
$$= F(1.50) - F(-1.00)$$

$$= F(1.50) - (1 - F(1.00))$$

$$= F(1.50) + F(1.00) - 1$$

$$= 0.9332 + 0.8413 - 1$$

$$= 0.7745$$



ie, approximately 77% of the ball bearings are effective, and this is the probability that any ball bearing selected at random will be effective.

III FAILURE DISTRIBUTION FUNCTION

DEFINITION

The **failure distribution function** (also called the failure density function) f(t) of a component specifies the time-dependent distribution of probability of failure over the operating life of the component, ie, f(t)dt is the probability that the component will fail in the time interval (t,t+dt) - see Figure 2.

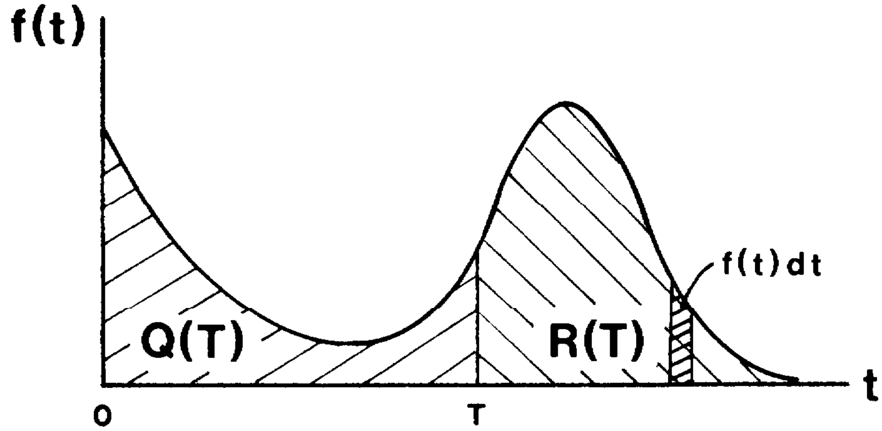


Figure 2

Hypothetical Failure Distribution Function

With reference to Figure 2, the probability Q(T) that the component has failed by time T, ie, the component unreliability at time T, is given by the area under the f(t) curve from t = 0 to t = T,

ie, $Q(T) = \int_0^T f(t)dt$.

Similarly, the component reliability at time T is given by

$R(T) = \int_T^\infty f(t)dt$.

Example 2

The lifetime of a certain type of light bulb is normally distributed with mean 1000 hours and standard deviation 200 hours. If 2000 such bulbs are installed at time zero, calculate:

a) the expected number of bulb failures during the first 700 hours

b) the expected number of failures between 900 hours and 1300 hours

c) the time by which the first 10% of the bulbs have failed.

Solution

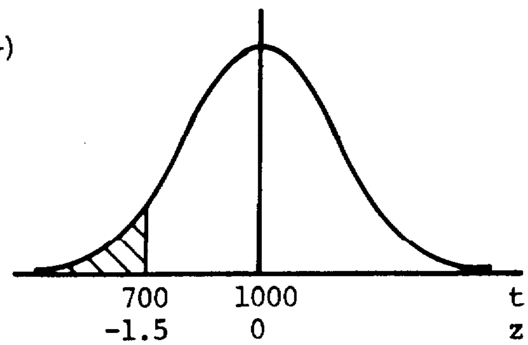
a) $P(t < 700) = P(z < \frac{700 - 1000}{200})$

$= P(z < -1.5)$

$= F(-1.5)$

$= 1 - F(1.5)$

$= 0.0668$



$\therefore$ Expected number of bulb failures in the interval (0,700h)

$= 2000 \times 0.0668$

$= \underline{\underline{134}}$

b) $P(900 \leqslant t \leqslant 1300) = P(\frac{900 - 1000}{200} \leqslant z \leqslant \frac{1300 - 1000}{200})$

$= P(-0.5 \leqslant z \leqslant 1.5)$

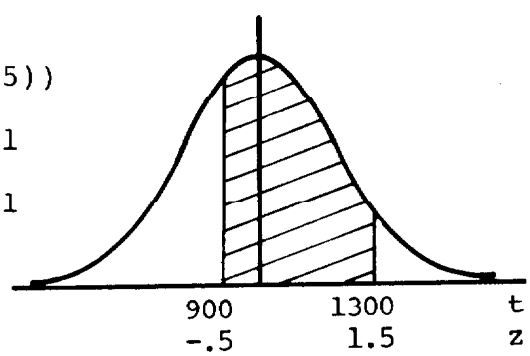
$= F(1.5) - F(-0.5)$

$= F(1.5) - (1 - F(0.5))$

$= F(1.5) + F(0.5) - 1$

$= 0.9332 + 0.6915 - 1$

$= 0.6247$



$\therefore$ Expected number of failures in the interval (900, 1300h)

$= 2000 \times 0.6247$

$= \underline{\underline{1250}}$

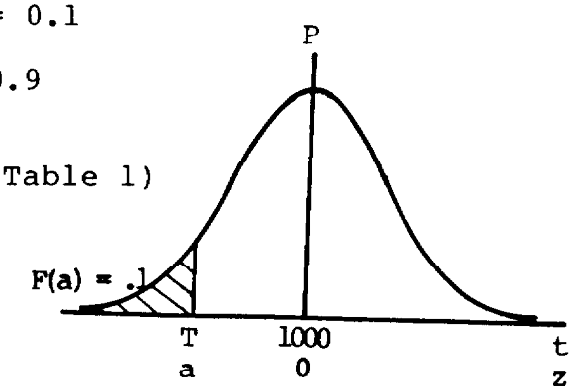
c) $F(a) = 0.1 \Rightarrow 1 - F(-a) = 0.1$

ie, $F(-a) = 0.9$

$\therefore$ $-a = 1.28$ (from Table 1)

ie, $\dfrac{T - 1000}{200} = -1.28$

ie, $T = 744$



$\therefore$ first 10% bulbs expected to fail during the interval (0,744h).

ASSIGNMENT

1. Specifications for a certain job call for washers with an inside diameter of 0.250 $\pm$ 0.005 inches. If the inside diameter of the washers made by a given manufacturer are normally distributed with $\mu = 0.251$ and $\sigma = 0.003$, what percentage of these washers will meet specifications?

2. A study showed that the lifetimes of a certain kind of automobile battery are normally distributed with a mean of 1248 days and a standard deviation of 185 days. If the manufacturer wishes to guarantee the battery for 36 months (a month is taken to be 30 days), what percentage of the batteries will have to be replaced under the guarantee?

L. Haacke